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ON CONTACT P -SPHERES

by Mathias ZESSIN

1. Introduction and generalities.

Contact circles and contact p -spheres are families of contact forms parametrized by the circle and the p -sphere respectively and have been introduced by H. Geiges and J. Gonzalo in 1995 (see [4]). These authors give a fundamental existence theorem for contact circles and closely study a special class of contact circles with additional geometrical properties (see also [5]). Their work is about 3-dimensional manifolds.

In the first part of this paper we are interested in *invariant* contact circles and contact p -spheres on principal \mathbb{S}^1 -bundles. In this specific situation we adapt some methods developed by R. Lutz which have been used to construct the first examples of contact forms on the 5-dimensional torus (see [9]). They will be particularly helpful for the construction of new classes of examples. In this work, we will see examples of invariant contact circles and contact spheres on 3-dimensional principal \mathbb{S}^1 -bundles. In the second part, we examine the general situation in higher dimensions and give examples of contact p -spheres there.

In paragraph 2 we start with some examples. In dimension 3, the most natural ones are found on \mathbb{S}^3 . In paragraph 3, we develop tools to

study invariant contact circles and contact spheres. In paragraphs 4 and 5 we state the main theorems of this first part. They solve the existence problem of invariant contact circles and contact spheres on principal \mathbb{S}^1 -bundles over closed, connected and orientable surfaces. In paragraph 6 we construct examples on all such principal \mathbb{S}^1 -bundles where contact circles might exist according to the theorems of paragraphs 4 and 5. In paragraph 7 we consider higher dimensions and prove that on manifolds of dimension $4n + 1$, contact circles and contact p -spheres do not exist. In paragraph 8 we study round contact p -spheres, that is, contact p -spheres whose Reeb vector fields are stable under linear combinations within the contact p -sphere, and in paragraph 9 we give examples of contact p -spheres on the spheres of dimension $4n - 1$. In paragraph 10 we make a connection to Sasakian geometry, which gives more examples of contact spheres.

Let us now give some definitions and preliminary results.

Let M be a $(2n + 1)$ -dimensional manifold. A contact form on M is a 1-form ω such that $\omega \wedge (d\omega)^n$ is a volume form on M . The Reeb vector field R associated to ω is defined by $\omega(R) = 1$ and $R \lrcorner d\omega = 0$.

DEFINITION 1. — *The set of normalized linear combinations of $p + 1$ contact forms $\omega_1, \dots, \omega_{p+1}$*

$$\left\{ \sum_{i=1}^{p+1} \lambda_i \omega_i \mid \sum_{i=1}^{p+1} \lambda_i^2 = 1 \right\}$$

is called a contact p -sphere, if every element of this family is a contact form. It will be denoted by $\mathcal{S}_c^p\{\omega_1, \dots, \omega_{p+1}\}$.

Note that Geiges and Gonzalo call contact circle a pair (ω_1, ω_2) of contact forms whose normalized linear combinations are still contact forms and contact sphere a triple $(\omega_1, \omega_2, \omega_3)$ of such forms. Here, however, contact p -sphere will denote the whole family generated by these forms.

In this paper, a contact 1-sphere will be called a *contact circle*, and a contact 2-sphere might simply be called *contact sphere*. Note that each contact sphere contains contact circles and that in general each contact m -sphere contains contact n -spheres for $n \leq m$.

Following Geiges and Gonzalo, a contact circle or a contact p -sphere will be called *taut*, if all its elements generate the same volume form, that is, if for any element $\omega = \lambda_1 \omega_1 + \lambda_2 \omega_2$ of, say, a contact circle in dimension 3, the form $\omega \wedge d\omega$ does not depend on its coefficients λ_1 and λ_2 .

2. Examples on \mathbb{S}^3 and on \mathbb{T}^3 .

We use a quaternionic representation of \mathbb{S}^3 to get the following natural example of a contact sphere:

In the space of quaternions \mathbb{H} , the forms

$$\begin{cases} \tilde{\alpha} = \langle qi, dq \rangle = q_1 dq_2 - q_2 dq_1 + q_4 dq_3 - q_3 dq_4 \\ \tilde{\beta} = \langle qj, dq \rangle = q_1 dq_3 - q_3 dq_1 + q_2 dq_4 - q_4 dq_2 \\ \tilde{\gamma} = \langle qk, dq \rangle = q_3 dq_2 - q_2 dq_3 + q_1 dq_4 - q_4 dq_1 \end{cases}$$

induce three contact forms α, β and γ on the unit sphere \mathbb{S}^3 , which are independent and complementary in the sense that their Reeb vector fields give an orthonormal basis of the tangent space $T_p\mathbb{S}^3$ at every point $p \in \mathbb{S}^3$ with respect to the induced Euclidean metric.

Any form

$$\tilde{\omega} = \lambda_1 \tilde{\alpha} + \lambda_2 \tilde{\beta} + \lambda_3 \tilde{\gamma} \quad \text{with} \quad \sum_{i=1}^3 \lambda_i^2 = 1$$

satisfies

$$\tilde{\omega} \wedge d\tilde{\omega} \wedge (q_1 dq_1 + q_2 dq_2 + q_3 dq_3 + q_4 dq_4) = dq_1 \wedge dq_2 \wedge dq_3 \wedge dq_4,$$

which is non-zero everywhere. Hence the induced form $\omega = \lambda_1 \alpha + \lambda_2 \beta + \lambda_3 \gamma$ is a contact form and $\omega \wedge d\omega = \alpha \wedge d\alpha$. This gives a taut contact sphere. It is also round in a sense defined in paragraph 8.

We also observe that the forms α, β and γ are invariant with respect to the vector field X induced by

$$q_1 \frac{\partial}{\partial q_2} - q_2 \frac{\partial}{\partial q_1} + q_3 \frac{\partial}{\partial q_4} - q_4 \frac{\partial}{\partial q_3},$$

which is the Reeb vector field of the contact form $\hat{\alpha}$ induced by $\langle iq, dq \rangle$ on \mathbb{S}^3 . The orbits of X are the fibres of a principal Hopf fibration of \mathbb{S}^3 over \mathbb{S}^2 with fibre \mathbb{S}^1 and connection form $\hat{\alpha}$. In this case, $\mathcal{S}_c^2\{\alpha, \beta, \gamma\}$ is an invariant contact sphere on this \mathbb{S}^1 -bundle.

$$\begin{array}{c} \mathbb{S}^3 \subset \mathbb{H} \\ \pi(q)=q i \bar{q} \downarrow \mathbb{S}^1 \\ \mathbb{S}^2 \end{array}$$

Another quite natural example of contact circles can be found on \mathbb{T}^3 , with pseudo-coordinates $(\theta_1, \theta_2, \theta_3)$. On this torus, consider the forms

$$\begin{cases} \omega_1 &= \cos(n\theta_1)d\theta_2 + \sin(n\theta_1)d\theta_3 \\ \omega_2 &= -\sin(n\theta_1)d\theta_2 + \cos(n\theta_1)d\theta_3, \end{cases}$$

for some non-zero integer n . Setting $\omega = \lambda_1 \omega_1 + \lambda_2 \omega_2$, with $\lambda_1^2 + \lambda_2^2 = 1$, we get

$$\omega \wedge d\omega = -n d\theta_1 \wedge d\theta_2 \wedge d\theta_3.$$

Thus for any n , ω_1 and ω_2 generate a taut contact circle. We can observe that ω_1 and ω_2 and thus all elements of the contact circle are invariant with respect to the vector fields $\frac{\partial}{\partial\theta_2}$ and $\frac{\partial}{\partial\theta_3}$. So these contact circles are even invariant with respect to the corresponding \mathbb{T}^2 -action.

3. Singular sets and knotted fibrations.

In this paragraph we develop tools to study invariant contact circles and contact p -spheres on principal \mathbb{S}^1 -bundles over connected closed manifolds. We first examine singular sets associated to invariant contact forms in this context and then consider these sets as knots of knotted fibrations. This approach allows great insight into the topology of invariant contact circles and contact p -spheres.

Consider a $(2n + 1)$ -dimensional manifold M , which is a principal \mathbb{S}^1 -bundle over a $(2n)$ -dimensional manifold B with connection form α . Assume that B is a compact, connected, orientable manifold without boundary. Then for each invariant form ω on M , there is a form η and a function φ on the base space B , such that

$$\omega = \pi^*(\eta) + \pi^*(\varphi)\alpha,$$

where π is the bundle map.

We have the following technical lemma:

LEMMA 1. — *Let $\mathcal{S}_c^p\{\omega_1, \dots, \omega_{p+1}\}$ be an invariant contact p -sphere on M and assume that for each i ,*

$$\omega_i = \pi^*(\eta_i) + \pi^*(\varphi_i)\alpha$$

for some forms and functions η_i and φ_i on B . Then for any coefficients $(\lambda_1, \dots, \lambda_{p+1})$ with $\sum_{i=1}^{p+1} \lambda_i^2 = 1$, the expressions

$$(1) \quad \sum_{i=1}^{p+1} \lambda_i \varphi_i \text{ and } \sum_{i=1}^{p+1} \lambda_i d\varphi_i$$

do not vanish simultaneously at any point of B .

Proof. — This is an immediate consequence of the contact property of linear combinations of the forms $\omega_1, \dots, \omega_{p+1}$. Indeed, for $\omega = \sum_{i=1}^{p+1} \lambda_i \omega_i$, we have

$$\omega \wedge (d\omega)^n = \left(\sum_{i=1}^{p+1} \lambda_i \eta_i + \sum_{i=1}^{p+1} \lambda_i \varphi_i \alpha \right) \wedge \left(\sum_{i=1}^{p+1} \lambda_i d\eta_i + \sum_{i=1}^{p+1} \lambda_i d\varphi_i \wedge \alpha + \sum_{i=1}^{p+1} \lambda_i \varphi_i d\alpha \right)^n.$$

This term equals

$$\left(\sum_{i=1}^{p+1} \lambda_i \eta_i \right) \wedge \left(\sum_{i=1}^{p+1} \lambda_i d\eta_i \right)^n,$$

at points where the expressions (1) vanish simultaneously. But a $(2n + 1)$ -form on B_{2n} is zero. This contradicts the assumption that $\mathcal{S}_c^p\{\omega_1, \dots, \omega_{p+1}\}$ is a contact p -sphere and proves the lemma. \square

Let us now define on the base manifold the singular set associated to a given invariant contact form on M , following R. Lutz (see [8]).

DEFINITION 2. — *Let $\omega = \pi^*(\eta) + \pi^*(\varphi)\alpha$ be an invariant contact form on an \mathbb{S}^1 -bundle M over a manifold B . The singular set associated to ω is the set $\Sigma_\omega = \varphi^{-1}(0)$ on B .*

We now prove some fundamental properties of the singular sets of invariant contact circles and contact p -spheres on a principal \mathbb{S}^1 -bundle M over a closed connected manifold B .

PROPOSITION 1. —

- i) *The singular sets are submanifolds of B .*
- ii) *If M admits an invariant contact circle, then each point of B lies on the singular set of some element of this contact circle.*
- iii) *The singular sets of two forms of the same contact circle or contact p -sphere are isotopic.*
- iv) *The singular set of an element of an invariant contact circle or contact p -sphere is non-empty.*
- v) *The singular sets of two different and non-opposite elements of an invariant contact circle or contact p -sphere intersect transversally.*

Proof. — According to R. Lutz ([8]) (or to Lemma 1 in the particular case of $p = 0$), $d\varphi$ does not vanish on $\varphi^{-1}(0)$, if $\omega = \pi^*(\eta) + \pi^*(\varphi)\alpha$ is an invariant contact form on M . Thus the corresponding singular set is a submanifold of B .

Let ω_1 and ω_2 generate an invariant contact circle on M . Writing, for $i = 1, 2$,

$$\omega_i = \pi^*(\eta_i) + \pi^*(\varphi_i)\alpha,$$

an arbitrary element of the contact circle will be

$$\begin{aligned} \omega &= \sin(\theta)\omega_1 + \cos(\theta)\omega_2 \\ &= \pi^*(\sin(\theta)\eta_1 + \cos(\theta)\eta_2) + \pi^*((\sin(\theta)\varphi_1 + \cos(\theta)\varphi_2))\alpha \\ &= \pi^*(\eta) + \pi^*(\varphi)\alpha. \end{aligned}$$

Thus, for any values $\varphi_1(p)$ and $\varphi_2(p)$, there is some $\theta \in [0, 2\pi]$ such that $\varphi(p) = 0$.

This argument shows also that in any contact circle there are elements whose singular sets are not empty. On the other hand, according to [6] (see also [10]), the contact structures associated to elements of one-parameter-families of contact forms are isomorphic. Obviously, contact circles are special one-parameter-families, so the corresponding structures are isomorphic. The isomorphisms are equivariant, as explained in [8], so the singular sets of elements of a contact circle are isotopic. Thus any element of a contact circle has a non-empty singular set.

In contact p -spheres, p being an arbitrary dimension, any two different and non-opposite elements generate a contact circle, so their singular sets are non-empty and isotopic, too.

Finally, at a point where the singular sets of two different and non-opposite elements $\omega_1 = \pi^*(\eta_1) + \pi^*(\varphi_1)\alpha$ and $\omega_2 = \pi^*(\eta_2) + \pi^*(\varphi_2)\alpha$ of an invariant contact circle intersect, we have $\varphi_1 = \varphi_2 = 0$, so by Lemma 1, the forms $d\varphi_1$ and $d\varphi_2$ are linearly independent, hence the intersection of Σ_{ω_1} and Σ_{ω_2} is transversal. \square

One way to study singular sets and thus invariant contact p -spheres is to consider their associated knotted fibrations. This viewpoint is due to R. Lutz and generalizes J. Milnor’s open book constructions in the context of invariant contact forms. It means the following (see [9]):

DEFINITION 3. — (B, φ, N) is a knotted fibration along N over \mathbb{S}^{k-1} , if

1. B is a connected, orientable, compact manifold
2. N is either empty or a closed codimension- k -submanifold in B
3. $\varphi : B \setminus N \longrightarrow \mathbb{S}^{k-1}$ is a locally trivial fibration
4. If N is non-empty, there exists an open neighborhood W of N and a diffeomorphism $h : N \times D^k \longrightarrow W$, where D^k is the unit disk of \mathbb{R}^k , such that $h(z, 0) = z$ on $N \times \{0\}$ and such that the following diagram commutes:

$$\begin{array}{ccc}
 N \times (D^k \setminus \{0\}) & \xrightarrow{h} & W \setminus N \\
 \text{proj.} \downarrow & & \varphi \downarrow \\
 D^k \setminus \{0\} & \xrightarrow{\|\cdot\|} & \mathbb{S}^{k-1}.
 \end{array}$$

N will be called the knot and $\varphi^{-1}(p)$ will be called a fibre of the knotted fibration, for any point $p \in \mathbb{S}^{k-1}$.

As an example in dimension 3, we can consider singular sets associated to invariant contact p -spheres on principal \mathbb{S}^1 -bundles over surfaces in different ways:

- Given one contact form $\omega = \pi^*(\eta) + \pi^*(\varphi)\alpha$, the singular set $\Sigma_\omega = \varphi^{-1}(0)$ is a curve on the base space and defines a knotted fibration over \mathbb{S}^0 with knot Σ_ω and two fibres determined by the sign of the function φ . These fibres are well defined (see [8]).
- For a contact circle generated by two invariant contact forms ω_1 and ω_2 , the intersection Σ of the associated singular sets Σ_{ω_1} and Σ_{ω_2} is

finite. Σ is the knot of a knotted fibration of B over \mathbb{S}^1 , whose fibres are curves starting and ending at points of Σ .

- In the case of an invariant contact sphere with generating forms ω_1, ω_2 and ω_3 , there is no common singular set, which is due to Lemma 1, but we can consider an associated knotted fibration of M over \mathbb{S}^2 with empty knot and fibres which are finite subsets of B .

These are special cases of the following fibration theorem which gives this decomposition in full generality:

THEOREM 1. — *Fibration theorem*

Let M be a $(2n+1)$ -dimensional principal \mathbb{S}^1 -bundle over a closed connected base manifold B_{2n} with connection form α and let $\mathcal{S}_c^p\{\omega_1, \dots, \omega_{p+1}\}$ be an invariant contact p -sphere on M . Write ω_i as $\omega_i = \pi^*(\eta_i) + \pi^*(\varphi_i)\alpha$, for $i=1, \dots, p+1$, and let $\Sigma = \bigcap_{i=1}^{p+1} \varphi_i^{-1}(0)$ be the intersection of the singular sets of the generating forms. Then for $r = \sqrt{\sum_{i=1}^{p+1} \varphi_i^2}$,

$$(B, (\frac{\varphi_1}{r}, \dots, \frac{\varphi_{p+1}}{r}), \Sigma)$$

defines a knotted fibration over \mathbb{S}^p .

Proof. — According to [9], it is enough to show that the map

$$\Phi = (\frac{\varphi_1}{r}, \dots, \frac{\varphi_{p+1}}{r}) : B \setminus \Sigma \rightarrow \mathbb{S}^p$$

is of rank p everywhere and that at any point of Σ , we have

$$d\varphi_1 \wedge \dots \wedge d\varphi_{p+1} \neq 0.$$

The first condition is a consequence of Lemma 1. Indeed, we show that if Φ is not of rank p , then there are coefficients $(\lambda_1, \dots, \lambda_{p+1})$ with $\sum_{i=1}^{p+1} \lambda_i^2 = 1$, such that

$$\sum_{i=1}^{p+1} \lambda_i \varphi_i = 0 \text{ and } \sum_{i=1}^{p+1} \lambda_i d\varphi_i = 0,$$

at some points of $B \setminus \Sigma$, which is excluded by Lemma 1.

On a point $x \in B \setminus \Sigma$, at least one of the functions φ_i does not vanish. Assume that φ_{p+1} is non-zero. Then the rank of Φ is given by the rank of the system $(d(\frac{\varphi_1}{r}), \dots, d(\frac{\varphi_p}{r}))$.

Assume now that there is some point $x \in B \setminus \Sigma$ and p real numbers μ_1, \dots, μ_p , such that $\sum_{i=1}^p \mu_i d(\frac{\varphi_i}{r}) = 0$. Differentiating this and using $dr = \frac{1}{r} (\sum_{i=1}^{p+1} \varphi_i d\varphi_i)$, we get

$$0 = \frac{1}{r} \left(\sum_{i=1}^p \mu_i d\varphi_i \right) - \sum_{i=1}^p \frac{\mu_i \varphi_i}{r^2} dr = \sum_{i=1}^{p+1} \lambda_i d\varphi_i,$$

where we set $\lambda_i = \frac{\mu_i}{r} - (\sum_{j=1}^p \frac{\mu_j \varphi_j}{r^2}) \frac{\varphi_i}{r}$, for $i = 1, \dots, p$ and $\lambda_{p+1} = -(\sum_{j=1}^p \frac{\mu_j \varphi_j}{r^2}) \frac{1}{r} \varphi_{p+1}$. With the same coefficients, we have now

$$\begin{aligned} \sum_{i=1}^{p+1} \lambda_i \varphi_i &= \sum_{i=1}^p \frac{\mu_i \varphi_i}{r} - \sum_{i=1}^p \varphi_i^2 \sum_{j=1}^p \frac{\mu_j \varphi_j}{r^3} - \sum_{i=1}^p \frac{\mu_i \varphi_i}{r^3} \varphi_{p+1}^2 \\ &= \sum_{i=1}^p \frac{\mu_i \varphi_i}{r^3} \left(r^2 - \sum_{j=1}^{p+1} \varphi_j^2 \right) \\ &= 0. \end{aligned}$$

The second condition is another consequence of Lemma 1. As on Σ the functions φ_i vanish, any expression $\sum_{i=1}^{p+1} \lambda_i d\varphi_i$ is non-zero by Lemma 1. Thus the forms $\{d\varphi_i, i = 1 \dots p + 1\}$ are linearly independent on Σ , hence $d\varphi_1 \wedge \dots \wedge d\varphi_{p+1} \neq 0$ on Σ . This completes the proof. \square

COROLLARY 1. — *Let $\mathcal{S}_c^1\{\omega_1, \omega_2\}$ be an invariant contact circle on an \mathbb{S}^1 -bundle M over a closed connected surface B with common singular set Σ . Then $B \setminus \Sigma$ fibres over \mathbb{S}^1 .*

This corollary is a useful tool in certain situations. It gives us some additional information about the singular sets associated to invariant contact circles on principal \mathbb{S}^1 -bundles over surfaces, for example the following properties.

PROPOSITION 2. —

- a) *The singular set Σ_ω of an element ω of a contact circle generated by ω_1 and ω_2 on a principal \mathbb{S}^1 -bundle over \mathbb{S}^2 is a topological circle and the singular sets of two different and non-opposite elements intersect in two points.*

- b) *The singular set Σ_ω of an element ω of a contact circle generated by ω_1 and ω_2 on a principal \mathbb{S}^1 -bundle over \mathbb{T}^2 is the union of an even number of topological circles and the singular sets of two different and non-opposite elements do not intersect.*

Proof. — In the case $B = \mathbb{S}^2$, it is enough to show that Σ_{ω_1} is a topological circle. As it is a non-empty submanifold of \mathbb{S}^2 , it is a union of topological circles. To see why Σ_{ω_1} can not have several connected components, we first observe that the common singular set $\Sigma = \Sigma_{\omega_1} \cap \Sigma_{\omega_2}$ is the union of two points, as $\mathbb{S}^2 \setminus \Sigma$ fibres over \mathbb{S}^1 , and that the fibres are non-closed curves starting and ending at different components of Σ . As the fibration map is $\Phi = (\frac{\varphi_1}{r}, \frac{\varphi_2}{r})$, where $r = \sqrt{\varphi_1^2 + \varphi_2^2}$, $\Sigma_{\omega_1} \setminus \Sigma$ is the union of two fibres which are given by $\Phi^{-1}(0, 1)$ and $\Phi^{-1}(0, -1)$ and which meet on Σ . Thus Σ_{ω_1} is a topological circle.

In the case $B = \mathbb{T}^2$, the common singular set Σ is empty, as $\mathbb{T}^2 \setminus \Sigma$ fibres over \mathbb{S}^1 . So the singular sets of linearly independent elements of an invariant contact circle over \mathbb{T}^2 do not intersect. As before, the singular set of an element ω is non-empty and the number of its components is even, because the sign of the corresponding function φ is different in adjacent regions of $\mathbb{T}^2 \setminus \Sigma_\omega$ and because the components of Σ_ω do not bound disks, as there is an isotopy which carries any such component into itself, filling the whole torus on the way. \square

4. Invariant contact circles in dimension 3.

We are now looking for principal \mathbb{S}^1 -bundles over orientable, connected closed surfaces which admit invariant contact circles. It is in fact the base manifold that carries all the information with respect to this question. We have the following theorem:

THEOREM 2. — *Let M be a principal circle-bundle over an orientable, connected closed surface B . There exists an invariant contact circle on M if and only if the base space B is either the 2-sphere or the 2-torus.*

Proof (necessary part). — Here we prove that only principal \mathbb{S}^1 -bundles over \mathbb{S}^2 or \mathbb{T}^2 might carry an invariant contact circle. The converse will be proved in a constructive way in paragraph 6.

Let ω_1 and ω_2 generate an invariant contact circle on M . The singular sets Σ_{ω_1} and Σ_{ω_2} are unions of topological circles on B . By Corollary 1, the complement of their intersection Σ fibres over \mathbb{S}^1 .

If Σ is empty, B is an orientable, connected closed surface which fibres over \mathbb{S}^1 , thus the torus \mathbb{T}^2 .

If Σ is non-empty, it is a finite union of points, as we have seen in paragraph 3. According to the Fibration theorem 1, each component of Σ has a neighborhood which is diffeomorphic to a disk and each radius of this disk corresponds to a different fibre of the associated knotted fibration over \mathbb{S}^1 . As each fibre starts and ends at two different components of Σ , Σ has at least two components. $B \setminus \Sigma$ is a locally trivial fibration over \mathbb{S}^1 , so for reasons of continuity all fibres starting at a given component of Σ end at the same component of Σ . With these restrictions the fibred surface can only be a twice punctured sphere, that is, $B = \mathbb{S}^2$. \square

5. Invariant contact spheres in dimension 3.

For invariant *contact spheres* on principal circle-bundles over orientable, connected closed surfaces, the possibilities are even more restricted, as any contact sphere contains contact circles. In fact, contact spheres do not exist on principal circle-bundles over 2-tori, only on those over 2-spheres. We have the following theorem, analogous to the one in section 4:

THEOREM 3. — *Let M be a principal circle-bundle over a connected, orientable closed surface B . There exists an invariant contact sphere on M if and only if the base space B is the 2-sphere.*

Here again, we will only prove that the condition is necessary, whereas the constructions in paragraph 6 will prove that it is also sufficient.

Proof of theorem 3 (necessary part). — By Theorem 2 it is clear that the base space can only be \mathbb{S}^2 or \mathbb{T}^2 . Let us see why it can not be the torus.

Assume that ω_1, ω_2 and ω_3 generate an invariant contact sphere on a principal circle bundle M over \mathbb{T}^2 , with $\omega_i = \pi^*(\eta_i) + \pi^*(\varphi_i)\alpha$, for $i = 1, 2, 3$. We show that there exist two forms in the contact sphere whose singular sets intersect, which contradicts Proposition 2.

Remember that the singular set of any element of the contact sphere is a non-empty union of topological circles on B , as we have seen in paragraph 3.

Let Γ be one connected component of Σ_{ω_1} . By Proposition 2, the singular sets of two elements of the contact sphere do not intersect, so the function φ associated to an arbitrary element of the contact sphere is everywhere non-zero on Γ , except for φ_1 and $-\varphi_1$. Let ω be an element of $\mathcal{S}_c^2\{\omega_1, \omega_2, \omega_3\}$ other than ω_1 and $-\omega_1$ and let

$$(2) \quad \varphi = \sum_{i=1}^3 \lambda_i \varphi_i, \quad \text{where } \sum_{i=1}^3 \lambda_i^2 = 1 \text{ and } \lambda_1 \notin \{-1, 1\},$$

be the associated function on the base space. We can assume that φ is positive on Γ . Then there is a path on the unit sphere of triples $(\lambda_1, \lambda_2, \lambda_3)$ relating the coefficients of φ in expression (2) to those of $-\varphi$ which does not take the values $(1, 0, 0)$ or $(-1, 0, 0)$, the coefficients of $\pm\varphi_1$. This path corresponds to a path in $\mathcal{S}_c^2\{\omega_1, \omega_2, \omega_3\}$ connecting ω and $-\omega$ and avoiding ω_1 and $-\omega_1$. Thus we continuously transform φ , which is positive on Γ , into $-\varphi$, which is negative on Γ . Then there is an intermediate function $\tilde{\varphi}$, which corresponds to some element $\tilde{\omega}$ of $\mathcal{S}_c^2\{\omega_1, \omega_2, \omega_3\}$ and which has zeros on Γ . Thus Γ and $\Sigma_{\tilde{\omega}}$ intersect. □

6. Construction of examples.

In the preceding paragraphs we proved necessary conditions for the existence of invariant contact circles and contact spheres on principal \mathbb{S}^1 -bundles over surfaces. We will now see that they are also sufficient. On any manifold mentioned in Theorems 2 and 3, that is, on principal \mathbb{S}^1 -bundles over \mathbb{S}^2 or \mathbb{T}^2 , examples can be constructed.

Consider a principal circle-bundle M over \mathbb{S}^2 with bundle map π and connection form α and three functions φ_1, φ_2 and φ_3 on \mathbb{S}^2 such that $\sum \varphi_i^2 = 1$ everywhere. We define the following 1-forms on M :

$$\begin{cases} \omega_1 &= \pi^*(\varphi_2 d\varphi_3 - \varphi_3 d\varphi_2) + \pi^*(\varphi_1) k \alpha \\ \omega_2 &= \pi^*(\varphi_3 d\varphi_1 - \varphi_1 d\varphi_3) + \pi^*(\varphi_2) k \alpha \\ \omega_3 &= \pi^*(\varphi_1 d\varphi_2 - \varphi_2 d\varphi_1) + \pi^*(\varphi_3) k \alpha, \end{cases}$$

where k is a real number.

LEMMA 2. — Consider the form $\omega = \lambda_1 \omega_1 + \lambda_2 \omega_2 + \lambda_3 \omega_3$ for some $\lambda_1, \lambda_2, \lambda_3$ such that $\sum \lambda_i^2 = 1$. Then we have the formula:

$$\omega \wedge d\omega = k \pi^* \left(\left(\sum_{i=1}^3 \lambda_i \varphi_i \right) (\lambda_1 d\varphi_2 \wedge d\varphi_3 + \lambda_2 d\varphi_3 \wedge d\varphi_1 + \lambda_3 d\varphi_1 \wedge d\varphi_2) \right) \wedge \alpha$$

$$+ k \pi^* (\Phi^*(\Omega)) \wedge \alpha + k^2 \pi^* \left(\left(\sum_{i=1}^3 \lambda_i \varphi_i \right)^2 \right) \alpha \wedge d\alpha$$

where Ω is the Euclidean volume form on \mathbb{S}^2 and $\Phi = (\varphi_1, \varphi_2, \varphi_3) : \mathbb{S}^2 \rightarrow \mathbb{S}^2$.

Proof. — This is an easy, though lengthy calculation using the identities $\sum \lambda_i^2 = 1$, $\sum \varphi_i^2 = 1$ and $\Phi^*(\Omega) = \varphi_3 d\varphi_1 \wedge d\varphi_2 + \varphi_1 d\varphi_2 \wedge d\varphi_3 + \varphi_2 d\varphi_3 \wedge d\varphi_1$ and the fact that $d\alpha$ is an horizontal form. The only form involved which is not a pullback from \mathbb{S}^2 is α . □

This lemma gives us the possibility to construct contact spheres on principal \mathbb{S}^1 -bundles over \mathbb{S}^2 . Indeed, up to an adjustment of the connection form α , the forms $(\omega_1, \omega_2, \omega_3)$ given above generate a contact sphere for the right choice of k and of the functions $(\varphi_1, \varphi_2, \varphi_3)$.

Consider a function triple $(\varphi_1, \varphi_2, \varphi_3)$ such that $\sum_{i=1}^3 \varphi_i^2 = 1$ and which defines a knotted fibration of \mathbb{S}^2 over \mathbb{S}^2 , i.e. the corresponding common singular set $\bigcap_{i=1}^3 \varphi_i^{-1}(0)$ is empty and $\Phi = (\varphi_1, \varphi_2, \varphi_3)$ is of rank two everywhere. As the structural group of the fibration is abelian, the form $d\alpha$ is horizontal, thus there is a 2-form ν on the base space B , such that $d\alpha = \pi^*(\nu)$. Since $\Phi^*(\Omega)$ is a volume form on \mathbb{S}^2 , there is a function f such that $\nu = f \Phi^*(\Omega)$ and some functions C_1, C_2, C_3 , such that $d\varphi_2 \wedge d\varphi_3 = C_1 \Phi^*(\Omega)$, $d\varphi_3 \wedge d\varphi_1 = C_2 \Phi^*(\Omega)$ and $d\varphi_1 \wedge d\varphi_2 = C_3 \Phi^*(\Omega)$. The right hand expression in Lemma 2 now becomes

(3)

$$k \left(\pi^* \left(\left(1 + k f \left(\sum_{i=1}^3 \lambda_i \varphi_i \right)^2 + \left(\sum_{i=1}^3 \lambda_i \varphi_i \right) (\lambda_1 C_1 + \lambda_2 C_2 + \lambda_3 C_3) \right) \Phi^*(\Omega) \right) \wedge \alpha \right).$$

To make this form a volume form, we observe that the function $\left(\sum_{i=1}^3 \lambda_i \varphi_i \right) (\lambda_1 C_1 + \lambda_2 C_2 + \lambda_3 C_3)$ is bounded on \mathbb{S}^2 and that it is thus sufficient to choose k large enough and of the right sign, if f is a non-vanishing function of constant sign. Next, we notice that adding the function $h = \frac{\int_{\mathbb{S}^2} f \Phi^*(\Omega)}{\int_{\mathbb{S}^2} \Phi^*(\Omega)} - f$ to f makes f a constant function.

This is possible, for the following reason: the form $\eta = h \Phi^*(\Omega)$ satisfies $\int_{\mathbb{S}^2} \eta = 0$, so $\pi^*(\eta)$ is an exact form on M and there is a 1-form ξ such that $d\xi = \pi^*(\eta)$. The fibration of M over \mathbb{S}^2 is characterized by the value of $\int_{\mathbb{S}^2} \nu$, where $\pi^*(\nu)$ is the differential of the connection form, so $\alpha + \xi$ is another connection form of the same fibration. Thus replacing α by $\alpha + \xi$ changes f into a constant function, without modifying the fibration.

Now assume f to be constant. If f is not identically zero, choosing k large enough and of the same sign as f makes the expression (3) a volume form. If $f \equiv 0$, then the manifold M is equivariantly diffeomorphic to $\mathbb{S}^2 \times \mathbb{S}^1$ and there are known examples of invariant contact spheres on such manifolds (see [5], p.274). This gives us examples of invariant contact spheres on all principal \mathbb{S}^1 -bundles over \mathbb{S}^2 .

Let us now construct examples of invariant contact circles on principal \mathbb{S}^1 -bundles over \mathbb{T}^2 . In this case we consider the following 1-forms on M :

$$\begin{cases} \omega_1 &= \pi^*(\cos \theta_1 d\theta_2) + \pi^*(\sin \theta_1) k \alpha \\ \omega_2 &= \pi^*(-\sin \theta_1 d\theta_2) + \pi^*(\cos \theta_1) k \alpha, \end{cases}$$

θ_1 and θ_2 being pseudo-coordinates on \mathbb{T}^2 , α a connection form for the fibration of M over \mathbb{T}^2 and k a real number. Setting $\omega = \lambda_1 \omega_1 + \lambda_2 \omega_2$, we get the formula

$$(4) \quad \omega \wedge d\omega = -k d\theta_1 \wedge d\theta_2 \wedge \alpha + k^2(\lambda_1 \sin \theta_1 + \lambda_2 \cos \theta_1)^2 \alpha \wedge d\alpha.$$

Similarly to the previously discussed case, there is a function f such that $d\alpha = f d\theta_1 \wedge d\theta_2$, and we can assume that f is constant, up to a modification of α by addition of a 1-form whose differential comes from a basic 2-form integrating to zero over the torus. Thus, (4) becomes

$$(5) \quad \omega \wedge d\omega = k(k f (\lambda_1 \sin \theta_1 + \lambda_2 \cos \theta_1)^2 - 2) d\theta_1 \wedge d\theta_2 \wedge \alpha.$$

It is now obvious that it is enough to choose k non-zero and such that $k \cdot f$ is not positive to make $\omega \wedge d\omega$ a volume form, for any coefficients (λ_1, λ_2) .

It may be interesting to see that a contact circle obtained by this construction is taut only if f is identically zero, that is, for $M = \mathbb{T}^3$.

The above construction leads to the following proposition, which gives examples of invariant contact circles and contact spheres on all manifolds

mentioned in Theorems 2 and 3. Thus it finishes the proves of these two theorems.

PROPOSITION 3. — *Let M be a circle-bundle over \mathbb{S}^2 with connection form α and φ_1, φ_2 and φ_3 three functions on \mathbb{S}^2 such that $\sum \varphi_i^2 = 1$ everywhere and which define a knotted fibration of \mathbb{S}^2 over \mathbb{S}^2 . Then there exist a real number k and a 1-form ξ on M , such that the 1-forms*

$$\begin{cases} \omega_1 &= \pi^*(\varphi_2 d\varphi_3 - \varphi_3 d\varphi_2) + \pi^*(\varphi_1)k(\alpha + \xi) \\ \omega_2 &= \pi^*(\varphi_3 d\varphi_1 - \varphi_1 d\varphi_3) + \pi^*(\varphi_2)k(\alpha + \xi) \\ \omega_3 &= \pi^*(\varphi_1 d\varphi_2 - \varphi_2 d\varphi_1) + \pi^*(\varphi_3)k(\alpha + \xi) \end{cases}$$

generate an invariant contact sphere on M .

Similarly, if M is a circle-bundle over \mathbb{T}^2 with connection form $\tilde{\alpha}$ and pseudo-coordinates θ_1, θ_2 on \mathbb{T}^2 , then there exist a real number \tilde{k} and a 1-form $\tilde{\xi}$ on M , such that the 1-forms

$$\begin{cases} \omega_1 &= \pi^*(\cos \theta_1 d\theta_2) + \pi^*(\sin \theta_1) \tilde{k}(\tilde{\alpha} + \tilde{\xi}) \\ \omega_2 &= \pi^*(-\sin \theta_1 d\theta_2) + \pi^*(\cos \theta_1) \tilde{k}(\tilde{\alpha} + \tilde{\xi}) \end{cases}$$

generate an invariant contact circle on M .

7. Non-existence of contact p -spheres in dimension $4n + 1$.

We now consider the situation in higher dimensions. It quickly becomes obvious that in dimension 5 it is not easy to construct contact circles, although there are, of course, 5-manifolds admitting contact forms. In fact, we prove the following:

THEOREM 4. — *On 5-dimensional manifolds, and more generally on $(4n + 1)$ -dimensional manifolds, contact circles and a fortiori contact p -spheres do not exist for $p \geq 1$.*

Proof. — Let (ω_1, ω_2) be a pair of contact forms on a 5-dimensional manifold M . It generates a contact circle if $\omega = \lambda_1 \omega_1 + \lambda_2 \omega_2$ is a contact form for any (λ_1, λ_2) with $\lambda_1^2 + \lambda_2^2 = 1$, that is, if

$$\omega \wedge (d\omega)^2 = \sum_{i,j,k=1}^2 \lambda_i \lambda_j \lambda_k (\omega_i \wedge d\omega_j \wedge d\omega_k)$$

is everywhere non-zero.

Now fix a point $x \in M$ and five linearly independent tangent vectors

$$(v_1, \dots, v_5) \in \mathcal{T}_x(M)$$

and consider the function

$$\begin{aligned} \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (\lambda_1, \lambda_2) &\longmapsto \sum_{i,j,k=1}^2 \lambda_i \lambda_j \lambda_k (\omega_i \wedge d\omega_j \wedge d\omega_k)_x(v_1, \dots, v_5), \end{aligned}$$

which is a homogeneous polynomial function of degree 3. It has zeros on the unit circle of \mathbb{R}^2 (if it is positive at some point of the circle, it is negative at its antipode), so $\mathcal{S}_c^1\{\omega_1, \omega_2\}$ cannot be a contact circle, as for the corresponding coefficients (λ_1, λ_2) , $\omega \wedge (d\omega)^2$ is not a volume form.

In general, pairs of contact forms in dimension $4n + 1$ give us polynomial functions of degree $2n + 1$, which is odd, so they necessarily vanish on the unit circle of \mathbb{R}^2 . Thus, contact circles do not exist in these dimensions.

On the other hand, in dimension $4n - 1$, these polynomial functions are of degree $2n$, which is even, so there is no restriction to the existence of contact circles in these dimensions. □

8. Round contact p -spheres.

Let us now consider contact p -spheres with a special regularity property. H. Geiges and J. Gonzalo call a contact p -sphere *taut* if all elements yield the same volume form. We introduce a more geometrical property which is equivalent to tautness in dimension 3, as we shall see.

8.1. Definition and examples.

DEFINITION 4. — *Let $\{\omega_1, \dots, \omega_{p+1}\}$ generate a contact p -sphere and let R_1, \dots, R_{p+1} be the corresponding Reeb vector fields. Then $\mathcal{S}_c^p\{\omega_1, \dots, \omega_{p+1}\}$ is called a round contact p -sphere if any form $\sum_{i=1}^{p+1} \lambda_i \omega_i$ with $\sum_{i=1}^{p+1} \lambda_i^2 = 1$ admits $\sum_{i=1}^{p+1} \lambda_i R_i$ as its Reeb vector field.*

So the Reeb vector field of a linear combination of elements of a round contact p -sphere is the corresponding linear combination of the Reeb vector

fields of these elements. This is a strong restriction and quite useful for computations.

We have the following characterization of round contact p -spheres:

LEMMA 3. — *Let $\{\omega_1, \dots, \omega_{p+1}\}$ generate a contact p -sphere on some manifold M and let R_1, \dots, R_{p+1} be the corresponding Reeb vector fields. Then $\mathcal{S}_c^p\{\omega_1, \dots, \omega_{p+1}\}$ is round if and only if the following conditions are satisfied:*

- i) $\omega_i(R_j) + \omega_j(R_i) = 0$, for $i, j \in \{1, \dots, p + 1\}$, $i \neq j$
- ii) $R_i \lrcorner d\omega_j + R_j \lrcorner d\omega_i = 0$, for $i, j \in \{1, \dots, p + 1\}$.

Proof. — Consider a contact form $\omega = \sum_{i=1}^{p+1} \lambda_i \omega_i$ with normalized coefficients. $R = \sum_{i=1}^{p+1} \lambda_i R_i$ is the corresponding Reeb vector field if and only if

- a) $(\sum_{i=1}^{p+1} \lambda_i \omega_i)(\sum_{i=1}^{p+1} \lambda_i R_i) = 1$
- b) $(\sum_{i=1}^{p+1} \lambda_i R_i) \lrcorner (\sum_{i=1}^{p+1} \lambda_i d\omega_i) = 0$.

Condition *a)* means that $\sum_{i < j} \lambda_i \lambda_j (\omega_i(R_j) + \omega_j(R_i)) = 0$. If $\mathcal{S}_c^p\{\omega_1, \dots, \omega_{p+1}\}$ is a round contact p -sphere, this equality is true for any normalized $(p + 1)$ -tuple of coefficients $(\lambda_1, \dots, \lambda_{p+1})$. Considering the left-hand side as a homogeneous polynomial on \mathbb{S}^p , we conclude that all coefficients must be zero, that is, condition *i)* is satisfied. The converse is of course true.

The equivalence of conditions *b)* and *ii)* is proved in the same way. \square

Remark. — This lemma shows that a contact p -sphere is round if and only if every contact circle it contains is round. The same is true for tautness. So to show that a contact p -sphere is round or taut, it is enough to examine contact circles for this property.

Examples. —

1. We have mentioned an example of a round contact 2-sphere on \mathbb{S}^3 in paragraph 2. Indeed, consider \mathbb{S}^3 as the unit sphere of the group of quaternions \mathbb{H} and the contact sphere generated by ω_1, ω_2 and ω_3 induced on \mathbb{S}^3 by

$$\begin{cases} \tilde{\omega}_1 &= < iq, dq > \\ \tilde{\omega}_2 &= < jq, dq > \\ \tilde{\omega}_3 &= < kq, dq > . \end{cases}$$

The corresponding Reeb vector fields are

$$\begin{cases} R_1 &= iq \\ R_2 &= jq \\ R_3 &= kq. \end{cases}$$

Using Lemma 3, we can check that this contact sphere is indeed round:

In this example, the Reeb vector fields of ω_1, ω_2 and ω_3 are also their dual vector fields, so $\omega_i(R_j) = \delta_{ij}$. So condition *i*) is satisfied.

On the other hand, as $d\tilde{\omega}_1 = -d\bar{q} \wedge idq$ and $d\tilde{\omega}_2 = -d\bar{q} \wedge jdq$, we have

$$\begin{aligned} R_{2\lrcorner}d\tilde{\omega}_1 &= \bar{q} j i dq + d\bar{q} i j q \\ R_{1\lrcorner}d\tilde{\omega}_2 &= \bar{q} i j dq + d\bar{q} j i q \\ &= -(\bar{q} j i dq + d\bar{q} i j q). \end{aligned}$$

So $R_{2\lrcorner}d\omega_1 + R_{1\lrcorner}d\omega_2 = 0$ and the remaining relations of condition *ii*) are obtained in the same way.

2. There is also an example of a round contact circle on \mathbb{T}^3 . Indeed, the forms

$$\begin{cases} \omega_1 &= \cos \theta_1 d\theta_2 + \sin \theta_1 d\theta_3 \\ \omega_2 &= -\sin \theta_1 d\theta_2 + \cos \theta_1 d\theta_3 \end{cases}$$

generate a contact circle and their Reeb vector fields are respectively

$$\begin{cases} R_1 &= \cos \theta_1 \frac{\partial}{\partial \theta_2} + \sin \theta_1 \frac{\partial}{\partial \theta_3} \\ R_2 &= -\sin \theta_1 \frac{\partial}{\partial \theta_2} + \cos \theta_1 \frac{\partial}{\partial \theta_3}. \end{cases}$$

Once again, we have $\omega_i(R_j) = \delta_{ij}$, for $i, j = 1, 2$. So condition *i*) of Lemma 3 is satisfied. Moreover, we have

$$\begin{cases} d\omega_1 &= -\sin \theta_1 d\theta_1 \wedge d\theta_2 + \cos \theta_1 d\theta_1 \wedge d\theta_3 \\ d\omega_2 &= -\cos \theta_1 d\theta_1 \wedge d\theta_2 - \sin \theta_1 d\theta_1 \wedge d\theta_3, \end{cases}$$

so $R_{1\lrcorner}d\omega_2 = d\theta_1 = -R_{2\lrcorner}d\omega_1$ and condition *ii*) is also satisfied.

8.2. Round \Leftrightarrow taut in dimension 3.

We now prove that in dimension 3, roundness is indeed equivalent to tautness. This gives us the possibility to have different viewpoints on the same property. Let us first prove the following lemma:

LEMMA 4. — *The Reeb vector fields of two generating elements of a contact circle are everywhere linearly independent.*

Proof. — Let ω_1 and ω_2 generate a contact circle and let R_1 and R_2 be the corresponding Reeb vector fields. Assume that in some point p , $(R_1)_p$ and $(R_2)_p$ are parallel. Then $d\omega_1(R_1)_p = 0$ and $d\omega_2(R_1)_p = 0$, so $d\omega(R_1)_p = 0$, for any linear combination ω of ω_1 and ω_2 . Thus the Reeb vector fields of all elements of the contact circle are parallel to R_1 . Now, the Reeb vector field of $-\omega_1$, which is an element of the contact circle, is $-R_1$, and as the Reeb vector field depends continuously on the coefficients, there must be some form in the contact circle with a Reeb vector field of length zero. This is of course impossible. \square

THEOREM 5. — *On a 3-dimensional manifold M , a contact circle (resp. a contact sphere) is taut if and only if it is round.*

Proof. — Let us first see that taut contact circles are round. Let $\mathcal{S}_c^1\{\omega_1, \omega_2\}$ be a taut contact circle, that is, satisfying

$$(6) \quad \begin{cases} \omega_1 \wedge d\omega_1 = \omega_2 \wedge d\omega_2 \\ \omega_1 \wedge d\omega_2 = -\omega_2 \wedge d\omega_1 \end{cases}$$

and let R_1 and R_2 be the corresponding Reeb vector fields. Applying the first equation of (6) to the couple of vectors (R_1, R_2) , we get

$$R_1 \lrcorner d\omega_2 + R_2 \lrcorner d\omega_1 = 0,$$

which is the second condition of roundness of Lemma 3. Applying the second equation of (6) to the couple (R_1, R_2) and using the previous relation, we get

$$(R_1 \lrcorner d\omega_2)(\omega_1(R_2) + \omega_2(R_1)) = 0,$$

which gives us the first condition of roundness of Lemma 3, if we are sure that $R_1 \lrcorner d\omega_2$ never vanishes. But this is granted by Lemma 6. So taut contact circles are round.

It is now immediate that taut contact spheres are round, because the roundness conditions of Lemma 3 carry on pairs of generating contact forms. So if two generating forms of a contact sphere satisfy the tautness condition (6), then they also satisfy the roundness condition, as we have just seen.

Let us now see that round contact spheres are taut. Let $(\omega_1, \omega_2, \omega_3)$ generate a round contact sphere and consider an element $\omega = \sum_{i=1}^3 \lambda_i \omega_i$. By assumption, the corresponding Reeb vector field is $R = \sum_{i=1}^3 \lambda_i R_i$, where R_i is the Reeb vector field of the form ω_i . Thus non-trivial linear combinations of R_1, R_2 and R_3 never vanish, so R_1, R_2 and R_3 are everywhere linearly independent vector fields, hence a coordinate system on M . To prove that $\mathcal{S}_c^2\{\omega_1, \omega_2, \omega_3\}$ is taut, it is enough to show that $\omega \wedge d\omega(R_1, R_2, R_3)$ is independent of the coefficients $(\lambda_1, \lambda_2, \lambda_3)$. An evaluation of the relation $R_2 \lrcorner d\omega_1 + R_1 \lrcorner d\omega_2 = 0$ on the vector field R_3 yields

$$d\omega_1(R_2, R_3) = d\omega_2(R_3, R_1),$$

and by an analogous evaluation we get

$$d\omega_1(R_2, R_3) = d\omega_3(R_1, R_2).$$

So we have

$$\begin{aligned} \omega \wedge d\omega(R_1, R_2, R_3) &= (\lambda_1 \omega_1 + \lambda_2 \omega_2 + \lambda_3 \omega_3) \\ &\quad \wedge (\lambda_1 d\omega_1 + \lambda_2 d\omega_2 + \lambda_3 d\omega_3)(R_1, R_2, R_3) \\ &= \lambda_1^2 d\omega_1(R_2, R_3) + \lambda_2^2 d\omega_2(R_3, R_1) + \lambda_3^2 d\omega_3(R_1, R_2) \\ &= d\omega_1(R_2, R_3), \end{aligned}$$

where the second step is due to the preceding relations and Lemma 3. This expression does not depend on the coefficients $(\lambda_1, \lambda_2, \lambda_3)$.

It remains to show that round contact circles are taut in dimension 3. So let ω_1 and ω_2 generate a round contact circle and let R_1 and R_2 be the corresponding Reeb vector fields. By Lemma 4, R_1 and R_2 are everywhere linearly independent, so we can find a third vector field X to make (R_1, R_2, X) a basis of the tangent bundle. An evaluation of the relation $R_1 \lrcorner d\omega_2 + R_2 \lrcorner d\omega_1 = 0$ on X gives us

$$d\omega_1(R_2, X) = d\omega_2(X, R_1).$$

Now an analogous calculation as in the case of contact spheres shows that the contact circle is taut. \square

8.3. Round vs taut in higher dimensions.

In dimensions higher than 3, roundness and tautness are not equivalent. In fact, from dimension 7 on, there is an essential difference between the two notions, due to the powers of the differentials which appear in the definition of tautness and not in the definition of roundness. To illustrate this difference, let us consider the situation in dimension 7.

A contact circle $\mathcal{S}_c^1\{\omega_1, \omega_2\}$ defined on a 7-dimensional manifold is taut if the following four equalities are satisfied:

$$(7) \quad \begin{cases} \omega_1 \wedge (d\omega_1)^3 - \omega_2 \wedge (d\omega_2)^3 & = 0 \\ 3\omega_2 \wedge d\omega_2 \wedge (d\omega_1)^2 + 3\omega_1 \wedge d\omega_1 \wedge (d\omega_2)^2 - 2\omega_1 \wedge (d\omega_1)^3 & = 0 \\ \omega_1 \wedge (d\omega_2)^3 + 3\omega_2 \wedge d\omega_1 \wedge (d\omega_2)^2 & = 0 \\ \omega_2 \wedge (d\omega_1)^3 + 3\omega_1 \wedge d\omega_2 \wedge (d\omega_1)^2 & = 0. \end{cases}$$

On the other hand, if $\mathcal{S}_c^1\{\omega_1, \omega_2\}$ is taut and round, we have the following equalities, which are necessary, but not sufficient:

$$(8) \quad \begin{cases} (d\omega_1)^3 - 3d\omega_1 \wedge (d\omega_2)^2 & = 0 \\ (d\omega_2)^3 - 3d\omega_2 \wedge (d\omega_1)^2 & = 0. \end{cases}$$

This can be seen as follows: Let $\omega_\theta = \cos\theta\omega_1 + \sin\theta\omega_2$ be an element of $\mathcal{S}_c^1\{\omega_1, \omega_2\}$ and note $\Omega = \omega_1 \wedge (d\omega_1)^3$. As $\mathcal{S}_c^1\{\omega_1, \omega_2\}$ is taut, we have $\omega_\theta \wedge (d\omega_\theta)^3 = \Omega$, and thus $(d\omega_\theta)^3 = R_\theta \lrcorner \Omega$, where R_θ is the Reeb vector field of ω_θ . This relation leads to the conditions (8).

The systems (7) et (8) are not of the same nature, as the equations of the first one are of degree 7 and those of the second one are of degree 6. So in general, a contact circle which satisfies (7) will not satisfy (8). In dimension 3, the corresponding equations of (8) are trivial, this is why we have equivalence of tautness and roundness in dimension 3.

The following counter-examples prove that in dimension 7 there is no implication between tautness and roundness:

1. The contact circle on \mathbb{R}^7 generated by the forms

$$\begin{cases} \omega_1 & = x_1 dx_2 + x_3 dx_4 + x_5 dx_6 + dx_7 \\ \omega_2 & = -(x_5 + x_6) dx_3 - x_5 dx_4 + (x_1 + x_3) dx_6 + x_1 dx_7 - dx_2 \end{cases}$$

is round and not taut.

2. The contact circle on \mathbb{R}^7 generated by the forms

$$\begin{cases} \omega_1 &= x_1 dx_2 + x_3 dx_4 + x_5 dx_6 + dx_7 \\ \omega_2 &= x_5 dx_4 - x_3 dx_6 + (x_1 + x_3) dx_7 - dx_2 \end{cases}$$

is taut and not round.

9. Examples of contact p -spheres in higher dimensions.

We have seen that contact circles do not exist on manifolds of dimension $4n + 1$. In the other odd dimensions, however, many interesting examples of contact p -spheres can be found. A first family of manifolds where contact p -spheres are likely to be found easily are the spheres of dimension $4n - 1$.

On the spheres, we have a natural upper bound of the size of contact p -spheres, given by Adams' formula. According to Adams, there do not exist more than $\rho(n)$ continuous unitary vector fields on \mathbb{S}^{n-1} , which are everywhere linearly independent, where

$$\rho(n) = 2^c + 8d - 1, \text{ with } n = \text{odd} \cdot 2^{c+4d}, c \leq 3.$$

This means of course that there can not be more than $\rho(n)$ everywhere linearly independent contact forms neither. So on \mathbb{S}^{4n-1} , contact p -spheres can only exist for $p \leq (\rho(4n) - 1)$.

On the other hand, there are works of B. Eckmann, relying on ideas of A. Hurwitz and J. Radon, leading to the following theorem:

THEOREM 6. — *On \mathbb{S}^{4n-1} , there exists a contact $(\rho(4n) - 1)$ -sphere, for $n \geq 1$.*

Proof. — According to B. Eckmann (see [7]), there exist, for each integer n , $\rho(4n)$ antisymmetric matrices $A_1, \dots, A_{\rho(4n)}$ of $\mathcal{O}(4n, \mathbb{R})$, such that the vectors $(A_1 x)_x, \dots, (A_{\rho(4n)} x)_x$ are linearly independent everywhere on the unit sphere of \mathbb{R}^{4n} , that is, Eckmann gives a realization of the maximum number of such vector fields given by Adam's formula. The same matrices can also be used to construct a contact $(\rho(4n) - 1)$ -sphere on \mathbb{S}^{4n-1} , as we will see now.

The matrices $A_1, \dots, A_{\rho(4n)}$ satisfy the relations

$$(9) \quad \begin{cases} A_i A_j + A_j A_i = 0, & i \neq j \\ A_i^2 = -Id, & i = 1, \dots, \rho(4n), \end{cases}$$

that is, they generate a Clifford algebra.

Now define $\rho(4n)$ 1-forms on \mathbb{R}^{4n} by

$$(\tilde{\omega}_i)_x = \langle A_i x, dx \rangle = \sum_{r,s=1}^{4n} a_{rs}^i x_s dx_r,$$

where $A_i = (a_{rs}^i)_{r,s=1 \dots 4n}$.

These forms induce some forms ω_i on \mathbb{S}^{4n-1} , which are contact forms: as each matrix A_i is orthogonal and antisymmetric, there is an orthogonal basis of \mathbb{R}^{4n} in which A_i becomes

$$(10) \quad \begin{pmatrix} 0 & -1 & & & & \\ 1 & 0 & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & 0 & -1 \\ & & & & 1 & 0 \end{pmatrix}.$$

In this basis, we have

$$\langle A_i x, dx \rangle = \sum_{i=1}^{2n} (x_{2i-1} dx_{2i} - x_{2i} dx_{2i-1}),$$

which induces a contact form on \mathbb{S}^{4n-1} .

Moreover, and quite surprisingly, these $\rho(4n)$ contact forms generate a $(\rho(4n) - 1)$ -contact sphere on \mathbb{S}^{4n-1} . Each matrix

$$\sum_{i=1}^{\rho(4n)} \lambda_i A_i \quad \text{with} \quad \sum_{i=1}^{\rho(4n)} \lambda_i^2 = 1$$

is indeed antisymmetric and it is orthogonal as well, as

$$\begin{aligned} {}^t \left(\sum_{i=1}^{\rho(4n)} \lambda_i A_i \right) \left(\sum_{i=1}^{\rho(4n)} \lambda_i A_i \right) &= - \sum_{i=1}^{\rho(4n)} \lambda_i^2 A_i^2 - \sum_{i < j} \lambda_i \lambda_j (A_i A_j + A_j A_i) \\ &= Id. \end{aligned}$$

So the corresponding 1-form $\langle \sum_{i=1}^{\rho(4n)} \lambda_i A_i x, dx \rangle$ induces a contact form, as we have already seen. □

This theorem gives us examples of contact p -spheres on \mathbb{S}^{4n-1} with $p \geq 2$ for any $n \geq 1$. According to Adams' formula, we have $\rho(4n) = 2^c + 8d - 1$, where $4n = \text{odd} \cdot 2^{c+4d}$, $c \leq 3$. In our situation, we have $c \geq 2$ or $d \geq 1$ and in both cases $\rho(4n) \geq 3$, which means that on \mathbb{S}^{4n-1} , there is a contact p -sphere generated by at least 3 contact forms.

It may be interesting to see that these examples are round and taut.

Indeed, if we consider a contact circle $\mathcal{S}_c^1\{\omega_1, \omega_2\}$ which is contained in one of these examples on some sphere \mathbb{S}^{4n-1} , the Reeb vector field of a form ω_i induced by $(\tilde{\omega}_i)_x = \langle A_i x, dx \rangle$ on \mathbb{S}^{4n-1} is $(R_i)_x = (A_i x)_x$. Then we have

$$\begin{cases} \tilde{\omega}_1(R_2) &= \langle A_1 x, A_2 x \rangle = \langle -A_2 A_1 x, x \rangle = 0 \\ \tilde{\omega}_2(R_1) &= \langle A_2 x, A_1 x \rangle = \langle -A_1 A_2 x, x \rangle = 0 \\ (R_2 \lrcorner d\tilde{\omega}_1)_x &= {}^t x A_1 A_2 dx = -{}^t x A_2 A_1 dx = -(R_1 \lrcorner d\tilde{\omega}_2)_x, \end{cases}$$

so $\mathcal{S}_c^1\{\omega_1, \omega_2\}$ is round.

Now consider an element of the contact circle

$$\omega = \lambda_1 \omega_1 + \lambda_2 \omega_2 \quad \text{with} \quad \lambda_1^2 + \lambda_2^2 = 1.$$

ω is induced by $(\tilde{\omega})_x = \langle (\lambda_1 A_1 + \lambda_2 A_2)x, dx \rangle$. There is an orthogonal, thus volume preserving coordinate change which transforms the matrix $\lambda_1 A_1 + \lambda_2 A_2$ into the matrix (10). Thus ω defines the same volume form as ω_1 and $\mathcal{S}_c^1\{\omega_1, \omega_2\}$ is taut.

10. Contact spheres and Sasakian 3-structures.

Another way to find examples of contact spheres on higher dimensional manifolds is to consider Sasakian geometry, and more precisely Sasakian 3-structures. As a Sasakian 3-structure is defined by three contact forms with special properties, we can ask if all non-trivial linear combinations of these forms are still contact forms, that is, if they define a contact sphere. In that case, we can study the regularity properties of such a contact sphere. Proposition 4, which has been suggested to me by the referee, answers these questions. Let us first recall the definition of a Sasakian 3-structure (see [2]).

Let (M, g) be a Riemannian manifold which carries a contact form ω and let R be the Reeb vector field associated to ω . We define the tensor field φ of type $(1, 1)$ by

$$g(X, \varphi(Y)) = \frac{1}{2} d\omega(X, Y).$$

ω is a metric contact form if it satisfies

$$g(\varphi(X), \varphi(Y)) = g(X, Y) - \omega(X)\omega(Y),$$

or equivalently

$$\begin{cases} \varphi^2 &= -I + \omega \otimes R \\ \omega(Y) &= g(R, Y), \text{ for any } Y. \end{cases}$$

(φ, R, ω) defines a Sasakian structure if it satisfies

$$[\varphi, \varphi] + d\omega \otimes R = 0,$$

where $[\cdot, \cdot]$ is the Nijenhuis bracket, defined by

$$[T, T](X, Y) = T^2[X, Y] + [TX, TY] - T[TX, Y] - T[X, TY].$$

Three given Sasakian structures ω_1, ω_2 et ω_3 on M define a Sasakian 3-structure if for even permutations (i, j, k) of $(1, 2, 3)$ the following properties are satisfied:

$$(11) \quad \begin{cases} \varphi_k &= \varphi_i \varphi_j - \omega_j \otimes R_i &= -\varphi_j \varphi_i + \omega_i \otimes R_j \\ R_k &= \varphi_i(R_j) &= -\varphi_j(R_i) \\ \omega_k &= \omega_i \circ \varphi_j &= -\omega_j \circ \varphi_i, \end{cases}$$

where R_i is the Reeb vector field associated to ω_i .

PROPOSITION 4. — *If a manifold M of dimension $4n - 1$ admits a Sasakian 3-structures given by three forms ω_1, ω_2 and ω_3 , then these forms generate a contact sphere which is both round and taut.*

Proof. — Let R_1, R_2 and R_3 be the Reeb vector fields associated to ω_1, ω_2 and ω_3 . We define the tensor fields φ_i by $g(X, \varphi_i(Y)) = \frac{1}{2} d\omega_i(X, Y)$. Then for $i = 1, 2, 3$, $(\varphi_i, R_i, \omega_i)$ defines a Sasakian structure and ω_1, ω_2

and ω_3 define a Sasakian 3-structure. Therefore we have:

$$\left\{ \begin{array}{l} g(X, \varphi_i(Y)) = \frac{1}{2} d\omega_i(X, Y) \\ g(\varphi_i(X), \varphi_i(Y)) = g(X, Y) - \omega_i(X)\omega_i(Y) \\ \varphi_i^2 = -I + \omega_i \otimes R_i \\ \omega_i(Y) = g(R_i, Y), \text{ for any } Y \\ [\varphi_i, \varphi_i] + d\omega_i \otimes R_i = 0, \end{array} \right.$$

for $i = 1, 2, 3$ and (11) is satisfied for even permutations (i, j, k) of $(1,2,3)$.

As $\omega_i(\varphi_i(X)) = g(R_i, \varphi_i(X)) = d\omega_i(R_i, X) = 0$, we know that the image of φ_i is in $Ker(\omega_i)$, for any i . Furthermore, as for any $X, Y \in Ker(\omega_i)$ we have $\varphi_i^2(X) = -X$ and $g(\varphi_i(X), \varphi_i(Y)) = g(X, Y)$, φ defines an isometry of $Ker(\omega_i)$ with $g(X, \varphi_i(X)) = 0$. For even permutations (i, j, k) of $(1,2,3)$ we also have $\omega_j(\varphi_i(X)) = -\omega_k(X)$, so φ preserves the intersection Σ of the kernels of ω_1, ω_2 and ω_3 .

The Reeb vector fields R_1, R_2 and R_3 are everywhere linearly independent. Indeed, for even permutations (i, j, k) of $(1,2,3)$, we have:

$$(12) \quad \omega_i(R_j) = \omega_j(\varphi_k(R_j)) = g(R_j, \varphi_k(R_j)) = \frac{1}{2} d\omega_k(R_j, R_j) = 0,$$

so R_1, R_2 and R_3 are orthogonal, as $\omega_i(R_j) = g(R_i, R_j)$. None of these vectors is in Σ and the dimension of Σ is $4n - 4$. Thus, for a given point $p \in M$, there is a basis of T_pM which can be written as $(R_1, R_2, R_3, X_4, \dots, X_{4n-1})$, where the X_i are elements of Σ . It is possible to choose them in such a way that for $k = 4m$ we have $X_{k+1} = \varphi_1(X_k)$, $X_{k+2} = \varphi_2(X_k)$ and $X_{k+3} = \varphi_3(X_k)$.

We set $\omega = \sum_{i=1}^3 \lambda_i \omega_i$, with $\sum_{i=1}^3 \lambda_i^2 = 1$. Then we have

$$\omega \wedge (d\omega)^{2n-1}(R_1, R_2, R_3, X_4, \dots, X_{4n-1}) = 2^{2n-2} (2n - 2)!.$$

This is due to the equations

$$\begin{aligned} \omega_i(Y_j) &= \delta_{ij}, & \text{for } R_i = Y_i \text{ and } X_j = Y_j \\ d\omega_i(R_j, R_k) &= 1, & \text{if } (i, j, k) \text{ is an even permutation of } (1, 2, 3) \\ d\omega_i(R_j, X_k) &= 0 \\ d\omega_i(X_j, X_k) &= 0, & \text{if } X_k \neq \varphi_i X_j \\ d\omega_i(X_j, \varphi_i X_j) &= -2. \end{aligned}$$

Thus, ω_1, ω_2 and ω_3 generate a taut contact sphere.

Furthermore, we have (12) and for any vector field X

$$R_{i\lrcorner}d\omega_j(X) = 2\omega_k(X) = -R_{j\lrcorner}d\omega_i(X).$$

So by Lemma 3, the contact sphere generated by ω_1 , ω_2 and ω_3 is round. \square

This proposition yields new classes of examples of round and taut contact spheres in dimension higher than 3. According to Boyer, Galicki and Mann (see [3]), the homogeneous manifolds which carry Sasakian 3-structures are the following:

$$\begin{array}{c} \frac{Sp(n+1)}{Sp(n)} \cong \mathbb{S}^{4n+3}, \quad \frac{Sp(n+1)}{Sp(n) \times \mathbb{Z}_2} \cong \mathbb{R}P^{4n+3}, \\ \frac{SU(m)}{S(U(m-2) \times U(1))}, \quad \frac{SO(k)}{SO(k-4) \times Sp(1)}, \\ \frac{G_2}{Sp(1)}, \quad \frac{F_4}{Sp(3)}, \quad \frac{E_6}{SU(6)}, \quad \frac{E_7}{Spin(12)}, \quad \frac{E_8}{E_7} \end{array}$$

for $m \geq 3$ and $k \geq 7$. The named authors also prove the existence of infinitely many compact, simply connected and strongly inhomogeneous manifolds of dimension 7 which carry Sasakian 3-structures and which are homotopically distinct.

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