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Antonios D. MELAS

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UNIVERSAL FUNCTIONS ON NONSIMPLY CONNECTED DOMAINS

by Antonios D. MELAS

1. Introduction.

Let $\Omega \subseteq \mathbb{C}$ be a connected open set. For every function $f : \Omega \rightarrow \mathbb{C}$ holomorphic in Ω and $\zeta_0 \in \Omega$ we let

$$(1.1) \quad S_N(f, \zeta_0)(z) = \sum_{n=0}^N \frac{f^{(n)}(\zeta_0)}{n!} (z - \zeta_0)^n$$

denote the N th partial sum of the Taylor development of f with center ζ_0 . This sequence of polynomials converges to f uniformly on compact subsets of the open disc $D(\zeta_0, \rho)$ where $\rho = \text{dist}(\zeta_0, \partial\Omega)$ and “generically” it will diverge on the complement of the closed disc $\overline{D(\zeta_0, \rho)}$. However certain subsequences of $(S_N(f, \zeta_0)(z))_{N=1}^{\infty}$ may have nice approximation properties outside Ω and this can be described by defining the following class.

DEFINITION 1. — *A holomorphic function $f : \Omega \rightarrow \mathbb{C}$ is said to be universal with respect to Ω and ζ_0 if the following holds:*

For every compact set $K \subseteq \mathbb{C}$ such that $K \cap \Omega = \emptyset$ and its complement K^c is connected, and every function $h : K \rightarrow \mathbb{C}$ continuous on K and holomorphic in the interior K° (if nonempty), there exists a sequence (λ_n)

of nonnegative integers such that $S_{\lambda_n}(f)(z) \rightarrow h(z)$ uniformly on K . We denote by $\mathcal{U}(\Omega, \zeta_0)$ the class of all such functions.

Here the compact set K is allowed to contain pieces of the boundary $\partial\Omega$. The existence of such functions in the case where Ω is simply connected has been established in [7] where it actually has been shown that $\mathcal{U}(\Omega, \zeta_0)$ is a G_δ -dense subset of the Frechet space $H(\Omega)$ of all functions holomorphic in Ω . Similar definition but with K not allowed to meet the boundary $\partial\Omega$ has been given by Luh [4], where also existence has been proven when Ω is simply connected.

To explain this definition let us assume that $\zeta_0 = 0 \in \Omega$ and for $f \in \mathcal{U}(\Omega, 0)$ write $a_n = \frac{f^{(n)}(0)}{n!}$. Then the Taylor series $\sum_{n=0}^{\infty} a_n z^n$ defines a germ of analytic functions that can be analytically continued throughout Ω and its partial sums approximate everything one might hope in $\mathbb{C} \setminus \Omega$.

Several properties of these classes $\mathcal{U}(\Omega, \zeta_0)$ have been established in [5] in the case Ω is assumed to be contained in a half plane (actually in the complement of an angle suffices). Under this assumption it has been shown in [5]:

- (a) That the class $\mathcal{U}(\Omega, \zeta_0)$ is empty if Ω is not simply connected.
- (b) That $\mathcal{U}(\Omega, \zeta_1) = \mathcal{U}(\Omega, \zeta_2)$ for all $\zeta_1, \zeta_2 \in \Omega$.
- (c) That for Ω simply connected any $f \in \mathcal{U}(\Omega, \zeta_0)$ has $\partial\Omega$ as its natural boundary, that is cannot be continued analytically across any portion of $\partial\Omega$, which answered a conjecture by J.-P. Kahane.

However nothing is known for the class $\mathcal{U}(\Omega, \zeta_0)$ in the case where Ω is not contained in the complement of an angle (except that it is G_δ -dense in $H(\Omega)$ if Ω is simply connected).

The only results known in this direction are that the corresponding class where the compact set is not allowed to meet the boundary $\partial\Omega$ is G_δ -dense in $H(\Omega)$ if Ω is connected and equal to the complement of a connected compact set (see [1]) and that for the special case where $\Omega = \mathbb{C} \setminus \{1\}$ the class $\mathcal{U}(\mathbb{C} \setminus \{1\}, 0)$ is nonempty (see [8]). It is not known whether the class $\mathcal{U}(\mathbb{C} \setminus A, 0)$ is nonempty if A is a finite set or if the result in [1] extends for the class $\mathcal{U}(\Omega, \zeta_0)$ and also whether some of the above mentioned properties (a)-(c) hold in these cases.

Here we will give some answers to the above questions. First we will prove the following.

THEOREM 1. — *Let $K \subset \mathbb{C}$ be a connected compact set such that $\Omega = \mathbb{C} \setminus K$ is also connected. Then for every $\zeta_0 \in \Omega$ the class $\mathcal{U}(\Omega, \zeta_0)$ is a G_δ -dense subset of $H(\Omega)$, hence nonempty.*

Thus the result in [1] actually does extend for the class $\mathcal{U}(\Omega, \zeta_0)$. We mention here that it is not known whether $\mathcal{U}(\Omega, \zeta_0)$ is nonempty when the compact set K in the above theorem is not assumed to be connected for example if Ω is equal to the complement of two disjoint closed discs and ζ_0 is any point of Ω .

To proceed with the other questions let E be any countable subset of $\mathbb{C} \setminus D$ (where D is the unit disc), let B be any discrete subset of $\mathbb{C} \setminus D$ such that $1 \in B$ and let $W = \mathbb{C} \setminus B$. We will denote by $\mathcal{V}(W, E)$ the class of all functions $f \in H(W)$ (if any) having the following property:

“For every finite subset $E' \subset E$ and any function $h : E' \rightarrow \mathbb{C}$ there exists a strictly increasing sequence of positive integers (λ_k) such that

$$(1.2) \quad S_{\lambda_k}(f, 0)(z) \rightarrow h(z) \text{ for every } z \in E'.”$$

Then we have the following.

THEOREM 2. — *For every E and B as above the class $\mathcal{V}(W, E)$ is a G_δ -dense subset of $H(W)$, hence nonempty.*

By taking $E = (\mathbb{C} \setminus D) \cap (\mathbb{Q} + i\mathbb{Q})$ and $B = \{1\}$ then Theorem 2 provides us with an $F \in \mathcal{V}(\mathbb{C} \setminus \{1\}, E)$. Consider also any infinite discrete subset L of E such that $1 \in L$ and let $\Omega = \mathbb{C} \setminus L$. Then on the one hand clearly $f = F|_\Omega \in \mathcal{U}(\Omega, 0)$ (compact subsets of $\mathbb{C} \setminus \Omega$ are finite). However on the other hand f extends analytically to the larger domain $\mathbb{C} \setminus \{1\}$. Also $f \notin \mathcal{U}(\Omega, \zeta_0)$ for every $\zeta_0 \in \Omega$ such that $D(\zeta_0, |\zeta_0 - 1|) \cap L \neq \emptyset$ since then $\{(S_N(f, \zeta_0)(z))\}_{N=1}^\infty$ will converge to $F(z)$ for any $z \in D(\zeta_0, |\zeta_0 - 1|) \cap L$, F being holomorphic in $\mathbb{C} \setminus \{1\}$ and so $\mathcal{U}(\Omega, 0) \neq \mathcal{U}(\Omega, \zeta_0)$ for every such ζ_0 . It is easy to see that we can arrange L so that every $\zeta_0 \in \Omega \setminus D$ has this property. Moreover the sequence $\{(S_N(f, 0)(z))\}_{N=1}^\infty$ will be dense for every $z \in E \setminus L$ which is a nonempty subset of Ω . Hence we have proved the following.

COROLLARY 1. — *There exists a nonsimply connected domain $\Omega \subseteq \mathbb{C}$ with $0 \in D \subseteq \Omega$ and $\mathbb{C} \setminus \Omega$ is infinite and discrete such that*

- (a) *The class $\mathcal{U}(\Omega, 0)$ is G_δ -dense in $H(\Omega)$.*

(b) $\mathcal{U}(\Omega, 0)$ contains functions that can be analytically continued on the larger domain $\mathbb{C} \setminus \{1\}$, hence Kahane's conjecture cannot be generalized in this case.

(c) $\mathcal{U}(\Omega, 0) \neq \mathcal{U}(\Omega, \zeta)$ for every $\zeta \in \Omega \setminus D \neq \emptyset$.

(d) There exists $f \in \mathcal{U}(\Omega, 0)$ such that sequence of complex numbers $\{(S_N(f, 0)(z))\}_{N=1}^\infty$ is dense for some points z in Ω .

Thus the situation is completely different if the complement of Ω is small. Property (d) appears to be completely new.

It is not known and natural to ask here how small must $\mathbb{C} \setminus \Omega$ be for Ω to share the above properties. For example if $\Omega = \mathbb{C} \setminus [1, +\infty)$, is there a function $f \in \mathcal{U}(\Omega, 0)$ that can be analytically continued across some subinterval of $(1, +\infty)$? Also in this case one may ask whether we can find $\zeta \in \Omega$ such that $\mathcal{U}(\Omega, 0) \neq \mathcal{U}(\Omega, \zeta)$.

In Section 2 we will prove Theorem 1. Then in Section 3 we will establish a technical proposition that will be used in Section 4 to prove Theorem 2.

2. Proof of Theorem 1.

Applying a translation, a rotation and a dilation we may assume that $\zeta_0 = 0$, $D \subset \Omega$, $1 \in K$ and that there exists a real number $\alpha \geq 1$ such that $\alpha \in K$ and $\alpha = \max\{|z| : z \in K\}$.

By Lemma 3.3 in [1], for every open neighborhood V of K there is a connected and simply connected open set U such that $K \subset U \subset V$. Hence it is easy to construct an exhausting sequence $(L_m)_{m=1}^\infty$ of compact subsets of $\Omega = \mathbb{C} \setminus K$ such that for every m , $\mathbb{C} \setminus L_m$ has exactly two connected components, one of which is bounded and contains K and the other unbounded.

Now for every polynomial p and every $\varepsilon > 0$ we define

$$\begin{aligned} \Gamma(p, \varepsilon, \Omega) \\ = \{f \in H(\Omega) : \text{there exists } n \text{ such that } \max_{z \in K} |S_n(f, 0)(z) - p(z)| < \varepsilon\}. \end{aligned}$$

Then letting $(p_j)_{j=1}^\infty$ denote an enumeration of all polynomials with coefficients in $\mathbb{Q} + i\mathbb{Q}$ we have the following.

LEMMA 1. — $\mathcal{U}(\Omega, 0) = \bigcap_{j,s=1}^{\infty} \Gamma(p_j, \frac{1}{s}, \Omega)$.

The proof of the above lemma is well known (see for example [5]), noticing that every compact subset of Ω is of course contained in K .

It is also easy to show that each $\Gamma(p, \varepsilon, \Omega)$ is an open subset of $H(\Omega)$ (see [7], [5]). Hence, in view of Baire's Theorem, to complete the proof of Theorem 1 it suffices to prove the following.

LEMMA 2. — *For every polynomial p and every $\varepsilon > 0$ the set $\Gamma(p, \varepsilon, \Omega)$ is dense in $H(\Omega)$.*

Proof. — It suffices to show that for every $g \in H(\Omega)$, $\delta > 0$ and $m \geq 1$ there exists $f \in \Gamma(p, \varepsilon, \Omega)$ such that $\max_{L_m} |f - g| < \delta$.

By taking m sufficiently large we may assume that $\overline{D(0, \frac{1}{2})} \subset L_m$. Fix such g, δ, m . Then there exists $\beta > \alpha$ such that β belongs to the bounded component of $\mathbb{C} \setminus L_m$ and hence by Runge's Theorem there exists a rational function Q with poles only at β such that

$$(2.1) \quad \max_{L_m} |g - Q| < \frac{\delta}{2} \quad \text{and} \quad \max_K |p - Q| < \frac{\varepsilon}{2}.$$

Since the Taylor development of Q with center 0 converges to Q uniformly on compact subsets of $D(0, \beta)$ and by the choice of α, β the compact set K is contained in $D(0, \beta)$ we conclude that

$$(2.2) \quad S_N(Q, 0)(z) \rightarrow Q(z) \text{ uniformly for } z \in K \text{ as } N \rightarrow \infty.$$

Hence in view of (2.1) there is N_0 sufficiently large such that

$$(2.3) \quad \max_{z \in K} |S_{N_0}(Q, 0)(z) - p(z)| < \frac{\varepsilon}{2}.$$

Fix such a N_0 . The only problem is that Q is not holomorphic in Ω as β does not belong to K . However by applying again Runge's Theorem we conclude that there exists a sequence $(R_M)_{M=1}^{\infty}$ of rational functions with poles only at $\alpha \in K$ such that $R_M \rightarrow Q$ uniformly in L_m .

Since $\overline{D(0, \frac{1}{2})} \subset L_m$ the Cauchy's estimates imply that $S_{N_0}(R_M, 0)(z) \rightarrow S_{N_0}(Q, 0)(z)$ uniformly for $z \in K$ as $M \rightarrow \infty$, since N_0 is fixed and K is compact.

Hence there is M_0 sufficiently large such that $f = R_{M_0} \in H(\Omega)$ satisfies

$$(2.4) \quad \max_{L_m} |f - Q| < \frac{\delta}{2} \text{ and } \max_{z \in K} |S_{N_0}(f, 0)(z) - S_{N_0}(Q, 0)(z)| < \frac{\varepsilon}{2}.$$

Therefore by (2.1), (2.3) and (2.4) we have $\max_{L_m} |f - g| < \delta$ and $f \in \Gamma(p, \varepsilon, \Omega)$. This completes the proof of the lemma and hence of Theorem 1. \square

3. A property of certain linear systems.

In this section we will prove a technical result that will be essential for the proof of Theorem 2.

For any positive integers m, N and any $z \in \mathbb{C}$ we define

$$(3.1) \quad A_N(m, z) = \sum_{n=1}^{N-1} n^m z^n.$$

If s is another positive integer and $z_1, \dots, z_s \in \mathbb{C}$ we also define

$$(3.2) \quad \Delta_N(m, z_1, \dots, z_s) = \det[A_N(m + r, z_j)]_{j,r=1}^s$$

the determinant of the corresponding $s \times s$ matrix.

Consider also the meromorphic function

$$(3.3) \quad \psi(\zeta) = \frac{1}{e^\zeta - 1}.$$

Then we have:

LEMMA 3. — *If $z = e^\zeta \in \mathbb{C}$ is such that $|z| \geq 1$, $z \neq 1$ then for every $m > s > 1$*

$$(3.4) \quad N^{-m} z^{-N} A_N(m, z) = \sum_{\lambda=0}^{s-1} \binom{m}{\lambda} N^{-\lambda} \psi^{(\lambda)}(\zeta) + \rho_N(m, s, z)$$

where $|\rho_N(m, s, z)| \leq CN^{-s}$ the constant C depending only on m, s and z .

Proof. — Since $e^\zeta \neq 1$ we have

$$\begin{aligned} A_N(m, z) &= \sum_{n=1}^{N-1} n^m e^{n\zeta} = \left(\frac{d}{d\zeta}\right)^m \left(\sum_{n=1}^{N-1} e^{n\zeta}\right) = \left(\frac{d}{d\zeta}\right)^m \left(\frac{e^{N\zeta} - 1}{e^\zeta - 1}\right) \\ &= \sum_{\lambda=0}^m \binom{m}{\lambda} \left(\frac{d}{d\zeta}\right)^{m-\lambda} (e^{N\zeta} - 1)\psi^{(\lambda)}(\zeta) \\ &= \sum_{\lambda=0}^{s-1} \binom{m}{\lambda} N^{m-\lambda} z^N \psi^{(\lambda)}(\zeta) + \sum_{\lambda=s}^{m-1} \binom{m}{\lambda} N^{m-\lambda} z^N \psi^{(\lambda)}(\zeta) \\ &\quad + (z^N - 1)\psi^{(m)}(\zeta) \end{aligned}$$

from which the lemma follows easily since $|z| \geq 1$. □

LEMMA 4. — Let $z_1 = e^{\zeta_1}, \dots, z_s = e^{\zeta_s} \in \mathbb{C} \setminus (D \cup \{1\})$ and $m \geq 1$. Then for all sufficiently large N we have

$$(3.5) \quad |\Delta_N(m, z_1, \dots, z_s)| \geq c |z_1 \dots z_s|^N N^{(m+1)s}$$

where $c > 0$ doesn't depend on N .

Proof. — By Lemma 2 we have

$$(3.6) \quad \begin{aligned} A_N(m+r, z_j) &= N^{m+r} z_j^N \left(\sum_{\lambda=0}^{s-1} \binom{m+r}{\lambda} N^{-\lambda} \psi^{(\lambda)}(\zeta_j) + \rho_N(m+r, s, z_j) \right) \end{aligned}$$

hence

$$\begin{aligned} (z_1 \dots z_s)^{-N} N^{-sm-1-2-\dots-s} \Delta_N(m, z_1, \dots, z_s) &= \det[\psi(\zeta_j)] \\ &\quad + (m+r) \frac{\psi'(\zeta_j)}{N} + \dots \\ &\quad + \binom{m+r}{s-1} \frac{\psi^{(s-1)}(\zeta_j)}{N^{s-1}} \\ &\quad + \rho_N(m+r, s, z_j)]_{j,r=1}^s. \end{aligned}$$

Now it is easy to see that each column of the above determinant is the sum of $s+1$ columns. The r th column is thus the sum of $[\psi(\zeta_j)]_{j=1}^s, \dots, \left[\binom{m+r}{s-1} \frac{\psi^{(s-1)}(\zeta_j)}{N^{s-1}}\right]_{j=1}^s$ and the column of “errors” $[\rho_N(m+r, s, z_j)]_{j=1}^s$.

We can thus expand this determinant into a sum of $(s + 1)^s$ simple determinants. This sum can be further partitioned into two sums. The first consists of all determinants that don't contain any column of errors $[\rho_N(m+r, s, z_j)]_{j=1}^s$ and thus this sum is equal to the following determinant:

$$(3.7) \quad \det \left[\sum_{\lambda=0}^{s-1} \binom{m+r}{\lambda} N^{-\lambda} \psi^{(\lambda)}(\zeta_j) \right]_{j,r=1}^s \\ = \det[\psi^{(k-1)}(\zeta_j)]_{j,k=1}^s \cdot \det \left[N^{-k+1} \binom{m+r}{k-1} \right]_{k,r=1}^s$$

according to the determinant of the product formula.

Now $\det \left[N^{-k+1} \binom{m+r}{k-1} \right]_{k,r=1}^s = N^{-(0+1+\dots+(s-1))} \det \left[\binom{m+r}{k-1} \right]_{k,r=1}^s$ and it is easy to see by induction (using the formula $\binom{m+r+1}{k-2} - \binom{m+r}{k-2} = \binom{m+r}{k-1}$) that $\det \left[\binom{m+r}{k-1} \right]_{k,r=1}^s = 1$.

The second sum consists of those determinants that contain at least one column of "errors" $[\rho_N(m+r, s, z_j)]_{j=1}^s$. Observing that such a determinant will be zero if it contains at least two columns of the same type i.e. $\left[\binom{m+r_1}{\lambda} \frac{\psi^{(\lambda)}(\zeta_j)}{N^\lambda} \right]_{j=1}^s$ and $\left[\binom{m+r_2}{\lambda} \frac{\psi^{(\lambda)}(\zeta_j)}{N^\lambda} \right]_{j=1}^s$ for $r_1 \neq r_2$ and for the same λ we conclude that each nonzero determinant in the second sum has absolute value at most

$$C' N^{-(0+1+\dots+(s-2))} \cdot N^{-s} = C' N^{-\frac{s(s-1)}{2} - 1}$$

where C' doesn't depend on N . Hence the second sum has absolute value at most $C N^{-\frac{s(s-1)}{2} - 1}$ where C depends only on m, s and the z_j 's.

Therefore

$$(3.8) \quad |\Delta_N(m, z_1, \dots, z_s)| \\ \geq |z_1 \dots z_s|^N N^{s(m+1)} \left[\left| \det[\psi^{(k-1)}(\zeta_j)]_{j,k=1}^s \right| - \frac{C}{N} \right].$$

Hence to complete the proof of the lemma it suffices to prove that $\det[\psi^{(k-1)}(\zeta_j)]_{j,k=1}^s \neq 0$. But it is easy to show by induction that for every $k \geq 1$ we have

$$(3.9) \quad \psi^{(k-1)}(\zeta) = a_{1,k} \psi(\zeta) + \dots + a_{k,k} \psi(\zeta)^k$$

for certain constants $a_{1,k}, \dots, a_{k,k} \in \mathbb{C}$ where $a_{k,k} = (-1)^k(k-1)! \neq 0$. Hence simplifying the corresponding determinant we get

$$(3.10) \quad \det[\psi^{(k-1)}(\zeta_j)]_{j,k=1}^s = a_{1,1} \dots a_{k,k} \det[\psi(\zeta_j)]_{j,k=1}^s \neq 0$$

since the last determinant is a Vandermonde one and the corresponding complex numbers $\psi(\zeta_1), \dots, \psi(\zeta_s)$ are distinct. This completes the proof of the lemma. □

Remark. — We can get a similar asymptotic behavior in the case that one of the z_j 's is equal to 1. However since we won't be needing this in the proof of Theorem 2 we don't include it.

Now we can prove the following.

PROPOSITION 1. — *Let $z_1, \dots, z_s \in \mathbb{C} \setminus (D \cup \{1\})$ be distinct and let $(\gamma_N^1), \dots, (\gamma_N^s)$ be s sequences of complex numbers such that*

$$(3.11) \quad \left| \gamma_N^j \right| \leq C |z_j|^N N^d$$

for all $N \geq 1$ and $1 \leq j \leq s$ where C, d are constants that do not depend on N . Then there exists an integer $m > d$ such that for all sufficiently large N the linear system

$$(3.12) \quad \sum_{r=1}^s A_N(m+r, z_j) t_{r,N} = \gamma_N^j \quad \text{for } j = 1, 2, \dots, s$$

has a unique solution $t_{1,N}, \dots, t_{s,N} \in \mathbb{C}$ and moreover

$$(3.13) \quad |t_{1,N}| + \dots + |t_{s,N}| \rightarrow 0 \text{ as } N \rightarrow \infty.$$

Proof. — We choose $m = d + s(s+3)$. Then for every sufficiently large N Lemma 3 implies that the determinant of the system (3.12) which is $\Delta_N(m, z_1, \dots, z_s)$ satisfies the estimate (3.5) in particular it is nonzero and hence the system has a unique solution $t_{1,N}, \dots, t_{s,N} \in \mathbb{C}$. Now each $t_{r,N}$ can be computed according to Cramer's rule and is equal to the ratio of two determinants, the one in the denominator being $\Delta_N(m, z_1, \dots, z_s)$ and the other in the numerator is produced by $\Delta_N(m, z_1, \dots, z_s)$ by replacing the r th column of $\Delta_N(m, z_1, \dots, z_s)$ by $[\gamma_N^j]_{j=1}^s$. Expanding such a determinant we get a sum of $s!$ terms of the form $\pm A_N(m+r_1, z_{j_1}) \dots A_N(m+r_{s-1}, z_{j_{s-1}}) \gamma_N^{j_s}$

where the j_1, \dots, j_s are distinct and $r_1 < \dots < r_{s-1}$ belong to $\{1, 2, \dots, s\}$. Since

$$(3.14) \quad |A_N(m+r, z)| \leq \sum_{n=1}^{N-1} n^{m+r} |z|^N < N^{m+r+1} |z|^N$$

whenever $|z| \geq 1$ we conclude that any such term has absolute value at most

$$N^{m+r_1+1} |z_{j_1}|^N \dots N^{m+r_{s-1}+1} |z_{j_{s-1}}|^N CN^d |z_{j_s}|^N \leq CN^{m(s-1)+3+\dots+(s+2)+d} |z_1 \dots z_s|^N.$$

Therefore using Lemma 4 we conclude that for every sufficiently large N we have

$$(3.15) \quad |t_{1,N}| + \dots + |t_{s,N}| \leq s \frac{Cs! N^{m(s-1)+3+\dots+(s+2)+d} |z_1 \dots z_s|^N}{cN^{(m+1)s} |z_1 \dots z_s|^N} = s \frac{Cs!}{c} N^{-m + \frac{s(s+3)}{2} + d}$$

which tends to 0 as $N \rightarrow \infty$ by the choice of m . This completes the proof of the proposition. □

4. Proof of Theorem 2.

In this section we will prove Theorem 2. For this purpose let $(h_\ell)_{\ell=1}^\infty$ be an enumeration of all functions $h : E' \rightarrow \mathbb{Q} + i\mathbb{Q}$ where $E' \subseteq E$ is finite. Then we define

$$\Gamma(\ell, k, W) = \left\{ f \in H(W) : \text{there exists } n \geq 1 \text{ such that } |S_n(f, 0)(z) - h_\ell(z)| < \frac{1}{k} \text{ for every } z \in \text{dom } h_\ell \right\}.$$

It is easy to see that $\bigcap_{\ell, k=1}^\infty \Gamma(j, k, W) = \mathcal{V}(W, E)$ and it is also easy to see that each $\Gamma(\ell, k, W)$ is an open subset of $H(W)$. Hence to prove Theorem 2 it suffices to prove the following.

PROPOSITION 2. — *Let $\{z_1, \dots, z_s\}$ be a finite subset of $\mathbb{C} \setminus D$, $a_1, \dots, a_s \in \mathbb{C}$ and $\varepsilon > 0$. Then the set*

$$\Gamma = \{f \in H(W) : \text{there exists } n \geq 1 \text{ such that } |S_n(f, 0)(z_j) - a_j| < \varepsilon \text{ for every } j = 1, 2, \dots, s\}$$

is a dense subset of $H(W)$.

Proof. — It suffices to prove that given any $\delta > 0$, $g \in H(W)$ and compact set $L \subseteq W$ there exists $f \in \Gamma$ such that $\max_L |g - f| < \delta$.

Since $B = \mathbb{C} \setminus W$ is discrete we may assume that L is of the form $\{z \in \mathbb{C} : |z| \leq R \text{ and } \text{dist}(z, B) \geq \eta\}$ for certain $R > 2$ and $\eta > 0$.

Then $\mathbb{C} \setminus L$ has finitely many bounded components each containing at least one point of B . Therefore by Runge's Theorem there exists a rational function Q with poles in $B \subseteq \mathbb{C} \setminus D$ such that $\max_L |g - Q| < \frac{\delta}{2}$. Expanding Q in simple fractions we may write

$$(4.1) \quad Q(z) = \sum_{\mu=1}^M \sum_{k=1}^K \frac{b_{\mu,k}}{(1 - z/w_{\mu})^k}$$

where $w_1, \dots, w_M \in B$ and so each $|w_{\mu}| \geq 1$.

Since each $\frac{1}{(1-z/w_{\mu})^k} = \sum_{n=0}^{\infty} \binom{n+k-1}{k-1} \left(\frac{z}{w_{\mu}}\right)^n$ has partial sums with absolute value

$$(4.2) \quad \left| \sum_{n=0}^N \binom{n+k-1}{k-1} \left(\frac{z}{w_{\mu}}\right)^n \right| \leq C_k N^{k-1} |z|^N$$

for every z with $|z| \geq 1$ we conclude that

$$(4.3) \quad |S_N(Q, 0)(z)| \leq C |z|^N N^{K-1}$$

for every $z \in \mathbb{C} \setminus D$ where C depends only on Q .

Now we define for any integer $p \geq 0$ the power series $\sum_{n=0}^{\infty} n^p z^n$. It is easy to see that there exist rational functions R_p with pole at 1 only such that

$$(4.4) \quad R_p(z) = \sum_{n=0}^{\infty} n^p z^n \text{ on } D.$$

In fact $R_0(z) = \frac{1}{1-z} - 1$ and $R_{p+1}(z) = \frac{d}{dz}(zR_p(z))$.

We fix a large integer m to be defined, choose a complex number of absolute value 1, $e^{i\theta} \notin \{z_1, \dots, z_s\}$ belonging to the same component of $\mathbb{C} \setminus L$ that contains 1 (using $e^{i\theta}$ will not be necessary unless $1 \in \{z_1, \dots, z_s\}$) and consider the function

$$(4.5) \quad G(z) = \sum_{r=1}^s \lambda_r R_{m+r}(ze^{-i\theta})$$

where $\lambda_1, \dots, \lambda_s$ are to be determined.

Clearly G is a rational function with pole at $e^{i\theta}$ only.

Then

$$(4.6) \quad S_N(G, 0)(z) = \sum_{r=1}^s A_N(m+r, ze^{-i\theta})\lambda_r$$

and so using (4.3) and choosing m as in Proposition 1 applied to the numbers $z_1 e^{-i\theta}, \dots, z_s e^{-i\theta} \in \mathbb{C} \setminus (D \cup \{1\})$ we conclude that given $\varepsilon_1 > 0$ there exist N sufficiently large and $\lambda_1, \dots, \lambda_s \in \mathbb{C}$ such that

$$(4.7) \quad S_N(G, 0)(z_j) + S_N(Q, 0)(z_j) = a_j \text{ for } j = 1, 2, \dots, s$$

and

$$(4.8) \quad |\lambda_1| + \dots + |\lambda_s| < \varepsilon_1.$$

We choose $\varepsilon_1 > 0$ such that (4.8) implies that

$$(4.9) \quad \max_{z \in L} |G(z)| < \frac{\delta}{2}$$

and fix the corresponding $N = N_0$ for which (4.7) and (4.8) hold. This is possible since L is compact and each $R_{m+r}(ze^{-i\theta})$ has no poles in L .

Then by Runge's Theorem there is a sequence $(F_q)_{q=1}^\infty$ of rational functions having poles only at 1 such that $F_q \rightarrow G$ uniformly on $T = \{z \in \mathbb{C} : |z| \leq R, |z-1| \geq \eta\}$. Since T contains a disc around 0 the Cauchy's estimates imply that

$$(4.10) \quad S_{N_0}(F_q, 0)(z_j) \rightarrow S_{N_0}(G, 0)(z_j) \text{ for } j = 1, 2, \dots, s$$

as $q \rightarrow \infty$.

Therefore we can choose q_0 sufficiently large such that the rational function $F = F_{q_0}$ that has pole only at 1 satisfies

$$(4.11) \quad \max_{z \in L} |F(z)| < \frac{\delta}{2}$$

and $|S_{N_0}(F+Q, 0)(z_j) - a_j| < \varepsilon$ for $j = 1, 2, \dots, s$.

Then by taking $f = F + Q \in H(W)$ we have

$$(4.12) \quad \max_L |g - f| \leq \max_L |g - Q| + \max_L |F| < \delta$$

and

$$(4.13) \quad |S_{N_0}(f, 0)(z_j) - a_j| < \varepsilon \text{ for } j = 1, 2, \dots, s$$

hence $f \in \Gamma$. This completes the proof of the proposition and hence of Theorem 2. \square

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Antonios D. MELAS,
University of Athens
Department of Mathematics
Panepistimiopolis 157-84
Athens (Greece).
amelas@math.uoa.gr