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NONRESONANCE CONDITIONS FOR ARRANGEMENTS

by D.C. COHEN[†], A. DIMCA & P. ORLIK[‡]

1. Introduction.

Let \mathcal{A} be an arrangement of hyperplanes in the complex projective space \mathbb{P}^n , with complement $M(\mathcal{A}) = \mathbb{P}^n \setminus \bigcup_{H \in \mathcal{A}} H$. Let \mathcal{L} be a complex local system of coefficients on $M(\mathcal{A})$. The need to calculate the local system cohomology $H^*(M(\mathcal{A}), \mathcal{L})$ arises in a variety of contexts, including the Aomoto-Gelfand theory of multivariable hypergeometric integrals [1], [15]; representation theory of Lie algebras and quantum groups and solutions of the Knizhnik-Zamolodchikov differential equation in conformal field theory [25]; and the determination of the cohomology groups of the Milnor fiber of the non-isolated hypersurface singularity at the origin in \mathbb{C}^{n+1} associated to the arrangement \mathcal{A} [5].

In light of these applications, and others, the cohomology $H^*(M(\mathcal{A}), \mathcal{L})$ has been the subject of considerable recent interest. Call the local system \mathcal{L} *nonresonant* if this cohomology is concentrated in dimension n , that is, $H^k(M(\mathcal{A}), \mathcal{L}) = 0$ for $k \neq n$. Necessary conditions for vanishing, or nonresonance, have been determined by a number of authors, including Esnault, Schectman, and Viehweg [13], Kohno [17], and Schechtman, Terao, and Varchenko [24]. Many of these results make use of Deligne's work [7],

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and thus require the realization of $M(\mathcal{A})$ as the complement of a normal crossing divisor in a complex projective manifold.

An edge is a nonempty intersection of hyperplanes. An edge is *dense* if the subarrangement of hyperplanes containing it is irreducible: the hyperplanes cannot be partitioned into nonempty sets so that after a change of coordinates hyperplanes in different sets are in different coordinates. This is a combinatorially determined condition which can be checked in a neighborhood of a given edge, see [24]. Consequently, this notion makes sense for both affine and projective arrangements. Let $D(\mathcal{A})$ denote the set of dense edges of the arrangement \mathcal{A} .

Let $N = \bigcup_{H \in \mathcal{A}} H$ be the union of the hyperplanes of \mathcal{A} . There is a canonical way to obtain an embedded resolution of the divisor N in \mathbb{P}^n . First, blow up the dense 0-dimensional edges of \mathcal{A} to obtain a map $p_1 : Z_1 \rightarrow \mathbb{P}^n$. Then, blow up all the proper transforms under p_1 of projective lines corresponding to dense 1-dimensional edges in $D(\mathcal{A})$. Continuing in this way, we get a map $p = p_{n-1} : Z_{n-1} \rightarrow \mathbb{P}^n$ which is an embedded resolution of the divisor N in \mathbb{P}^n . Let $Z = Z_{n-1}$. Then, $D = p^{-1}(N)$ is a normal crossing divisor in Z , with smooth irreducible components D_X corresponding to the edges $X \in D(\mathcal{A})$. Furthermore, the map p induces a diffeomorphism $Z \setminus D = M(\mathcal{A})$, see [23], [24], [25] for details.

Let \mathcal{L} be a complex local system of rank r on the complement $M(\mathcal{A})$ associated to a representation

$$\rho : \pi_1(M(\mathcal{A}), a) \rightarrow \mathrm{GL}_r(\mathbb{C}).$$

To each irreducible component D_X of the normal crossing divisor D corresponds a well-defined conjugacy class T_X in $\mathrm{GL}_r(\mathbb{C})$, obtained as the monodromy of the local system \mathcal{L} along a small loop turning once in the positive direction about the hypersurface D_X . In this note, we prove the following nonresonance theorem.

THEOREM 1. — *Assume that there is a hyperplane $H \in \mathcal{A}$ such that for any dense edge $X \in D(\mathcal{A})$ with $X \subseteq H$ the corresponding monodromy operator T_X does not admit 1 as an eigenvalue. Then $H^k(M(\mathcal{A}), \mathcal{L}) = 0$ for any $k \neq n$.*

In the case when \mathcal{L} is a rank one complex local system arising in the context of the Milnor fiber associated to \mathcal{A} (see [5] and Section 5 below), this result was obtained by Libgober [20]. We do not see a simple way to

extend the topological proof given by Libgober in this special case to the general case stated above.

Outside the range of vanishing results, the interested reader can find new explicit computations for the dimensions of the cohomology groups $H^k(M(\mathcal{A}), \mathcal{L})$ in [12].

The structure of this note is as follows. Theorem 1 is proved in Section 2. Local systems which arise from flat connections on trivial vector bundles are considered in Section 3. The implications of Theorem 1 in this special case are compared with other nonresonance theorems in Section 4. A brief application to Milnor fibers associated to line arrangements in \mathbb{P}^2 , strengthening a result of Massey [21], is given in Section 5. A strategy to handle arrangements of more general hypersurfaces is presented in Section 6.

2. Proof of Theorem 1.

Before giving the proof of Theorem 1, we say a few words concerning the monodromy operators T_X . Let $\tilde{\mathcal{A}}$ denote the central arrangement in \mathbb{C}^{n+1} corresponding to the arrangement \mathcal{A} in \mathbb{P}^n . The representation $\rho : \pi_1(M(\mathcal{A}), a) \rightarrow \mathrm{GL}_r(\mathbb{C})$ induces a representation

$$\tilde{\rho} : \pi_1(M(\tilde{\mathcal{A}}), \tilde{a}) \rightarrow \mathrm{GL}_r(\mathbb{C}),$$

where \tilde{a} is any lift of the base point a .

For such a representation $\tilde{\rho}$, associated to a local system $\tilde{\mathcal{L}}$ on the complement of $\tilde{\mathcal{A}}$, there is a well-defined *total turn monodromy operator* $T(\tilde{\mathcal{A}}) = \tilde{\rho}(\gamma)$, where $\gamma(t) = \exp(2\pi i t)\tilde{a}$ for $t \in [0, 1]$. Choosing a generic line passing through \tilde{a} and close to (but not through) the origin yields, in the usual way, $m = |\tilde{\mathcal{A}}|$ elementary loops which generate the fundamental group $\pi_1(M(\tilde{\mathcal{A}}), \tilde{a})$. The product in a certain natural order of the associated monodromy operators in $\mathrm{GL}_r(\mathbb{C})$ is easily seen to be exactly the total turn monodromy operator $T(\tilde{\mathcal{A}})$. In particular, if $\tilde{\mathcal{L}}$ (resp., $\tilde{\rho}$) is abelian, the order is irrelevant and $T(\tilde{\mathcal{A}}) = T_1 \cdots T_m$, where T_j is the monodromy around the hyperplane H_j .

Let X be a dense edge of $\tilde{\mathcal{A}}$ and let V be an affine subspace of complementary dimension in \mathbb{C}^{n+1} , transverse to X at a generic point x in X . Consider V as a vector space with origin x and let $\tilde{\mathcal{B}}$ denote the central arrangement in V induced by $\tilde{\mathcal{A}}$. Let \mathcal{B} be the corresponding projective arrangement in $\mathbb{P}(V)$. The following result can be checked by the reader.

LEMMA 2. — *The following conjugacy classes in $GL_r(\mathbb{C})$ coincide:*

- (a) *The monodromy T_X ;*
- (b) *The total turn monodromy $T(\tilde{\mathcal{B}})$;*
- (c) *The monodromy of the restriction of the local system \mathcal{L} to any of the fibers of the projection $M(\tilde{\mathcal{B}}) \rightarrow M(\mathcal{B})$.*

Now we prove Theorem 1. In this proof, we use a partial resolution similar to the resolution $p : Z \rightarrow \mathbb{P}^n$ mentioned in the Introduction.

First, blow up all the dense 0-dimensional edges contained in the chosen hyperplane H . This yields a map $q_1 : W_1 \rightarrow \mathbb{P}^n$. Then, blow up all the proper transforms under q_1 of projective lines corresponding to dense 1-dimensional edges in $D(\mathcal{A})$ which are contained in H . Continuing in this way, we get an embedded resolution of the divisor N in \mathbb{P}^n along H . This yields a map $q = q_{n-1} : W = W_{n-1} \rightarrow \mathbb{P}^n$ such that $E = q^{-1}(N)$ is a normal crossing divisor at any point of $H' = q^{-1}(H)$. Moreover, H' has smooth irreducible components E_X corresponding to the edges $X \in D(\mathcal{A})$, $X \subseteq H$, and q induces a diffeomorphism $W \setminus H' = \mathbb{P}^n \setminus H$. Note that the conjugacy classes T_X for $X \in D(\mathcal{A})$, $X \subseteq H$ constructed from the resolutions Z and W coincide.

Let $U = W \setminus H' = \mathbb{P}^n \setminus H$, and let $i : M(\mathcal{A}) \rightarrow U$ and $j : U \rightarrow W$ be the corresponding inclusions. Denote the derived category of constructible bounded complexes of sheaves of \mathbb{C} -vector spaces on U by $\mathbf{D}_c^b(U)$, and let $\mathcal{F} = Ri_*\mathcal{L}[n] \in \mathbf{D}_c^b(U)$. Since $M(\mathcal{A})$ is smooth, the shifted local system $\mathcal{L}[n]$ is a perverse sheaf on $M(\mathcal{A})$. Moreover, since i is a quasi-finite affine morphism, it preserves perverse sheaves, and hence $\mathcal{F} \in \text{Perv}(U)$, see [16, (10.3.27)].

Since U is a smooth affine variety, by the Artin Theorem and Verdier Duality, see [16, (10.3.5) and (10.3.8)], we have the following vanishing results:

$$(1) \quad \mathbb{H}^k(U, \mathcal{F}) = 0 \text{ for all } k > 0, \text{ and } \mathbb{H}_c^k(U, \mathcal{F}) = 0 \text{ for all } k < 0.$$

Note that we can write $\mathbb{H}^k(U, \mathcal{F}) = H^k(Rs_*Rj_*\mathcal{F})$ and $\mathbb{H}_c^k(U, \mathcal{F}) = H^k(Rs_!Rj_!\mathcal{F})$, where $s : W \rightarrow \text{pt}$ is the constant map to a point. Since \mathbb{P}^n is compact, s is a proper map, and hence $Rs_* = Rs_!$. Consequently, in light of the Leray-type isomorphism

$$H^{k+n}(M(\mathcal{A}), \mathcal{L}) = \mathbb{H}^k(U, \mathcal{F}),$$

to prove Theorem 1, it is enough to establish the following result.

LEMMA 3. — *With the above notation, if for any dense edge $X \in \mathbf{D}(\mathcal{A})$ with $X \subseteq H$ the corresponding monodromy operator T_X does not admit 1 as an eigenvalue, then the canonical morphism $Rj_! \mathcal{F} \rightarrow Rj_* \mathcal{F}$ in $\mathbf{D}_c^b(\mathbb{P}^n)$ is an isomorphism.*

Proof. — The canonical morphism is an isomorphism if and only if the induced morphisms on the level of stalk cohomology are isomorphisms. This local property is clearly satisfied for the stalks at $x \in U$ since U is open.

Consider the case $x \in H'$. Then $\mathcal{H}^*(Rj_! \mathcal{F})_x = 0$ using the proper base change, see [16, (2.6.7)]. To show that $\mathcal{H}^*(Rj_* \mathcal{F})_x = 0$, we have to compute the cohomology groups $\mathcal{H}^k(Rj_* \mathcal{F})_x = H^{k+n}(M(\mathcal{A}) \cap B, \mathcal{L})$, where B is a small open ball in W centered at x .

Since E is a normal crossing divisor at x , it follows that the fundamental group of $M(\mathcal{A}) \cap B = (W \setminus E) \cap B$ is abelian. Using the methods of [14], we can decrease the rank of the local system \mathcal{L} . Repeating this process yields a rank one local system, where the result follows using the Künneth formula, since at least one of the irreducible components of E passing through x corresponds to a dense edge $X \subseteq H$.

This completes the proof of Lemma 3, and hence that of Theorem 1 as well. \square

Remark 4. — A vanishing result similar to Theorem 1 for hyperplane arrangements over algebraically closed fields of positive characteristic may be obtained by using [3] as a reference instead of [16].

Remark 5. — Assume that there is a hyperplane $H \in \mathcal{A}$ such that for any dense edge $X \in \mathbf{D}(\mathcal{A})$ with $X \subseteq H$ and $\text{codim } X \leq c$ the corresponding monodromy operator T_X does not admit 1 as an eigenvalue. Then $H^p(M(\mathcal{A}), \mathcal{L}) = 0$ for any p with $0 \leq p < c$. Indeed, by intersecting with a generic affine subspace E with $\dim E = c$, we obtain a c -homotopy equivalence $M(\mathcal{A}) \cap E \rightarrow M(\mathcal{A})$ induced by the inclusion, and hence isomorphisms $H^p(M(\mathcal{A}) \cap E, \mathcal{L}) = H^p(M(\mathcal{A}), \mathcal{L})$ for $0 \leq p < c$. The assertion follows by applying Theorem 1 to the arrangement in E induced by the arrangement \mathcal{A} .

3. A special case.

In this section, we consider the special case of local systems which arise from flat connections on trivial vector bundles. Write $\mathcal{A} = \{H_1, \dots, H_m\}$ and for each j , let f_j be a linear form with zero locus H_j . Let $\omega_j = d \log(f_j)$, and choose $r \times r$ matrices $P_j \in \text{End}(\mathbb{C}^r)$ which satisfy $\sum_{j=1}^m P_j = 0$. For an edge X of \mathcal{A} , set $P_X = \sum_{X \subseteq H_j} P_j$. Consider the connection on the trivial vector bundle of rank r over $M(\mathcal{A})$ with 1-form $\omega = \sum_{j=1}^m \omega_j \otimes P_j$. The connection is flat if $\omega \wedge \omega = 0$. This is the case if the endomorphisms P_j satisfy

(2)

$$[P_j, P_X] = 0 \text{ for all } j \text{ and edges } X \text{ such that } \text{codim } X = 2 \text{ and } X \subseteq H_j,$$

see [17]. Let \mathcal{L} be the rank r complex local system on $M(\mathcal{A})$ corresponding to the flat connection on the trivial vector bundle over $M(\mathcal{A})$ with 1-form ω .

Remark 6. — An arbitrary local system \mathcal{L} on $M(\mathcal{A})$ need not arise as the sheaf of horizontal sections of a trivial vector bundle equipped with a flat connection as described above. The existence of such a connection is related to the Riemann-Hilbert problem for \mathcal{L} , see Beauville [2], Bolibrukh [4], and Kostov [18]. Even in the simplest case, when $n = 1$ and $|\mathcal{A}| > 3$, there are local systems \mathcal{L} of any rank $r \geq 3$ on $M(\mathcal{A})$ for which the Riemann-Hilbert problem has no solution, see [4, Theorem 3].

For a local system which may be realized as the sheaf of horizontal sections of a trivial vector bundle equipped with a flat connection, Theorem 1 has the following consequence.

COROLLARY 7. — *Assume that there is a hyperplane $H \in \mathcal{A}$ such that*

(3) *none of the eigenvalues of P_X lies in \mathbb{Z} for every dense edge $X \subseteq H$.*

Then

$$H^k(M(\mathcal{A}), \mathcal{L}) = 0 \text{ for } k \neq n.$$

This result is a refinement of the vanishing theorem of Kohno [17], where condition (3) is required to hold for all edges. Next, we recall the following well-known nonresonance theorem of Schechtman, Terao, and Varchenko [24].

THEOREM 8 ([24, Corollary 15]). — *Assume that none of the eigenvalues of P_X lies in $\mathbb{Z}_{\geq 0}$ for every dense edge $X \in \mathcal{D}(\mathcal{A})$. Also suppose that*

$P_i P_j = P_j P_i$ for all i, j . Then

$$H^k(M(\mathcal{A}), \mathcal{L}) = 0 \text{ for } k \neq n.$$

Note that this result pertains only to abelian local systems. This assumption is not necessary.

THEOREM 9. — Assume that

(4) none of the eigenvalues of P_X lies in $\mathbb{Z}_{\geq 0}$ for every dense edge $X \in D(\mathcal{A})$.

Then

$$H^k(M(\mathcal{A}), \mathcal{L}) = 0 \text{ for } k \neq n.$$

Sketch of Proof. — Let $B^\bullet(\mathcal{A})$ denote the algebra of global differential forms on $M(\mathcal{A})$ generated by the 1-forms ω_j , the Brieskorn algebra of \mathcal{A} . Since the endomorphisms P_j satisfy (2), the tensor product $B^\bullet(\mathcal{A}) \otimes \mathbb{C}^r$, with differential given by multiplication by $\omega = \sum_{j=1}^m \omega_j \otimes P_j$, is a complex, which may be realized as a subcomplex of the twisted de Rham complex of $M(\mathcal{A})$ with coefficients in \mathcal{L} .

By work of Esnault, Schectman, and Viehweg [13], refined by Schechtman, Terao, and Varchenko [24], the above assumptions on the eigenvalues of P_X imply that there is an isomorphism

$$H^*(M(\mathcal{A}); \mathcal{L}) \cong H^*(B^\bullet(\mathcal{A}), \omega).$$

Thus it suffices to show that $H^q(B^\bullet(\mathcal{A}), \omega) = 0$ for $q \neq n$.

For an abelian local system, this was established by Yuzvinsky [26]. To extend his argument to an arbitrary local system, it is enough to show that the complex $(B^\bullet(\mathcal{A}), \omega)$ is acyclic for a central arrangement \mathcal{A} . This may be accomplished using the Euler derivation to produce a chain contraction, a straightforward modification of the proof given by Yuzvinsky. \square

4. Comparison.

The purpose of this section is to compare the nonresonance results of the previous section. A local system (resp., a collection (P_1, \dots, P_m) of endomorphisms satisfying (2) and $\sum_{j=1}^m P_j = 0$) will be called \mathcal{A} -nonresonant if it satisfies condition (4), and will be called (\mathcal{A}, H) -nonresonant if it satisfies condition (3). Let \mathbb{I}_r denote the $r \times r$ identity matrix, and note that

if k_1, \dots, k_m are integers with $\sum_{j=1}^m k_j = 0$, then the collections of endomorphisms (P_1, \dots, P_m) and $(P_1 + k_1 \cdot \mathbb{1}_r, \dots, P_m + k_m \cdot \mathbb{1}_r)$ give rise to the same representation and local system. Furthermore, if the endomorphisms P_j satisfy the conditions of (2), then so do the endomorphisms $P_j + k_j \cdot \mathbb{1}_r$. Hence, if the connection with 1-form $\sum_{j=1}^m \omega_j \otimes P_j$ is flat, then so is the connection with 1-form $\sum_{j=1}^m \omega_j \otimes (P_j + k_j \cdot \mathbb{1}_r)$.

Surprisingly, the monodromy condition (3) of Corollary 7 is more stringent than the condition (4) of Theorem 9.

PROPOSITION 10. — *Let \mathcal{L} be a rank r complex local system on $M(\mathcal{A})$ induced by a collection of (\mathcal{A}, H) -nonresonant endomorphisms (P_1, \dots, P_m) . Then there are integers k_1, \dots, k_m so that the collection $(P_1 + k_1 \cdot \mathbb{1}_r, \dots, P_m + k_m \cdot \mathbb{1}_r)$ is \mathcal{A} -nonresonant.*

Proof. — Without loss, assume that the collection of endomorphisms (P_1, \dots, P_m) is (\mathcal{A}, H_1) -nonresonant. Then by (3), for every dense edge X of \mathcal{A} for which $X \subseteq H_1$, the eigenvalues of P_X are not integers.

Let q be a positive integer, greater than the maximum of the absolute values of the eigenvalues of the endomorphisms P_Y , where Y ranges over all dense edges of \mathcal{A} for which $Y \not\subseteq H_1$. Let

$$\widehat{P}_j = \begin{cases} P_1 + (m - 1) \cdot q \cdot \mathbb{1}_r & \text{if } j = 1, \\ P_j - q \cdot \mathbb{1}_r & \text{if } j \neq 1, \end{cases}$$

and note that $\sum_{j=1}^m \widehat{P}_j = \sum_{j=1}^m P_j = 0$.

We assert that $(\widehat{P}_1, \widehat{P}_2, \dots, \widehat{P}_m)$ is an \mathcal{A} -nonresonant collection of endomorphisms. For this, let X be a dense edge of \mathcal{A} . If $X \subseteq H_1$, then the eigenvalues of $\widehat{P}_X = P_X + (m - |X|) \cdot q \cdot \mathbb{1}_r$ are not integers since the eigenvalues of P_X are not integers. If $X \not\subseteq H_1$, then the eigenvalues of $\widehat{P}_X = P_X - |X| \cdot q \cdot \mathbb{1}_r$ are not in $\mathbb{Z}_{\geq 0}$ by the choice of q . Hence the collection of endomorphisms $(\widehat{P}_1, \widehat{P}_2, \dots, \widehat{P}_m)$ satisfies (4), and is thus \mathcal{A} -nonresonant. □

In light of this result, one might speculate that the \mathcal{A} -nonresonance condition (4) and the (\mathcal{A}, H) -nonresonance condition (3) are equivalent. This is not the case, as the following examples illustrate. For simplicity, these examples involve rank 1 local systems. In this context, it is customary to refer to the collection of endomorphisms (P_1, \dots, P_m) as *weights*, and to write $\lambda = (\lambda_1, \dots, \lambda_m) = (P_1, \dots, P_m)$.

Example 11. — Let \mathcal{A} be the arrangement of five lines in \mathbb{P}^2 defined by the polynomial $Q = x(x - z)y(y - z)z$. Order the hyperplanes of \mathcal{A} as indicated by the order of the factors of Q . The dense edges of \mathcal{A} are the hyperplanes, $X_{125} = H_1 \cap H_2 \cap H_5$, and X_{345} . For this arrangement, the weights $\lambda = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -2)$ satisfy the \mathcal{A} -nonresonance condition (4), but there is no integer vector k for which $\lambda + k$ satisfies the (\mathcal{A}, H) -nonresonance condition (3).

Note that the rank 1 local system \mathcal{L} corresponding to the weights λ in the previous example has trivial monodromy about one of the hyperplanes of \mathcal{A} . There are examples where the monodromy about each hyperplane is nontrivial, and (4) holds, but (3) does not hold for any hyperplane of \mathcal{A} .

Example 12. — Let \mathcal{A} be the arrangement of six lines in \mathbb{P}^2 defined by the polynomial $Q = x(x - z)y(y - 2z)(x - y)z$. Order the hyperplanes of \mathcal{A} as indicated by the order of the factors of Q . The dense edges of \mathcal{A} are the hyperplanes, X_{126} , X_{135} , and X_{346} . For this arrangement, the weights $\lambda = (-\frac{5}{3}, \frac{1}{3}, -\frac{5}{3}, \frac{1}{3}, \frac{7}{3}, \frac{1}{3})$ satisfy the \mathcal{A} -nonresonance condition (4), but there is no integer vector k for which $\lambda + k$ satisfies the (\mathcal{A}, H) -nonresonance condition (3).

In both examples, the monodromy about each rank 2 dense edge is trivial, so the (\mathcal{A}, H) -nonresonance condition (3) cannot hold for any integer translate of the weights, but the weights nevertheless satisfy the \mathcal{A} -nonresonance condition (4).

5. An application to Milnor fibers of line arrangements.

Let $m = |\mathcal{A}|$ be the number of hyperplanes in the arrangement $\mathcal{A} \subset \mathbb{P}^n$ and let $Q = 0$ be a reduced equation for the corresponding central arrangement $\tilde{\mathcal{A}}$ in \mathbb{C}^{n+1} . The smooth affine hypersurface F in \mathbb{C}^{n+1} given by the equation $Q = 1$ is called the Milnor fiber of the arrangement \mathcal{A} (resp., $\tilde{\mathcal{A}}$). There is a naturally associated monodromy operator $h : F \rightarrow F$ given by multiplication by $\tau = \exp(2\pi i/m)$, satisfying $h^m = 1$. For $k, p \in \mathbb{Z}$, let $b_p(F)_k$ denote the dimension of the τ^k -eigenspace corresponding to the monodromy action on $H^p(F, \mathbb{C})$. It is known that

$$b_p(F)_k = \dim H^p(M(\mathcal{A}), \mathcal{L}_k)$$

where \mathcal{L}_k is the rank one local system on $M(\mathcal{A})$ corresponding to the monodromies $T_1 = T_2 = \dots = T_m = \tau^k$, see [5]. In this section, we prove the following strengthening of a result of Massey [21].

THEOREM 13. — *Let \mathcal{A} be a line arrangement in \mathbb{P}^2 , with associated Milnor fiber F . Then for any integer $0 < k < m$ and any line H in the arrangement \mathcal{A} we have*

$$b_1(F)_k \leq \sum_x (m_x - 2)$$

where the sum is over all points $x \in H$ such that the multiplicity of \mathcal{A} at x is $m_x > 2$ and m divides km_x .

Proof. — In this proof, we work directly on \mathbb{P}^2 (without using the partial resolution W from Section 2). Let $i : M(\mathcal{A}) \rightarrow \mathbb{P}^2 \setminus H$ and $j : \mathbb{P}^2 \setminus H \rightarrow \mathbb{P}^2$ denote the natural inclusions, and set $\mathcal{F} = Ri_*\mathcal{L}[2]$. Recall that \mathcal{F} is perverse, and extend the canonical morphism $Rj_!\mathcal{F} \rightarrow Rj_*\mathcal{F}$ in $\mathbf{D}_c^b(\mathbb{P}^2)$ to a distinguished triangle

$$(5) \quad Rj_!\mathcal{F} \rightarrow Rj_*\mathcal{F} \rightarrow \mathcal{G} \rightarrow .$$

Let x be a point of multiplicity m_x on the chosen line H . Applying the functors \mathcal{H}_x^k to the triangle (5), we obtain a long exact sequence, which yields in particular

$$\begin{aligned} \mathcal{H}^{-1}(\mathcal{G})_x &= \mathcal{H}^{-1}(Rj_*\mathcal{F})_x = H^1(M_x, \mathcal{L}) \\ \text{and } \mathcal{H}^0(\mathcal{G})_x &= \mathcal{H}^0(Rj_*\mathcal{F})_x = H^2(M_x, \mathcal{L}). \end{aligned}$$

Here $M_x = M(\mathcal{A}) \cap B_x$, where B_x is a small open ball centered at x . It follows that M_x is homeomorphic to the complement of the central line arrangement in \mathbb{C}^2 defined by $f(y) = y_1^{m_x} + y_2^{m_x}$. There are two cases. If m does not divide km_x , then $H^q(M_x, \mathcal{L}) = 0$ for all q . On the other hand, if m divides km_x , then $\dim H^1(M_x, \mathcal{L}) = \dim H^2(M_x, \mathcal{L}) = m_x - 2$, see for instance [8, p. 109] and [5], [21].

Now apply the functors \mathbb{H}^k to the triangle (5) and use the vanishing results (1). This yields an exact sequence

$$0 \rightarrow H^1(M(\mathcal{A}), \mathcal{L}) \rightarrow \mathbb{H}^{-1}(\mathbb{P}^2, \mathcal{G}) \rightarrow H_c^2(M(\mathcal{A}), \mathcal{L}) \rightarrow \dots$$

To compute the middle term in this sequence, we use the spectral sequence

$$E_1^{p,q} = H^p(S, \mathcal{H}^q\mathcal{G})$$

where S is the support of \mathcal{G} , a finite set. It follows that $\mathbb{H}^{-1}(\mathbb{P}^2, \mathcal{G}) = H^0(S, \mathcal{H}^{-1}\mathcal{G})$ is a \mathbb{C} -vector space of dimension $\sum_x (m_x - 2)$ where the sum is over all points $x \in H$ such that the multiplicity of \mathcal{A} at x is $m_x > 2$ and m divides km_x . □

Remark 14. — The proof of Theorem 13 uses only the fact that the support of the sheaf \mathcal{G} is finite. This may happen for arrangements in \mathbb{P}^n for $n > 2$, yielding a more general version of the theorem.

An alternative proof of Theorem 13 (as well as a proof of a related result), using only basic facts on the topology of polynomial functions and their monodromy, can be found in [10].

6. Generalizations.

Our main results can be stated (and proved in the same way, either using partial resolutions or working directly in the projective space) for arrangements of hypersurfaces $N = \bigcup_{i=1}^m V_i$ in a projective space \mathbb{P}^n . See [6] and the references there for other results in this setting. It is not necessary to assume that the individual hypersurfaces are smooth. It is enough to impose local vanishing assumptions, both for the intersections contained in a fixed hypersurface, say V_1 , and at all singular points of V_1 itself.

In the hyperplane arrangement case, we can treat the local cohomology groups whose vanishing is necessary in the proof of Lemma 3 in terms of complements of central arrangements. This allows us to decrease the dimension by one and proceed by induction.

In the general hypersurface arrangement case, this induction is no longer available, since the complement need not be locally a cone over a projective arrangement of smaller dimension. However, the following approach may be used to obtain vanishing results in this generality.

Let $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ be an analytic function germ and let $\mathcal{F} = \mathbb{C}[n + 1]$ be the perverse sheaf on \mathbb{C}^{n+1} obtained by shifting the constant sheaf \mathbb{C} . It is known that perverse sheaves are preserved by the perverse vanishing cycle functor, [16, Corollary 10.3.13]. Thus ${}^p\phi_f(\mathcal{F}) \in \text{Perv}(X)$, where $X = f^{-1}(0)$. There is a natural monodromy automorphism $\mu : {}^p\phi_f(\mathcal{F}) \rightarrow {}^p\phi_f(\mathcal{F})$. For any $a \in \mathbb{C}$, we can consider the eigenspace $\mathcal{F}_a = \ker(\mu - a \cdot \text{Id})$, which is a well-defined perverse sheaf on X , since the category $\text{Perv}(X)$ is abelian, [16, Proposition 10.1.11].

For any point $x \in X$, $\mathcal{H}^m({}^p\phi_f(\mathcal{F}))_x = H^{m+n}(F_x)$, where F_x is the local Milnor fiber of f at the point x . Moreover, the induced action of μ on $\mathcal{H}^m({}^p\phi_f(\mathcal{F}))_x$ corresponds exactly to the usual monodromy action on the local Milnor fiber F_x .

Let S_a be the support of the sheaf \mathcal{F}_a and let $s_a = \dim S_a$, with the convention $\dim \emptyset = -1$. Note that the integer s_a depends only on the hypersurface germ $(X, 0)$: indeed, any two reduced equations for this germ are topologically equivalent (since the contact equivalence classes are connected), see [8, Remark 3.1.8].

It follows that $\mathcal{F}_a \in \text{Perv}(S_a)$, see [9, Section (5.2)]. Hence the support condition in the definition of perverse sheaves gives $\mathcal{H}^m({}^p\phi_f(\mathcal{F}))_x = 0$ for any $m < -s_a$. This implies that

$$H^{n-s_a-j}(F_0)_a = 0$$

for all $j > 0$.

Using the Milnor fibration of f at the origin, we can identify the corresponding Milnor fiber F_0 with an infinite cyclic covering of U_0 , the local complement of X in $(\mathbb{C}^{n+1}, 0)$. For $a \in \mathbb{C}^*$, we denote by \mathcal{L}_a the rank one local system on U_0 whose monodromy around each irreducible component of X is multiplication by a .

Then it is well-known that

$$\dim H^q(U_0, \mathcal{L}_a) = \dim H^{q-1}(F_0)_a + \dim H^q(F_0)_a,$$

see for instance [19], [11]. It follows that

$$\dim H^q(U_0, \mathcal{L}_a) = 0,$$

for all $q \leq n - 1 - s_a$. Applying this local vanishing result to the global setting of hypersurface arrangements as in Section 2 above, we obtain the following (note that $n + 1$ is replaced by $n!$).

THEOREM 15. — *Let $N = \bigcup_{i=1}^m V_i$ be a hypersurface arrangement in \mathbb{P}^n , with associated Milnor fiber F . Let $d = d_1 + \cdots + d_m$ be the degree of N . For each point $x \in V_1$, denote by $s(x, k)$ the number s_{τ^k} associated to the hypersurface germ (N, x) as above, with $\tau = \exp(2\pi i/d)$. Let $s_k = \max_{x \in V_1} s(x, k)$. Then for any integer $0 < k < d$, we have*

$$b_q(F)_k = 0$$

for all $q \leq n - 2 - s_k$.

COROLLARY 16. — *If $N = \bigcup_{i=1}^m V_i$ is a normal crossing divisor at any point $x \in V_1$, then the monodromy action on $H^q(F)$ is trivial for $q \leq n - 1$.*

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