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**A COUNTEREXAMPLE TO  
SMOOTH LEAFWISE HODGE DECOMPOSITION  
FOR GENERAL FOLIATIONS AND  
TO A TYPE OF DYNAMICAL TRACE FORMULAS**

by C. DENINGER and W. SINGHOF

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**0. Introduction.**

For a smooth foliation  $\mathcal{F}$  on a closed Riemannian manifold  $X$  consider the de Rham complex

$$(\Gamma(X, \Lambda^\bullet T^* \mathcal{F}), d_0)$$

of forms along the leaves. It is a complex of Fréchet spaces. The Riemannian metric defines a canonical scalar product on  $\Gamma(X, \Lambda^\bullet T^* \mathcal{F})$  and we denote by

$$\Delta_0 = d_0 d_0^* + d_0^* d_0$$

the associated Laplace operator.

The cohomology groups

$$H^\bullet(\mathcal{F}) = H^\bullet(\Gamma(X, \Lambda^\bullet T^* \mathcal{F}), d_0)$$

are called the leafwise cohomology of  $\mathcal{F}$ . The maximal Hausdorff quotient

$$\overline{H}^\bullet(\mathcal{F}) = \text{Ker } d_0 / \overline{\text{Im } d_0}$$

is the reduced leafwise cohomology of  $\mathcal{F}$ .

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Let

$$(\Gamma_{0,\infty}(X, \Lambda^\bullet T^* \mathcal{F}), d_0)$$

be the complex of forms in the  $L^2$ -completion of  $\Gamma(X, \Lambda^\bullet T^* \mathcal{F})$  whose leafwise derivatives of any order exist and are also in  $L^2$ . Then  $\Gamma_{0,\infty}(X, \Lambda^\bullet T^* \mathcal{F})$  is a Fréchet space and  $d_0, d_0^*, \Delta_0$  extend canonically to operators  $d_{0,\infty}, d_{0,\infty}^*, \Delta_{0,\infty}$ . In [AT] Theorems A, B, Álvarez López and Tondeur prove the following  $L^2$ -Hodge decomposition:

THEOREM 0.1 ([AT]).

$$\Gamma_{0,\infty}(X, \Lambda^\bullet T^* \mathcal{F}) = \text{Ker } \Delta_{0,\infty} \oplus \overline{\text{Im } d_{0,\infty}} \oplus \overline{\text{Im } d_{0,\infty}^*}.$$

One of the ingredients in the proof, [AT] Theorem B, is stated only for Riemannian foliations and bundle like metrics in [AT]. It was pointed out to us by Álvarez López however that Theorem B remains true in general: For the argument, instead of  $\Delta_0$ , one uses the Laplace operator  $\Delta_{\mathcal{F}}$  whose restriction to any leaf is the Laplace operator on that leaf. For bundle like metrics we have  $\Delta_0 = \Delta_{\mathcal{F}}$ . In general  $\Delta_0$  and  $\Delta_{\mathcal{F}}$  differ by an operator of order at most one which is enough for the purpose.

In fact this kind of Hodge decomposition holds for any leafwise elliptic complex if one takes adjoints on the ambient manifold and not on the leaves.

Under a certain technical condition on the foliation, Álvarez López and Kordyukov, [AK1] 1.1, have been able to prove a Hodge decomposition theorem for smooth forms on  $X$ . This is quite surprising since  $\Delta_0$  is only elliptic along the leaves. The main application is to Riemannian foliations where they obtain for example the following smooth Hodge decomposition:

THEOREM 0.2 ([AK1], 1.2). — *Assume the foliation is Riemannian and the metric bundle like. Then one has*

$$\Gamma(X, \Lambda^\bullet T^* \mathcal{F}) = \text{Ker } \Delta_0 \oplus \overline{\text{Im } d_0} \oplus \overline{\text{Im } d_0^*}.$$

*In particular there is an isomorphism*

$$\overline{H}^\bullet(\mathcal{F}) = \text{Ker } \Delta_0.$$

The first purpose of our note is to show that this result does not remain valid for general foliations. More precisely we construct a one-dimensional foliation on a three-dimensional Heisenberg nilmanifold for which smooth leafwise Hodge decomposition fails. As a second application of our example we show that a certain type of dynamical trace formula

for flows on foliated manifolds discussed in [G], [P], [D1], [D2] for example cannot hold in complete generality. Apparently a different example for this phenomenon was found earlier by V. Guillemin but not published.

We would like to thank J. Álvarez López, A. Deitmar and R. Weis-sauer for helpful remarks.

### 1. The example.

Let  $G$  be the 3-dimensional simply connected real Heisenberg group and let  $\Gamma$  be its standard lattice. Concretely  $G$  can be realized as the subgroup of matrices

$$[x, y, z] = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \text{ in } \text{GL}_3(\mathbb{R}).$$

Then  $\Gamma$  consists of those matrices  $[n, m, k]$  with integer entries. The quotient  $X = \Gamma \backslash G$  is a compact manifold.

We can identify the Lie algebra  $\mathfrak{g}$  of  $G$  with the Lie algebra of matrices

$$[[a, b, c]] = \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \text{ in } M_3(\mathbb{R}).$$

For an immersed subgroup  $P$  of  $G$  with Lie algebra  $\mathfrak{p}$ , consider the foliation  $\mathcal{F}$  of  $X$  with leaves  $\Gamma gP, g \in G$ . For the leafwise cohomology of  $\mathcal{F}$  we have a natural isomorphism as topological vector spaces:

$$(1) \quad H^i(\mathcal{F}) = H^i(\mathfrak{p}, C^\infty(X)).$$

Namely let  $L = \mathfrak{p} \otimes C^\infty(X)$  be the Lie algebra of vector fields tangent to the leaves. Then

$$(2) \quad \Gamma(X, \Lambda^* T^* \mathcal{F}) = \text{Hom}_{C^\infty(X)}(\Lambda^*_{C^\infty(X)}(L), C^\infty(X))$$

as complexes where the differential on the right is given by the formula (3A.5) of [MS], Ch. III, Appendix.

Since

$$(3) \quad \begin{aligned} \text{Hom}_{C^\infty(X)}(\Lambda^*_{C^\infty(X)}(L), C^\infty(X)) &= \text{Hom}_{C^\infty(X)}(\Lambda^* \mathfrak{p} \otimes C^\infty(X), C^\infty(X)) \\ &= \text{Hom}_{\mathbb{R}}(\Lambda^* \mathfrak{p}, C^\infty(X)) \end{aligned}$$

as complexes, formula (1) follows.

The “regular” unitary representation  $R$  of  $G$  on the complex Hilbert space  $L^2(X)$  given by right translation

$$((Rg)f)(\Gamma g') = f(\Gamma g'g)$$

decomposes as follows:

$$(4) \quad L^2(X) = \widehat{\bigoplus_{\chi} \mathbb{C}_{\chi}} \oplus \widehat{\bigoplus_{m \in \mathbb{Z} \setminus 0} |m| U_m}.$$

Here  $\chi$  runs over those characters of  $G$  which factor over  $\Gamma^{\text{ab}} \setminus G^{\text{ab}}$  and  $U_m$  is the (induced) representation of  $G$  on  $L^2(\mathbb{R})$  given by the formula

$$(5) \quad ((U_m[x, y, z]f)(\tau) = \exp(2\pi im(z + \tau y)) \cdot f(x + \tau).$$

This is a classical result. The general theory is described in [H] or [R1]. For the smooth vectors in these representations we have

$$L^2(X)^{\infty} = C^{\infty}(X, \mathbb{C})$$

and

$$(L^2(\mathbb{R}), U_m)^{\infty} = \mathcal{S}(\mathbb{R})_m := U_m \text{ restricted to the Schwartz space } \mathcal{S}(\mathbb{R}) \\ \text{of (complex valued) rapidly decreasing} \\ \text{functions on } \mathbb{R}.$$

(See [Go], p. 65.) By an elementary argument we now get:

LEMMA 1.1. — *The algebraic direct sum*

$$\bigoplus_{\chi} H^i(\mathfrak{p}, \mathbb{C}_{\chi}) \oplus \bigoplus_{m \in \mathbb{Z} \setminus 0} |m| \overline{H}^i(\mathfrak{p}, \mathcal{S}(\mathbb{R})_m)$$

is a dense subspace of

$$\overline{H}^i(\mathcal{F}, \mathbb{C}) = \overline{H}^i(\mathcal{F}) \otimes_{\mathbb{R}} \mathbb{C},$$

the reduced cohomology of  $\mathcal{F}$  with complex coefficients. Here  $\overline{H}^i(\mathfrak{p}, \mathcal{S}(\mathbb{R})_m)$  denotes the maximal Hausdorff quotient of  $H^i(\mathfrak{p}, \mathcal{S}(\mathbb{R})_m)$ .

Now assume that  $P$  is one-dimensional and such that  $\mathcal{F}$  has dense leaves. Equivalently, the induced linear foliation on the torus  $\Gamma^{\text{ab}} \setminus G^{\text{ab}}$  should have dense leaves, [AGH], Ch. IV. In terms of a fixed generator  $v = [|\alpha, \beta, \gamma|]$  of  $\mathfrak{p}$  this means that  $\alpha$  and  $\beta$  should be  $\mathbb{Q}$ -linearly independent. Then we have

$$(6) \quad \bigoplus_{\chi} H^1(\mathfrak{p}, \mathbb{C}_{\chi}) = H^1(\mathfrak{p}, \mathbb{C}) = \mathbb{C}.$$

Moreover [R2] Theorem (4.2), case 2B and its proof imply that

$$(7) \quad \overline{H}^1(\mathfrak{p}, \mathcal{S}(\mathbb{R})_m) = H^1(\mathfrak{p}, \mathcal{S}(\mathbb{R})_m) \cong \mathbb{C}.$$

We recall the argument since an explicit description of the isomorphism (7) is required in the next section.

The action of  $v \in \mathfrak{p}$  on  $\mathcal{S}_m(\mathbb{R})$  is given by the formula

$$(v \cdot f)(\tau) = \lim_{t \rightarrow 0} \frac{1}{t} ((U_m(\exp tv)f)(\tau) - f(\tau)).$$

Since

$$\exp(tv) = \left[ t\alpha, t\beta, t\gamma + \frac{1}{2}t^2\alpha\beta \right]$$

we find

$$(8) \quad v \cdot f = \Theta_m(f)$$

with the first order differential operator on  $\mathcal{S}(\mathbb{R})$ :

$$\Theta_m = 2\pi im(\gamma + \tau\beta) + \alpha \frac{d}{d\tau}.$$

Set

$$A_m(\tau) = \exp\left(\frac{2\pi im}{\alpha}(\tau\gamma + \frac{1}{2}\tau^2\beta)\right)$$

and let  $\mathfrak{F}$  be the Fourier transform on  $\mathcal{S}(\mathbb{R})$ . Then the following diagram commutes:

$$(9) \quad \begin{array}{ccc} \mathcal{S}(\mathbb{R}) & \xrightarrow{\Theta_m} & \mathcal{S}(\mathbb{R}) \\ \cdot A_m \downarrow \wr & & \downarrow \wr \cdot \frac{1}{\alpha} A_m \\ \mathcal{S}(\mathbb{R}) & \xrightarrow{\frac{d}{d\tau}} & \mathcal{S}(\mathbb{R}) \\ \mathfrak{F} \downarrow \wr & & \downarrow \wr \mathfrak{F} \\ \mathcal{S}(\mathbb{R}) & \xrightarrow{\cdot \frac{s}{v}} & \mathcal{S}(\mathbb{R}). \end{array}$$

Here in the last line the variable on  $\mathbb{R}$  is denoted by  $s$ . The image of multiplication by  $s$  on  $\mathcal{S}(\mathbb{R})$  is the subspace of functions vanishing in  $0 \in \mathbb{R}$  because of the formula:

$$g(s) - g(0) = sh(s) \text{ where } h(s) = \int_0^1 g'(ts) dt.$$

Hence the image of  $\frac{d}{d\tau}$  is closed and consists of those functions whose integral over  $\mathbb{R}$  vanishes. It follows that we have isomorphisms

$$(10) \quad \overline{H}^1(\mathfrak{p}, \mathcal{S}_m(\mathbb{R})) = H^1(\mathfrak{p}, \mathcal{S}_m(\mathbb{R})) = \text{Coker } \Theta_m \xrightarrow{\Lambda_m} \mathbb{C}$$

where

$$(11) \quad \Lambda_m(f) = \int_{\mathbb{R}} A_m(\tau) f(\tau) d\tau.$$

Combining Lemma 1.1 with (7) we get in particular:

PROPOSITION 1.2. — *The reduced cohomology  $\overline{H}^1(\mathcal{F})$  is infinite dimensional.*

*Remarks.* — Theorem (3.1) of [R2] gives information on the non-reduced cohomology  $H^1(\mathcal{F})$ . For generalizations to other nilmanifolds see [CR].

Let us now turn to  $\Delta_0$ -harmonic forms. We consider on  $\mathfrak{g}$  the scalar product

$$\langle [|a, b, c|], [|a', b', c'|] \rangle = aa' + bb' + cc'.$$

It induces a left-invariant Riemannian metric on  $G$ . The corresponding volume element on  $G$  is left invariant and hence a Haar measure. Under the diffeomorphism  $G \cong \mathbb{R}^3$  sending  $[x, y, z]$  to  $(x, y, z)$  it corresponds to the standard Lebesgue measure  $dx dy dz$  on  $\mathbb{R}^3$ . We endow  $X$  with the induced Riemannian metric. For purposes of integration note that a fundamental domain for the operation of  $\Gamma$  on  $G$  is the unit cube in  $\mathbb{R}^3$ . In particular  $X$  has volume equal to 1.

We fix a generator  $v$  of  $\mathfrak{p}$  with  $|v| = 1$ . Then the chain complex  $(\Gamma(X, \Lambda^\bullet T^* \mathcal{F}), d_0)$  of pre-Hilbert spaces is isometrically isomorphic to the complex

$$C^\infty(X) \xrightarrow{D_v} C^\infty(X)$$

concentrated in degrees 0 and 1. Here  $D_v$  is induced by the left invariant vector field  $Y_v$  on  $G$  corresponding to  $v$ . Explicitly

$$D_v(f)(\Gamma g) = \langle L_{g*}(v), f \rangle$$

for all  $f \in C^\infty(X)$  where  $L_g$  denotes left translation by  $g$  on  $G$ .

Since the scalar product on  $C^\infty(X)$  is the ordinary  $L^2$ -inner product coming from the volume element on  $X$ , a short calculation shows that  $D_v^* = -D_v$ . Hence the leafwise Laplace operator  $\Delta_0^1$  in degree one corresponds to  $-D_v^2$  on  $C^\infty(X)$ . We therefore see that if the leaves of  $\mathcal{F}$  are dense we have

$$(12) \quad \text{Ker } \Delta_0^1 = \mathbb{R}\omega_v \cong \mathbb{R}.$$

Here

$$\omega_v \in \Gamma(X, T^* \mathcal{F})$$

is the 1-form along the leaves dual to the vector field induced by  $Y_v$ .

A smooth leafwise Hodge decomposition as in Theorem 0.2 would induce an isomorphism

$$\overline{H}^1(\mathcal{F}) \cong \text{Ker } \Delta_0^1.$$

Together with the proposition we therefore see that if  $\mathcal{F}$  has dense leaves then at least for our metric there can be no smooth Hodge decomposition.

Using diagram (9) it follows that on  $\mathcal{S}(\mathbb{R})_m \subset C^\infty(X)$  we have

$$(13) \quad \text{Ker}(\Delta_0^1 | \mathcal{S}(\mathbb{R})_m) = 0$$

if we identify  $\Delta_0^1$  with  $-D_v^2$ . Therefore these parts of the decomposition of  $C^\infty(X)$  are detected by the reduced cohomology  $\overline{H}^1(\mathfrak{p}, -)$  but not by  $\Delta_0$ .

## 2. A remark on dynamical trace formulas.

Consider a closed manifold  $X$  with a smooth flow  $\phi^t$  which is non-degenerate in the sense of [GS], p. 310 (a condition on the compact orbits). Let  $E$  be a smooth vector bundle with a smooth action  $\psi^t : \phi^{t*} E \rightarrow E$ . In [GS], Ch. VI, Guillemin and Sternberg define the trace of the induced linear flow on  $\Gamma(X, E)$  as a distribution on  $\mathbb{R}^+$ :

$$\text{Tr}(\psi^* | \Gamma(X, E)) \in \mathcal{D}'(\mathbb{R}^+).$$

Moreover they show that this trace can be expressed as a sum of distributions indexed by the periodic orbits and the stationary points of the flow [GS], p. 311. In particular if the flow does not have compact orbits their trace is defined and equal to zero for any bundle  $E$ .

If  $\mathcal{F}$  is a foliation of  $X$  whose leaves are mapped to leaves by the flow, i.e. if  $T\phi^t(T\mathcal{F}) \subset T\mathcal{F}$  for all  $t$ , there is interest in the alternating sum

$$(14) \quad \sum_i (-1)^i \text{Tr}(\psi^* | \Gamma(X, \Lambda^i T^* \mathcal{F})) \text{ in } \mathcal{D}'(\mathbb{R}^+)$$

where

$$\psi^t : \phi^{t*}(\Lambda^i T^* \mathcal{F}) \longrightarrow \Lambda^i(T^* \mathcal{F})$$

is induced by  $T^* \phi^t$ . Namely for the purposes of establishing a dynamical Lefschetz trace formula with respect to the foliation, one would like to



replace the sum (14) by an alternating sum over suitable distributional traces of  $\psi^*$  on the reduced leafwise cohomologies  $\overline{H}^i(\mathcal{F})$ . For applications to the closed orbits of the geodesic flow see [G], Lecture 3, §3, [P], §5. Analogies with number theory are discussed in [D1], [D2]. If the foliation is one-codimensional and everywhere transversal to the flow this passage from (14) to cohomology is possible, c.f. [AK2], [DS]. In this case the foliation is Riemannian. The proofs in loc. cit. are based on the smooth leafwise Hodge decomposition 0.2.

We will now show by an example that in general (14) is not equal to the alternating sum of traces on  $\overline{H}^\bullet(\mathcal{F})$ .

Let  $P$  be as in the previous section with a generator  $v = [|\alpha, \beta, \gamma|]$  of  $\mathfrak{p}$ . Recall that  $\alpha, \beta$  are  $\mathbb{Q}$ -linearly independent. For any  $c \in \mathbb{R}$  the flow  $\phi^t$  on  $X = \Gamma \backslash G$  generated by

$$v + [0, 0, c] = [|\alpha, \beta, \gamma + c|]$$

respects the foliation  $\mathcal{F}$  since  $[0, 0, c]$  lies in the center of  $\mathfrak{g}$ . For the restriction of  $\phi^{t*}$  to  $\mathcal{S}(\mathbb{R})_m \subset C^\infty(X)$  one finds

$$\begin{aligned} (\phi^{t*} f)(\tau) &= (U_m(\exp t[|\alpha, \beta, \gamma + c|])f)(\tau) \\ &= \exp\left(2\pi im\left(t(\gamma + c) + \frac{1}{2}t^2\alpha\beta + t\tau\beta\right)\right) f(\tau + t\alpha). \end{aligned}$$

It follows after some calculation that the diagram

$$\begin{array}{ccc} \mathcal{S}_m(\mathbb{R}) & \xrightarrow{\Lambda_m} & \mathbb{C} \\ \phi^{t*} \downarrow & & \downarrow \cdot \exp(2\pi im(\gamma + c)t) \\ \mathcal{S}_m(\mathbb{R}) & \xrightarrow{\Lambda_m} & \mathbb{C} \end{array}$$

commutes. Here  $\Lambda_m$  was defined in formula (11). Because of (10) we therefore find:

LEMMA 2.1. —  $\phi^{t*}$  acts on  $\overline{H}^1(\mathfrak{p}, \mathcal{S}_m(\mathbb{R}))$  by multiplication with  $\exp(2\pi im(\gamma + c)t)$ .

All orbits of  $\phi^t$  are dense since  $\alpha$  and  $\beta$  are  $\mathbb{Q}$ -linearly independent. In particular  $\phi^t$  does not have compact orbits so that as explained above,

$$(15) \quad \sum_i (-1)^i \text{Tr}(\psi^* | \Gamma(X, \Lambda^i T^* \mathcal{F})) = 0 \text{ in } \mathcal{D}'(\mathbb{R}^+).$$

Since  $\overline{H}^0(\mathcal{F}) = H^0(\mathcal{F}) = \mathbb{C}$  with trivial  $\phi^{t*}$  action we have

$$(16) \quad \text{Tr}(\phi^* | \overline{H}^0(\mathcal{F})) = 1.$$

On the other hand, because of the  $\phi^{t*}$ -invariant decomposition in Lemma 1.1, the only reasonable candidate for a distributional trace of  $\phi^{t*}$  on  $\overline{H}^1(\mathcal{F})$  is the following sum in  $\mathcal{D}'(\mathbb{R}^+)$ :

$$\begin{aligned} \text{Tr}(\phi^*|\overline{H}^1(\mathcal{F})) &:= \sum_x \text{Tr}(\phi^{t*} | H^1(\mathfrak{p}, \mathbb{C}_x)) + \sum_{m \in \mathbb{Z} \setminus 0} |m| \text{Tr}(\phi^{t*} | \overline{H}^1(\mathfrak{p}, \mathcal{S}(\mathbb{R})_m)) \\ (17) \qquad \qquad &= 1 + \sum_{m \in \mathbb{Z} \setminus 0} |m| \exp(2\pi im(\gamma + c)t). \end{aligned}$$

Here the individual traces are smooth functions of  $t$  viewed as distributions on  $\mathbb{R}^+$  and we have used Lemma 2.1 and formula (6). For  $\gamma + c = 0$  the sum does not converge. For  $\gamma + c \neq 0$  however, it does converge easily since for any test-function  $\varphi$  on  $\mathbb{R}^+$  one has by partial integration for real  $r$ :

$$\begin{aligned} \langle \exp(irt), \varphi \rangle &= \int_0^\infty \exp(irt)\varphi(t) dt \\ &= (ir)^{-N} \int_0^\infty \exp(irt)\varphi^{(N)}(t) dt \\ &= O(r^{-N}) \text{ as } r \rightarrow \infty. \end{aligned}$$

From (16), (17) we see that if  $\gamma + c \neq 0$  we have in  $\mathcal{D}'(\mathbb{R}^+)$ :

$$\sum_i (-1)^i \text{Tr}(\phi^* | \overline{H}^i(\mathcal{F})) = - \sum_{m \in \mathbb{Z} \setminus 0} |m| \exp(2\pi im(\gamma + c)t).$$

Formula (18) below which was mentioned to us by A. Deitmar shows that this sum is non-zero in  $\mathcal{D}'(\mathbb{R}^+)$ . Because of (15) we therefore see that passage to cohomology in (14) is not allowed in our example. On the bright side let us remark however that

$$\sum_i (-1)^i \text{Tr}(\phi^* | \text{Ker } \Delta_0^i) = 1 - 1 = 0$$

in accordance with (15). Here we have used that  $\phi^{t*}$  fixes  $\omega_v$  in formula (12). Possibly, dynamical trace formulas as envisaged in [G] Lecture 3, §2, [P] §3, [D1] §4 and [D2] should in general involve alternating sums over traces on the space of  $\Delta_0$ -harmonic forms rather than on cohomology.

*Fact.* — For  $A \in \mathbb{R}^*$  we have as distributions on  $\mathbb{R} \setminus A^{-1}\mathbb{Z}$ :

$$(18) \qquad \sum_{m \in \mathbb{Z} \setminus 0} |m| \exp(2\pi imAt) = -\frac{1}{2}(\sin \pi At)^{-2}.$$

*Proof.* — As distributions on  $\mathbb{R} \setminus A^{-1}\mathbb{Z}$  one checks that

$$\sum_{m=0}^\infty \exp(2\pi imAt) = (1 - e^{2\pi iAt})^{-1}.$$

Since differentiation of distributions commutes with limits this implies

$$\sum_{m=1}^{\infty} m \exp(2\pi i m A t) = (e^{\pi i A t} - e^{-\pi i A t})^{-2} \quad \text{in } \mathcal{D}'(\mathbb{R} \setminus A^{-1}\mathbb{Z})$$

and hence the formula.  $\square$

*Remark.* — In [G] Lecture 2, §2, Guillemin states a specific case in which he expects that passage to cohomology in (14) should be possible:  $X$  should be a contact manifold,  $\mathcal{F}$  should be a polarization i.e. a foliation by Legendre manifolds and the flow should be generated by a contact vector field. In regard to this, note the following: The 1-form  $dz - xdy$  defines a contact structure on  $X$ . The foliation corresponding to  $P$  is a polarization if and only if  $\mathfrak{p}$  contains a generator  $v = [|\alpha, \beta, \gamma|]$  with  $\gamma = 0$ . However the flows  $\phi^t$  considered above are not generated by contact vector fields.

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