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Jacques HURTUBISE & Thomas NEVINS

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THE GEOMETRY OF CALOGERO-MOSER SYSTEMS

by Jacques HURTUBISE & Thomas NEVINS (*)

1. Introduction.

The Calogero-Moser (or Calogero-Moser-Sutherland) [Ca], [Su], [Mo] system is a Hamiltonian system which is disconcertingly easy to define: as originally written down in its rational version, it has, in canonical coordinates, the Hamiltonian:

$$H = \frac{1}{2}\sum p_i^2 - \sum_{i<j}(q_i - q_j)^{-2}. \tag{1.1}$$

We are thus dealing with a system of particles on a line interacting with an inverse-square potential. Adding one periodicity condition creates a trigonometric version, and adding two, an elliptic version:

$$H = \frac{1}{2}\sum p_i^2 - \sum_{i<j}\mathfrak{p}(q_i - q_j). \tag{1.2}$$

Here $\mathfrak{p}$ is the Weierstrass p-function. One can think of the rational and trigonometric cases as degenerations of the elliptic case, and so we will concentrate on the latter.

The functions $(q_i - q_j)$ naturally lead one to think of the roots of the type A root systems, and so one defines the Calogero-Moser Hamiltonian for other root systems:

$$H = \frac{1}{2}\sum p_i^2 - \sum_{\rho\in\Delta_+}\mathfrak{p}(\rho(q)). \tag{1.3}$$

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Here the ρ belong to the set Δ_+ of positive roots of a root system Δ in a space $\mathbb{C}^r$ (so that r is the rank) and we have a Weyl group W acting on the system.

While of interest in its own right, what distinguishes the system is the disconcerting habit it has of popping up in a variety of places. One of the early surprises was the role the A_N systems play in the motion of poles of rational (or trigonometric or elliptic) solutions of the KdV and KP hierarchies. (The original proof was a computation [AMM]. The KP/CM correspondence was centered in a more geometric context, and directly related to an infinite-dimensional Grassmannian, the adelic Grassmannian, by G. Wilson [Wi]; it has recently been given a conceptual explanation, see [BN].) One feature of Wilson's approach in [Wi] is that it shows how the correspondence extends to the case when the particles collide.

Further occurrences of the system (and its quantum counterpart) appear in the representation theory of noncommutative algebras [EG], the topology of moduli spaces of Hilbert schemes [CG], and the Seiberg-Witten theory of integrable supersymmetric field theories [Do].

Meanwhile, however, the global geometry of the system remains somewhat obscure; indeed, the general integrability of the systems was only shown recently [KhS]. In the A_N case, a global geometric construction was given by Krichever [Kr], and expanded in Donagi[Do]: the phase space is the moduli space of pairs (E, ϕ), where E is a rank n vector bundle over the elliptic curve, and ϕ is a section of $\mathrm{End}(E)$ with a single pole, located at the origin, with residue of rank one and eigenvalues $(1, \ldots, 1, -N+1)$. The Calogero-Moser system is thus a generalised Hitchin system [Ma, Bo]. As pointed out by Donagi [Do], similar ideas for the other root systems simply cannot work; there are no available phase spaces of the right dimension. There are some ways of attacking the B, C, D cases which have been known for a while [OP]; more recently d'Hoker and Phong [dHP] produced ansatze for building Lax pairs with spectral parameter for many of the Calogero-Moser systems, but while their approach illuminates several aspects of the problem, it does not really attack the geometry, nor is it entirely systematic.

Recently, Bordner, Corrigan and Sasaki [BCS] produced a universal Lax pair for the Calogero-Moser system. The Lax matrices do not take values in the usual loop algebras, which would allow us to relate them to spectral curves and line bundles, and therefore link them to geometry; indeed, they are not matrices at all, but rather elements of an algebra which is a semidirect (smash) product of (a) the meromorphic functions on the product of the curve with the root space with (b) the group algebra of the Weyl group.

It is our purpose in this paper to produce a geometric phase space which completes the phase space on which the system is defined, exhibits the algebraically integrable structure of the Calogero-Moser system, and explains the formulation of Bordner, Corrigan and Sasaki. Along the way, we will find a curious duality of the Calogero-Moser phase spaces for dual root systems, which gives some insight into the global structure of the phase space.

A talk on this material was given by the first author at the conference in Poitiers celebrating the 60th birthday of Pierre van Moerbeke. The work of Pierre and his collaborator Mark Adler has had a wide influence on the subject of integrable systems. It is a pleasure to dedicate this paper to Pierre, and wish him ever more success for many years to come.

Both authors would like to thank Eyal Markman for very useful discussions.

2. A moduli space of regular semistable bundles on $A = \Sigma^r$.

2.1. Bundles on A.

Let Λ be a non-degenerate lattice in $\mathbb{C}$ with generators $2\omega_1, 2\omega_2$ and let $\Sigma = \mathbb{C}/\Lambda$ be the corresponding elliptic curve. Denote the origin by p_0. We have on $\mathbb{C}$ the standard elliptic functions $\sigma(z), \zeta(z)$ with expansions at $z = 0$:

$$\sigma(z) = z + O(z^5), \qquad \zeta(z) = \frac{1}{z} + O(z^3), \tag{2.1}$$

and periodicity relations

$$\sigma(z + 2\omega_i) = -\sigma(z)\exp(2\eta_i(z + 2\omega_i)), \qquad \zeta(z + 2\omega_i) = \zeta(z) + 2\eta_i, \tag{2.2}$$

with $\eta_i = \zeta(\omega_i)$. We define the function

$$x(q, \xi) = \frac{\sigma(q - \xi)}{\sigma(q)\sigma(\xi)}. \tag{2.3}$$

This function represents a section $s(\xi)$ of the line bundle L_q with divisor $(q)-(0)$, as functions on $\mathbb{C}$ with automorphy factors $\exp(-2\eta_i q)$ for the period $2\omega_i$.

Let H be an algebraic torus. Let $\mathfrak{h}$ denote its Lie algebra. Let $\mathcal{L}$ denote the lattice $\exp^{-1}(\mathbb{I})$ in $\mathfrak{h}$, so that $H=\mathfrak{h}/\mathcal{L}$. H is isomorphic to $\mathbb{C}^{*r}$, for some r; choose one such isomorphism. Let

$$A := \Sigma^r = \mathbb{C}^r/(\mathcal{L}\otimes_{\mathbb{Z}}\Lambda). \tag{2.4}$$

Now let $A^*=(\Sigma^r)^*$ denote the dual variety $\mathrm{Pic}^0(A)$. One has the Poincaré line bundle $\mathcal{P}$ on $A\times A^*$, and one has the Fourier-Mukai transform taking (the derived category of) coherent sheaves on A^* to (the derived category of) coherent sheaves on A: one pulls back from A^* to $A\times A^*$, tensors with $\mathcal{P}$, then pushes down to A. In particular, the structure sheaf of a point gets transformed into a line bundle. Explicitly, one has that $\mathfrak{h}^*$ is the covering space of A^*; a point $q=(q_1,q_2,\ldots,q_r)$ of $\mathfrak{h}^*$ defines a line bundle of degree zero over the variety $A=\mathbb{C}^r/(\mathcal{L}\otimes_{\mathbb{Z}}\Lambda)$ by giving for each period $\alpha\otimes 2\omega_i$ the automorphy factor $\exp(2\eta_i\langle q,\alpha\rangle)$.

More generally, given a zero-dimensional subscheme S of A^*, we can define a vector bundle on A. Indeed, lifting again to $\mathfrak{h}^*$, each coordinate q_i defines an element Q_i of $\mathcal{O}_S$ and so for each period $\alpha\otimes 2\omega_i$ an automorphism $\exp(2\eta_i\langle Q,\alpha\rangle)$. We then define a bundle E_S with this automorphy factor for the period $\alpha\otimes 2\omega_i$.

Remark. — We underline that this gives us a bit more than the vector bundle E_S; indeed, the bundle comes defined with a natural reduction to the abelian structure group given by the multiplicative group $\mathcal{O}_S^*$ of invertible elements of the ring $\mathcal{O}_S$. Thus we can view the transform as giving us a principal $\mathcal{O}_S^*$-bundle P_S.

We assume that we have a root system on $\mathfrak{h}$, with roots $\rho\in\Delta$, and dual roots $\rho^\vee$, along with an isomorphism $\mathfrak{h}\simeq\mathfrak{h}^*$. Now $\mathcal{L}$ will be the weight lattice. Let W be the Weyl group of this system, generated by the reflections $s_\rho(q)=q-(\rho\cdot q)\rho^\vee$ in the root planes $\rho=0$ ("walls"). The root planes project to A, giving the components through the origin of the divisors $D_\rho=(\hat\rho^\vee)^{-1}(0)$ where

$$\begin{aligned}\hat\rho^\vee : A &\to \Sigma,\\ \xi &\mapsto \rho^\vee\cdot\xi.\end{aligned} \tag{2.6}$$

D_ρ has a single component iff $\rho^\vee$ is primitive in $\mathcal{L}$. Set

$$D=\sum_{\rho\in\Delta^+} D_\rho. \tag{2.7}$$

Similarly, on A^*, one has the divisors $D^*_\rho = \hat{\rho}^{-1}(0)$ where

$$\begin{aligned}\hat{\rho} : A^* &\to \Sigma, \\ q &\mapsto \rho \cdot q.\end{aligned} \tag{2.8}$$

D^*_ρ has a single component iff ρ is primitive in $\mathcal{L}$. As above, set

$$D^* = \sum_{\rho \in \Delta^+} D^*_\rho. \tag{2.9}$$

Finally, we note that the Weyl group acts on A^*, in particular by the reflections $s_\rho(x)$. If one is interested in the divisor in A^* over which the Weyl group does not act freely, this is given by the union of the planes which are fixed under the reflections. These are cut out by setting

$$D^r_\rho = \{q \in A^* | (\rho \cdot q)\rho^\vee \in (\mathcal{L} \otimes_{\mathbb{Z}} \Lambda)\}. \tag{2.10}$$

We put

$$D^r = \sum_{\rho \in \Delta^+} D^r_\rho. \tag{2.11}$$

This is the ramification locus for the projection $A^* \mapsto A^*/W$.

Now we suppose our scheme S is invariant under W, so that W is a subgroup of $Aut(\mathcal{O}_S)$. The bundle E_S is W-equivariant under the diagonal action of W on $\mathcal{O}_S$ and on A. We emphasize that the action of W on E_S is non-trivial on the base; this also holds for the actions on other induced bundles such as $\mathrm{End}(E_S)$.

If we choose the scheme to be of length $|W|$, the generic S consists of a free W-orbit of a point (if the point is real, this gives a point in each Weyl chamber). The moduli space of these W-invariant schemes (and so of bundles E_S) is A^*/W; it parametrises regular bundles bijectively [FMW]. Notice that the structure group $\mathcal{O}^*_S$ of these bundles can vary, as one moves in the moduli space: for the generic S, the structure group is a torus of dimension $|W|$; for more exceptional S, one has unipotent elements. One has

2.12. Proposition ([Lo]). — *The moduli space A^*/W is a weighted projective space $\mathbb{P} = \mathbb{P}(1, n_1, n_2, \ldots, n_r)$, where $(n_1, n_2, \ldots, n_r)$ are the coefficients of the co-root associated to the highest root expressed as a sum of simple co-roots.*

2.13. Definition. — *We will say that a bundle is toric if its structure group is a torus, that is, it corresponds to an S in $(A^* - D^r)/W$.*

2.2. Induced bundles.

To finite schemes S of length $|W|$ in A^*, we have associated rank $|W|$ bundles E_S or, more properly, a principal $\mathcal{O}_S^*$-bundle P_S. Using this principal bundle, one associates vector bundles to $\mathcal{O}_S^*$-modules:

– A first example is naturally the bundle E_S, associated to the $\mathcal{O}_S^*$-module $\mathcal{O}_S$, with the standard left action.

– A second example is the endomorphism bundle $\mathrm{End}(E_S)$, associated to the module $\mathrm{Hom}_{\mathbb{C}}(\mathcal{O}_S,\mathcal{O}_S)$.

– One has, inside $\mathrm{End}(E_S)$, the trivial bundle $P_S(\mathcal{O}_S)$. This induced from the trivial conjugation action of $\mathcal{O}_S^*$ on $\mathcal{O}_S$. It induces an isomorphism on global sections

$$H^0(A, \mathrm{End}(E_S)) = H^0(A, P_S(\mathcal{O}_S)) = \mathcal{O}_S. \tag{2.14}$$

In the toric case, this is the diagonal subbundle of $\mathrm{End}(E_S)$.

– Assume now that the scheme is W-invariant. The group W acts as automorphisms of $\mathcal{O}_S$, and so the algebra $\mathcal{O}_S[W] := \mathcal{O}_S \otimes \mathbb{C}[W]$ acts as endomorphisms of $\mathcal{O}_S$. This algebra is an $\mathcal{O}_S$-module, and so one has the bundle $P_S(\mathcal{O}_S[W])$; since $\mathcal{O}_S[W]$ acts on $\mathcal{O}_S$, one has a map $P_S(\mathcal{O}_S[W]) \to \mathrm{End}(E_S)$. As we want to distinguish the W-action, which is non-trivial on the base, with the role of W in defining fibrewise endomorphisms on the bundle, we will substitute V for W in the latter context. We then have:

$$P_S(\mathcal{O}_S[V]) \to \mathrm{End}(E_S). \tag{2.15}$$

For toric S (*i.e.* not touching the walls), this map is an isomorphism; this follows from the fact that the Weyl group permutes the elements of S simply transitively. The map is not always an isomorphism, as we shall see later.

– Let $w \in W$; one has, associated to the $\mathcal{O}_S^*$-module $\mathcal{O}_S[w]$, the rank $|W|$ bundle $P_S(\mathcal{O}_S[w])$; it is a subbundle of $\mathrm{End}(E_S)$. In the toric case, when S consists of distinct points, this subbundle consist of endomorphisms which map each line bundle corresponding a point p to the line bundle corresponding to the point $w(p)$. Indeed, as we are dealing with a free W-orbit, in the toric case the bundle $\mathrm{End}(E_S)$ decomposes as a direct sum $\oplus_{w\in W} P_S(\mathcal{O}_S[w])$. We will see that this is not the case when the bundle is not toric.

2.3. Pull-backs.

If s_ρ is the reflection associated to the root ρ, let us consider the subbundle $P_S(\mathcal{O}_S[s_\rho])$ of $\mathrm{End}(E_S)$. This subbundle is a pull-back from Σ under the map $\hat\rho^\vee : A \to \Sigma$. The same holds for general S: indeed, one needs that $\langle Q - s_\rho(Q), \alpha\rangle = 0$ for all linear functions Q and all α such that $\langle \rho^\vee, \alpha\rangle = 0$, which is immediate, as $Q - s_\rho(Q) = \langle \rho, Q\rangle\ \rho^\vee$.

In the toric case, we define the subbundle End_ρ as the rank $2|W|$ subbundle $P_S(\mathcal{O}_S \oplus P_S(\mathcal{O}_S[s_\rho])$; when D_ρ is connected, it is, generically, the subbundle of E_S that is lifted from Σ via $\hat\rho^\vee$, though this latter bundle can be bigger.

In the general, non-toric case, we shall see that $P_S(\mathcal{O}_S)$ and $P_S(\mathcal{O}_S[s_\rho])$ can intersect non-trivially, and so their sum is then of rank less than $2|W|$. On the other hand, we can define the push-down End_ρ of $\mathrm{End}(E_S)$ as a bundle on Σ whose pull-back $\hat\rho^* \,\mathrm{End}_\rho$ is the flat subbundle of $\mathrm{End}(E_S)$ with automorphy factors that are pull-backs under $\hat\rho^\vee$. This will be of rank at least $2|W|$. We will return to this later.

3. Moduli of bundles and Higgs fields in the toric case.

We now have our configuration space, a moduli space of bundles over the variety Σ^r. We now define the phase space, which will be a set of pairs (E_S, ϕ), where E_S is a bundle lying in our moduli space (for the time being, a toric bundle), and ϕ is a "Higgs field" (a meromorphic section of $\Omega^1 \otimes \mathrm{End}(E_S))$ which can be written as a sum $\phi_0 + \sum_{\rho\in\Delta} \phi_\rho$, where ϕ_0 is holomorphic and ϕ_ρ is a pull-back (as a form-valued section) under $\hat\rho^\vee$ of a form-valued section ψ of $\mathrm{End}_\rho(p_0) \otimes K_\Sigma$ over Σ. The forms ψ have simple well defined residues in End_ρ at the origin p_0, and so ϕ has residues in $\mathrm{End}_\rho \subset \mathrm{End}(E_S)$ over D_ρ. One asks, that for suitable constants depending only on the length of the root ρ:

– that ϕ be W-invariant,

– that at the origin, the residues $R_\rho = m_{|\rho|}^{-1} \,\mathrm{res}_{D_\rho}(\phi)$ generate a representation of the Weyl group inside $\mathrm{End}(E_S)$ under $s_\rho \mapsto R_\rho$; in particular $(m_{|\rho|}^{-1} \,\mathrm{res}_{D_\rho}(\phi))^2$ be the identity along D_ρ.

– that $\phi \wedge \phi = 0$, as form-valued endomorphisms.

Let $\mathcal{M}$ be the moduli space of such pairs (E_S, ϕ), where again, for the time being, E_S is toric.

Let $S \in A^*/W$ correspond to a toric bundle; we then choose a point $q = (q_1, \ldots, q_r)$ representing S in $\mathfrak{h}^*$. The orbit $q_w, w \in W$ of q in $\mathfrak{h}^*$ determines a bundle E_S. We will think of sections of this bundle as vector valued functions on $\mathfrak{h}$ with appropriate covariance under translation by periods, as explained above. The q are configuration space coordinates. Correspondingly, let us choose a W-invariant orbit $p^w, w \in W$ of the momenta $p = (p^1, \ldots, p^r)$ in $\mathfrak{h} \simeq \mathfrak{h}^*$. We note that $T^*A = A \times \mathfrak{h}^*$, so that p^w, as well as the roots $\rho^\vee$, can be thought of as constant 1-forms on A. Let $\xi = (\xi_1, \ldots, \xi_r)$ denote a point on the universal cover $\mathfrak{h}$ of A. The bundle E_S is trivial when lifted to $\mathfrak{h}$, and the reflection s_ρ act naturally on the fibers of the trivialised bundles; let $\hat{s_\rho}$ denote their action. Let $m_{|\rho|}$ be a constant depending on the norm of the root ρ; inspired by the formula of [BCS], we set

$$\phi(q_w, p^w; \xi) = \mathrm{diag}_{w\in W}(p^w) + \sum_{\rho\in\Delta_+} m_{|\rho|}\rho^\vee \otimes \mathrm{diag}_{w\in W}(x(\rho\cdot q_w, \rho^\vee\cdot \xi))\hat{s}_\rho. \tag{3.1}$$

3.2. Proposition.

– *ϕ is W-invariant, under the diagonal action of W built from the natural action of W on the base A and the permutation action of W on the fiber of E_S.*

– *As a form (that is thinking of $p^w, \rho^\vee$ as one-forms), $\phi = \phi_0 + \sum_{\rho\in\Delta} \phi_\rho$, with ϕ_0 holomorphic and ϕ_ρ lifted under $\hat{\rho}^\vee$, with residue at D_ρ equal to the reflection s_ρ.*

– *As a form, again, $\phi \wedge \phi = 0$.*

Proof. — The first two items are simple verifications. For the third, one first notes that the p-term above commutes with itself and so its square in $\phi \wedge \phi$ vanishes. The cross term

$$\mathrm{diag}_w(p^w) \wedge [\sum_{\rho\in\Delta_+} m_{|\rho|}\rho^\vee \otimes \mathrm{diag}_w(x(\rho\cdot q_w, \rho^\vee\cdot\xi))\hat{s}_\rho] + [\sum_{\rho\in\Delta_+} m_{|\rho|}\rho^\vee \otimes \mathrm{diag}_w(x(\rho\cdot q_w, \rho^\vee\cdot\xi))\hat{s}_\rho] \wedge \mathrm{diag}_w(p^w)$$

gives

$$\sum_{\rho\in\Delta_+} \mathrm{diag}_w((p^w - p^{s_\rho(w)}) \wedge m_{|\rho|}\rho^\vee \otimes x(\rho\cdot q_w, \rho^\vee\cdot\xi))\hat{s}_\rho$$

which vanishes, as $p^w - p^{s_\rho(w)}$ is parallel to $\rho^\vee$. The product $\phi \wedge \phi$ then becomes:

$$\sum_{\rho,\sigma\in\Delta_+} m_{|\rho|}m_{|\sigma|}(\rho^\vee \wedge \sigma^\vee) \otimes \mathrm{diag}_w(x(\rho \cdot q_w, \rho^\vee \cdot \xi)x(\hat{s}_\rho(\sigma) \cdot q_w, \sigma^\vee \cdot \xi))\hat{s}_\rho\hat{s}_\sigma.$$

The product $\hat{s}_\rho\hat{s}_\sigma$ is of order 2, 3, 4 or 6.

If it is of order 2, then $\hat{s}_\rho\hat{s}_\sigma = \hat{s}_\sigma\hat{s}_\rho$, and the other terms in the product above are distinct from these two. The two terms then cancel.

If the order is three, then one has three products of reflections which are equal: relabeling if necessary, one has

$$\hat{s}_\rho\hat{s}_\sigma = \hat{s}_\sigma\hat{s}_{\sigma+\rho} = \hat{s}_{\sigma+\rho}\hat{s}_\rho.$$

Collating the terms in the product above which correspond to these, one has:

$$\begin{aligned}(3.3)\quad (\rho^\vee \wedge \sigma^\vee) \otimes \mathrm{diag}_w(&x(\rho \cdot q_w, \rho^\vee \cdot \xi)x(\hat{s}_\rho(\sigma) \cdot q_w, \sigma^\vee \cdot \xi)\\ &- x(\sigma \cdot q_w, \sigma^\vee \cdot \xi)x(\hat{s}_\sigma(\sigma+\rho) \cdot q_w, (\sigma+\rho)^\vee \cdot \xi)\\ &- x((\sigma+\rho) \cdot q_w, (\sigma+\rho)^\vee \cdot \xi)x(\hat{s}_{\sigma+\rho}(\rho) \cdot q_w, \rho^\vee \cdot \xi))\end{aligned}$$

applying the reflections to the root vectors, one gets:

$$\begin{aligned}(3.4)\quad (\rho^\vee \wedge \sigma^\vee) \otimes \mathrm{diag}_w(&x(\rho \cdot q_w, \rho^\vee \cdot \xi)x((\sigma+\rho) \cdot q_w, \sigma^\vee \cdot \xi)\\ &- x(\sigma \cdot q_w, \sigma^\vee \cdot \xi)x(\rho \cdot q_w, (\sigma+\rho)^\vee \cdot \xi)\\ &- x((\sigma+\rho) \cdot q_w, (\sigma+\rho)^\vee \cdot \xi)x(-\sigma \cdot q_w, \rho^\vee \cdot \xi)).\end{aligned}$$

Now evaluate the residues, along $D_\rho : \rho^\vee \cdot \xi = 0$:

$$(3.5)\qquad x((\sigma+\rho) \cdot q_w, \sigma^\vee \cdot \xi) - x((\sigma+\rho) \cdot q_w, (\sigma+\rho)^\vee \cdot \xi) = 0,$$

using $(\sigma+\rho)^\vee \cdot \xi = \sigma^\vee \cdot \xi$ along D^ρ. Similarly, along D_σ, one has the residue

$$x(\rho \cdot q_w, \rho^\vee \cdot \xi) - x(\rho \cdot q^w, (\sigma+\rho)^\vee \cdot \xi) = 0,$$

and along $D_{\sigma+\rho}$,

$$-x(\sigma \cdot q_w, \sigma^\vee \cdot \xi) - x(-\sigma \cdot q_w, \rho^\vee \cdot \xi)) = 0$$

where now we use as well the fact that the function x is odd when we change the signs of both the arguments. The sum of the three terms has no poles, and so it represents a holomorphic section of a line bundle; this line bundle is non-trivial, and so the section vanishes. The cases when the order of the product is four or six is handled in the same way. □

We thus have existence, for each toric bundle, of an r-dimensional plane of Higgs fields; one has that this is all, simply as a consequence of the fact that the space of W-invariant holomorphic sections of $\Omega^1 \otimes \mathrm{End}(E_S)$, which is the same as the space of holomorphic sections of $\Omega^1 \otimes P_S(\mathcal{O}_S)$, is r-dimensional:

3.6. Proposition. — *For each toric bundle, the space of sections ϕ in $H^0(A, (\Omega^1 \otimes \mathrm{End}(E_S)(D))$ with simple poles over D, satisfying:*

- *ϕ is W-invariant,*
- *$\phi = \sum_\rho \phi_\rho$, as above, with $(\mathrm{res}_{D_\rho}(\phi_\rho)) = m_{|\rho|} s_\rho$.*

is an r-dimensional plane. The sections all satisfy $\phi \wedge \phi = 0$, and are given by the formula (3.1).

4. Poisson structures and an integrable system.

4.1. Duality.

For F a flat bundle on Σ^r, we have a version of Serre duality which uses the natural Kähler structure on Σ^r, and provides a perfect pairing between $H^1(\Sigma^r, F)$ and $H^0(\Sigma^r, \Omega^1 \otimes F^*)$; one takes the product, to obtain an element of $f \in H^1(\Sigma^r, \Omega^1)$, then takes the cup product with the $(r-1)$-th power of the Kähler class, suitably normalised. Alternately, one can restrict f to each of the coordinate axes, and apply the isomorphism $H^1(\Sigma, \Omega^1) = \mathbb{C}$ there, then sum.

In particular, if F is the trivial line bundle $\mathcal{O}$, we note that elements of $\mathfrak{h}^*$ are sections of Ω^1. One then finds that if ω is a weight, the dual to ω (under the Killing form) is represented in $H^1(\Sigma^r, \mathcal{O})$ in Cech terms by the cocycle $\zeta(\omega(\xi))$, with respect to the covering of Σ^r by two open sets: V_0 = the complement in Σ^r to the polar divisor of $\zeta(\omega(\xi))$, and V_1 = a tubular neighbourhood of the polar divisor. We note that if one is looking at a multiple $n\omega$, we do have that the residue of $\zeta(nz)$ on an elliptic curve at the origin is n^{-1} times the residue of $\zeta(z)$, but, on the other hand one has n^2 poles instead of one, so that the total residue is n, as it should be.

We would like to be able to add these cocycles, and in particular express them with respect to a common covering. In particular, we will be dealing with the duals to the positive roots. To do this, we choose a covering of Σ^r indexed by the subsets of the set of positive roots. If D_ρ is the divisor of Σ cut out by $\zeta(\rho(\xi)) = 0$, and A is a subset of the set of positive roots, we set

$$U_A = \bigcap_{\rho\in A}(\text{tubular neighbourhoods of } D_\rho) - \bigcup_{\rho\notin A} D_\rho. \tag{4.1}$$

Let U_0 denote U_A, when A is the empty set, so that $U_0 = \Sigma^r - D$, and let U_1 denote U_A when A is the full set of positive roots.

We can then represent the dual to $\rho^\vee$ by the cocycle represented by $f_{A,B}(\rho^\vee)$ over $U_A \cap U_B$

$$f_{A,B}(\rho^\vee) = \begin{cases} \zeta(\rho^\vee(\xi)) & \text{if } \rho\in A \text{ and } \rho\notin B \\ -\zeta(\rho^\vee(\xi)) & \text{if } \rho\in B \text{ and } \rho\notin A \\ 0 & \text{otherwise.} \end{cases} \tag{4.2}$$

Since, by Schur's lemma, the W- invariant map

$$\alpha = \sum_{\rho\in\Delta_+} \langle\alpha,\rho^\vee\rangle\ \rho^\vee \tag{4.3}$$

is a (non-zero)-multiple kId of the identity, an alternate description of the same class can also be given by the cocycle

$$g_{A,B}(\rho) = k^{-1}\sum_{\sigma\in\Delta_+}\langle\rho^\vee,\sigma^\vee\rangle\ f_{A,B}(\sigma^\vee), \tag{4.4.}$$

4.2. Poisson structures.

We can use the pairing to define a Poisson structure. We will concentrate on the toric case.

We take infinitesimal variations of the pair (bundle, Higgs field). The deformations of the bundles are given by $H^1(\Sigma^r, P_S(\mathcal{O}_S))^W$; this is isomorphic to $H^1(\Sigma^r, \mathrm{End}(E_S))^W$. This can be seen by using the duality, and reducing to the corresponding statement on H^0.

On the other hand, we have for toric bundles a decomposition

$$\mathrm{End}(E_S) = \bigoplus_{w\in W} P_S(\mathcal{O}_S[w]). \tag{4.5}$$

The Higgs fields are W-invariant sections of

$$\Omega^1\otimes\Big(P_S(\mathcal{O}_S)\oplus\Big(\bigoplus_\rho P_S(\mathcal{O}_S[s_\rho])[D_\rho]\Big)\Big),$$

with poles along the divisor $D = \sqcup_\rho D_\rho$. We note, for example from the explicit formulas, that the $\oplus_\rho P_S(\mathcal{O}_S[s_\rho])$ portions of the Higgs fields are determined by the bundles, and that the only liberty for deformations of the Higgs field lie in the $P_S(\mathcal{O}_S)$ component (that is, the diagonal component) once one has fixed the bundle. Varying both bundle and Higgs field, one has a trivial deformation complex $\mathcal{C}_0$, given by

$$P_S(\mathcal{O}_S) \xrightarrow{0} \Omega^1 \otimes P_S(\mathcal{O}_S), \tag{4.6}$$

Deformations of the pairs (bundles, Higgs fields) correspond to W-invariant elements of the first hypercohomology of this complex. As the complex is trivial, this hypercohomology is simply

$$H^1(\Sigma^r, P_S(\mathcal{O}_S))^W \oplus H^0(\Sigma^r, \Omega^1 \otimes P_S(\mathcal{O}_S))^W, \tag{4.7}$$

As, by our duality, these spaces are dual to each other, there is a natural skew form on the sum, and this will be the symplectic structure for our space. One has, in a straightforward fashion:

4.8. Lemma. — *The coordinates p, q are Darboux coordinates for this symplectic form; the form is therefore closed and non-degenerate.*

We define this form in a more complicated fashion: using the duality, the deformation complex (4.6) is also the deformation complex for the cotangent space. The Poisson structure, as a map from the cotangent space to the tangent space, is the map on the first hypercohomology induced by the identity map from the complex (4.6) to itself:

$$\begin{array}{ccc} P_S(\mathcal{O}_S) & \xrightarrow{0} & \Omega^1 \otimes P_S(\mathcal{O}_S) \\ \uparrow & & \uparrow \\ P_S(\mathcal{O}_S) & \xrightarrow{0} & \Omega^1 \otimes P_S(\mathcal{O}_S). \end{array} \tag{4.9}$$

It is straightforward to show that this induces the symplectic form given above; we refer the reader to similar calculations found in [HuMa].

Our reason for doing this is that we will embed our complex into a larger one, which comprises more deformations, allowing the polar parts of the Higgs fields to vary, while staying within the bundles. We will keep the same family of bundles, but now allow the polar parts of the Higgs fields to vary:

$$P_S(\mathcal{O}_S) \xrightarrow{[\phi,\cdot]} \Omega^1 \otimes (P_S(\mathcal{O}_S) \oplus (\oplus_\rho P_S(\mathcal{O}_S[s_\rho])). \tag{4.10}$$

The (invariant part of the) first hypercohomology of this will be the tangent space to a larger moduli space, as this drops requirements on the Higgs fields.

Next, as we have noted, we could allow deformations of the bundle in $H^1(\Sigma^r, \mathrm{End}(E_S))^W$, without increasing the dimension. For the Higgs fields, we can enlarge the deformations again. We twist the bundle $\mathrm{End}(E_S)$, to allow for some poles: specifically, we allow a single pole along D_ρ in the s_ρ factor, no poles when w is the identity, and poles along D for the other factors. Let us call the result $\mathrm{End}(E_S)_D$. We then have the deformation complex for an even larger moduli space (with the same bundles)

$$\mathrm{End}(E_S) \xrightarrow{[\phi,\cdot]} \Omega^1 \otimes \mathrm{End}(E_S)_D. \tag{4.11}$$

The deformations of our original pairs (bundle, Higgs field) are represented by elements of the first hypercohomology of all three complexes (4.6), (4.10), (4.11).

We can define a pairing between the first hypercohomology of this complex (4.11), with that of the first hypercohomology of

$$\mathrm{End}(E_S)^*_D \xrightarrow{[\phi,\cdot]} \Omega^1 \otimes \mathrm{End}(E_S). \tag{4.12}$$

In terms of a covering by open sets U_i, the first hypercohomology of the first complex (4.11) is represented by cocycles $a_{ij} \in H^0(U_i \cap U_j, \mathrm{End}(E_S))$, $b_i \in H^0(U_i, \Omega^1 \otimes \mathrm{End}(E_S)_D)$, with $b_i - b_j = [\phi, a_{i,j}]$ on overlaps. Similarly, the first hypercohomology of the second complex (4.12) is represented by cocycles $\alpha_{ij} \in H^0(U_i \cap U_j, \mathrm{End}(E_S)^*_D)$, $\beta_i \in H^0(U_i, \Omega^1 \otimes \mathrm{End}(E_S)_D)$, with $\beta_i - \beta_j = [\phi, \alpha_{i,j}]$ on overlaps. The pairing is given by taking

$$\mu_{ij} = \langle a_{ij}, \beta_i + \beta_j\rangle + \langle \alpha_{i,j}, b_i + b_j\rangle \in H^1(\Sigma^r, \mathcal{O})$$

and cupping it with the $(r-1)$th power of the Kähler class, as above.

Thus elements τ of the first hypercohomology of this complex pair with elements of the tangent space of our moduli, though of as elements of the first hypercohomology of (4.11). On the other hand, one has a diagram of complexes:

$$\begin{array}{ccc}
\mathrm{End}(E_S)^*_D & \xrightarrow{[\phi,\cdot]} & \Omega^1 \otimes \mathrm{End}(E_S) \\
\downarrow & & \downarrow \\
\mathrm{End}(E_S) & \xrightarrow{[\phi,\cdot]} & \Omega^1 \otimes \mathrm{End}(E_S)_D \\
\uparrow & & \uparrow \\
P_S(\mathcal{O}_S) & \xrightarrow{[\phi,\cdot]} & \Omega^1 \otimes (P_S(\mathcal{O}_S) \oplus (\oplus_\rho P_S(\mathcal{O}_S[s_\rho])) \\
\downarrow & & \downarrow \\
P_S(\mathcal{O}_S) & \xrightarrow{0} & \Omega^1 \otimes P_S(\mathcal{O}_S).
\end{array} \tag{4.14}$$

The image of τ under the Poisson tensor is obtained by "moving" a cocycle through this diagram, first mapping a hypercohomology cocycle for (4.12) to a cocycle for (4.11), then using the fact that $H^1(\Sigma^r, P_S(\mathcal{O}_S))^W \simeq H^1(\Sigma^r, \mathrm{End}(E_S))^W$ to modify the cocycle by a coboundary, and then project the Higgs field so that the modified cocycle lives in (4.10), then finally projecting it to (4.6). We use this procedure in the next section.

4.3. An integrable system.

One can define Hamiltonians, derived from the invariant polynomials for the root system, as follows. Let P denote an invariant polynomial of degree k on $\mathfrak{h}$

$$P : \mathfrak{h} \to \mathbb{C}. \tag{4.15}$$

As the components of ϕ commute, we can define $P(\phi)$, and then define Hamiltonians

$$\widetilde{P}(E_S, \phi) = \text{constant term of } \operatorname{tr}(P(\phi)). \tag{4.16}$$

The differential of these functions in $H^1(A, \mathrm{End}(E_S)(-D))^W = (H^0(A, \Omega^1 \otimes \mathrm{End}(E_S)(D))^W)^*$ is given by the cocycle

$$P_{A,B} = \sum_{\sigma\in\Delta_+} \langle dP(\phi), \sigma^\vee\rangle\ g_{A,B}(\sigma^\vee), \tag{4.17}$$

where $g_{A,B}(\sigma)$ is the cocycle of (4.4).

One has to extend this to a cocycle for the complex (4.12). The invariance of P tells us that dP commutes with ϕ, and so one can extend the cocycle by zero, giving us a representative $(P_{A,B}, P'_A = 0)$ for the Hamiltonian vector field. Considering another of our Hamiltonians Q, with cocycle $(Q_{A,B}, Q'_A = 0)$, one has that the Poisson bracket is zero, since the pairing on the level of cocycles matches the $P_{A,B}$-terms with the $Q'_A = 0$, and the $Q_{A,B}$-terms with the $P'_A = 0$. Therefore:

4.18. Proposition. — *If P, Q are two invariant polynomials, the Hamiltonians $\widetilde{P}, \widetilde{Q}$ Poisson commute.*

Let ${}_2P$ be the quadratic invariant polynomial. As $d_2P(\phi) = \frac{1}{2}\phi$, $d_2\widetilde{P}$ is represented by the cocycle

$${}_2P_{A,B} = \frac{1}{2} \sum_{\sigma\in\Delta_+} \langle \phi, \sigma^\vee\rangle\ g_{A,B}(\sigma). \tag{4.19}$$

If we are in the generic set of solutions given by the formula (3.1), this gives:

$$
\begin{aligned}
(4.20)\qquad {}_2P_{A,B} = \frac{1}{2}\sum_{\sigma\in\Delta_+} &\mathrm{diag}_{w\in W}(\langle p^w,\sigma\rangle\ g_{A,B}(\sigma).) \\
&+\frac{1}{2}\sum_{\rho\in\Delta_+} g_{A,B}(\rho)\otimes \mathrm{diag}_{w\in W}(x(\rho\cdot q_w,\rho^\vee\cdot\xi))\hat{s}_\rho.
\end{aligned}
$$

We note that a sum $\sum_{\sigma\in\Delta_+} a_\sigma\zeta(\sigma\cdot\xi)$ defines a global meromorphic function on Σ^r if $\sum_{\sigma\in\Delta_+} a_\sigma\sigma = 0$. Since

$$
(4.21)\qquad \sum_{\sigma\in\Delta_+}\langle\rho^\vee,\sigma^\vee\rangle\ \sigma^\vee = k\rho^\vee,
$$

we can modify (4.20) by the coboundary of

$$
\begin{aligned}
(4.22)\qquad h_A = \delta_{A,0}\frac{1}{2}\sum_{\rho\in\Delta_+} &\mathrm{diag}_{w\in W}(x(\rho\cdot q_w,\rho^\vee\cdot\xi))\hat{s}_\rho \\
&\cdot\Big[k^{-1}\sum_{\sigma\in\Delta_+}\langle\rho^\vee,\sigma^\vee\rangle\ \zeta(\sigma^\vee\cdot\xi)-\zeta(\rho^\vee\cdot\xi)\Big],
\end{aligned}
$$

to obtain a new cocycle, whose $0,A$ term is:

$$
\begin{aligned}
(4.23)\qquad {}_2P_{0,A} = \frac{1}{2}\sum_{\sigma\in\Delta_+} &\mathrm{diag}_{w\in W}(\langle p^w,\sigma\rangle\ g_{A,0}(\sigma).) \\
&+\frac{1}{2}\sum_{\rho\in\Delta_+} f_{A,0}(\rho)\otimes \mathrm{diag}_{w\in W}(x(\rho\cdot q_w,\rho^\vee\cdot\xi))\hat{s}_\rho.
\end{aligned}
$$

Proceeding further, setting

$$
(4.24)\qquad \mu(y,z) = e^{-y\zeta(z)}\frac{\partial(e^{-y\zeta(z)}x(y,z))}{\partial y}, \nu(y,z) = \frac{\partial x(y,z)}{\partial y}
$$

using the identity

$$
(4.25)\qquad \zeta(z)x(y,z) = \mu(y,z)-\nu(y,z),
$$

and noting that $\nu(y,z)$ is holomorphic at $z=0$, while $\mu(y,z)$ is well defined over the elliptic curve (in z), away from the origin, we modify again the cocycle by the coboundary of

$$
(4.26)\qquad h_0 = \frac{1}{2}\sum_{\rho\in\Delta_+}\mathrm{diag}_{w\in W}(\mu(-\rho\cdot q_w,\rho^\vee\cdot\xi))\hat{s}_\rho
$$

$$
(4.27)\qquad h_1 = \frac{1}{2}\sum_{\rho\in\Delta_+}\mathrm{diag}_{w\in W}(\nu(-\rho\cdot q_w,\rho^\vee\cdot\xi))\hat{s}_\rho
$$

$$
(4.28)\qquad h_A = 0,\ \text{otherwise}
$$

to obtain yet another cocycle, whose $0,1$ term is

$$ {}_2P_{0,1} = \frac{1}{2}\sum_{\sigma\in\Delta_+} \mathrm{diag}_{w\in W}(\langle p^w,\sigma\rangle\ g_{0,1}(\sigma)). \tag{4.29} $$

Of course, one must also keep track in the hypercomology of how the fields ϕ vary, as one adds in the coboundaries; from $\dot{\phi} = 0$, the addition of coboundaries modifies the flows over U_1 to

$$ \dot{\phi} = [M,\phi] \tag{4.30} $$

where

$$ M = \frac{1}{2}\sum_{\rho\in\Delta_+} \mathrm{diag}_{w\in W}(\mu(-\rho\cdot q_w,\rho^\vee\cdot\xi))\hat{s}_\rho \tag{4.31} $$

which, mutatis mutandis, is the Lax equation of Bordner, Corrigan and Sasaki.

5. The general case; non-toric bundles.

There are some difficulties in extending the definitions of Higgs fields to the full moduli space of bundles. To get some idea of what this involves, we look first into the case of Sl(2).

5.1. An example: Sl(2).

As we noted, to get a closed moduli space of bundles, we have to consider bundles which are not toric. To get an idea of what this involves, we consider the case of $\mathrm{Sl}(2,\mathbb{C})$ in some detail. In this case, the non trivial element w of the Weyl group $\mathbb{Z}/2$ acts on the elliptic curve Σ by $z\mapsto -z$. There is one positive root, which in a standard basis is given by $\rho(z) = \rho^\vee(z) = 2z$.

The moduli of bundles one considers is parametrised by $\mathbb{P}^1 = \Sigma^*/(\mathbb{Z}/2)$. The bundles one obtains are of two types:

– A sum $L\oplus L^*$, where L, L^* are of degree 0 and $L\neq L^*$. This is the generic case, and corresponds to the complement of the branch locus in $\mathbb{P}^1$.

– Non-trivial extensions $0\to L\to E\to L\to 0$, with $L^2 = \mathcal{O}$. There are four of these, corresponding to the branch locus of $\Sigma\to\mathbb{P}^1$.

As noted above, there is already a construction of the Calogero-Moser system, due to Krichever, as a moduli space of stable pairs (E, ψ), where E is an $\mathbf{SL}(2)$-bundle over Σ, and ψ is a section of $\mathrm{End}(E)$ with a single pole at the origin, with residue conjugate to $\mathrm{diag}(1, -1)$. We will see that our construction gives the same result.

The toric case. The non trivial element w of the Weyl group acts on Σ by $z \mapsto -z$, and also permutes L and L^*.

For the induced bundles, one has an isomorphism

$$P_S(\mathcal{O}_S[V]) \simeq \mathrm{End}(E_S).$$

The global endomorphisms $\mathcal{O}_S$ (living in the diagonal subbundle of $\mathrm{End}(E_S)$) have a basis e_1, e_2: e_1 acts by the identity on L, by zero on L^*; e_2 acts by zero on L, by the identity on L^*. As well, s_ρ acts by permutations on L, L^* and so by permutations on e_1, e_2; the non-identity element w of W acts by permutations on L, L^* and by $z \mapsto -z$ on the curve.

Let h be a basis of $\mathfrak{h}$; the element w acts by $h \mapsto -h$.

On $\mathfrak{h} \otimes \mathcal{O}_S \otimes \mathbb{C}[V]$, the invariant elements are spanned by $h \otimes (e_1 - e_2) \otimes 1, h \otimes (e_1 - e_2) \otimes s_\rho$, and the anti-invariant ones by $h \otimes (e_1 + e_2) \otimes 1, h \otimes (e_1 + e_2) \otimes s_\rho$.

Looking for invariant global sections, then, one wants the constant terms to lie in the invariant piece, and the first order polar parts to lie in the anti-invariant piece; this gives, if $L \neq L^*$, a constant term in $h \otimes (e_1 - e_2) \otimes 1$, and a polar part in $h \otimes (e_1 + e_2) \otimes s_\rho$. If q corresponds to the bundle L, our formula:

$$p \cdot h \otimes (e_1 - e_2) + x(2q, 2\xi) \cdot h \otimes e_1 \otimes s_\rho + x(-2q, 2\xi) \cdot h \otimes e_2 \otimes s_\rho,$$

where p is constant over Σ. In matricial terms,

$$\phi = \begin{pmatrix} p & x(2q, 2\xi) \\ x(-2q, 2\xi) & -p \end{pmatrix}$$

which are the fields of [Kr]

The exceptional case. Taking the Fourier Mukai transform of the double point at the origin in Σ^* give an irreducible bundle $\mathcal{O} \to E_S \to \mathcal{O}$. For the induced bundles, one has, instead of an isomorphism, an exact sequence

$$0 \to \mathcal{O} \to P_S(\mathcal{O}_S[V]) \to \mathrm{End}(E_S) \to \mathcal{O} \to 0,$$

with $P_S(\mathcal{O}_S[V])$ embedding as the upper triangular subbundle. One has a basis e, f of $\mathcal{O}_S$ on which the Weyl group acts by $w(e) = e, w(f) = -f$.

The action of e on E_S is by the identity, and the action of f is nilpotent, projecting onto the $\mathcal{O}$ subbundle of E_S.

On $\mathfrak{h}\otimes\mathcal{O}_S[V]$, the invariant elements are spanned by $h\otimes f\otimes 1, h\otimes f\otimes s_\rho$, and the anti-invariant ones by $h\otimes e\otimes 1, h\otimes e\otimes s_\rho$.

One can compute the space of global sections of $P_S(\mathcal{O}_S[R])(D)$, in the trivialisations corresponding to the automorphy factors

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Let us define $\hat{\zeta} = r\zeta(z) + sz$ for suitable constants r, s by the requirement that translation by the two basic periods $2\omega_1, 2\omega_2$ of the elliptic curve changes the value of $\hat{\zeta}$ by one: $\hat{\zeta}(z+2\omega_i) = \hat{\omega}(z) + 1$. One obtains sections of $\mathfrak{h}\otimes P_S(\mathcal{O}_S[R])(D)$

$$a(h\otimes e\otimes 1) + b(h\otimes e\otimes s_\rho) + c(h\otimes f\otimes 1) + (d - a\hat{\zeta}(z))(h\otimes f\otimes s_\rho),$$

where a, b, c, d are constants. Mapping to $\mathfrak{h}\otimes \mathrm{End}(E_S)(D)$, one has

$$h\cdot\begin{pmatrix} a+b & c-d+a\hat{\zeta}(z) \\ 0 & a-b \end{pmatrix}.$$

On the other hand, sections of $\mathfrak{h}\otimes\mathrm{End}(E_S)(D)$ are given by

$$h\cdot\begin{pmatrix} a+b-e\hat{\zeta}(z) & c+a\hat{\zeta}(z) - e(\hat{\zeta}^2(z) - \mathfrak{p}(z)) \\ e & a-b+e\hat{\zeta}(z) \end{pmatrix},$$

where a, b, c, e are constants. Restricting to the invariant sections, one has, for $\mathfrak{h}\otimes P_S(\mathcal{O}_S[R])(D)$

$$h\cdot\begin{pmatrix} 0 & c-d \\ 0 & 0 \end{pmatrix}.$$

and, for $\mathfrak{h}\otimes\mathrm{End}(E_S)$,

$$h\cdot\begin{pmatrix} -e\hat{\zeta}(z) & c - e(\hat{\zeta}^2(z) - \mathfrak{p}(z)) \\ e & -e\hat{\zeta}(z) \end{pmatrix}.$$

Comparing, one sees that there are no sections of $P_S(\mathcal{O}_S[R])(D)$ with the correct poles; on the other hand, $\mathrm{End}(E_S)(D)$ does admit a section with residue $\mathrm{diag}(1,-1)$, that is s_ρ, which is exactly the correct residue. What this tells us in general is that we must enlarge $P_S(\mathcal{O}_S[R])(D)$ if we are to obtain the correct fields.

5.2. The subbundles End_ρ.

In the general case, one can define, generically, a subbundle End_ρ which includes both $P_S(\mathcal{O}_S)$ and $P_S(\mathcal{O}_S[s_\rho])$, is, in the toric case, isomorphic to their sum, but can be larger than their sum when the bundle ceases to be toric. To do this, one can first take a pushdown to Σ using the map $\hat\rho_{\mathrm{prim}} : \xi \mapsto \rho_{\mathrm{prim}} \cdot \xi$, either in terms of holomorphic sections, or in terms of sections flat along the fibers; here ρ_{prim} is the primitive vector which divides ρ in the lattice $\mathcal{L}$. Then, as covering maps induce surjections on the moduli of flat bundles, one can choose a bundle End_ρ on Σ such that $(\hat\rho^\vee)^* \mathrm{End}_\rho = (\hat\rho_{\mathrm{prim}})^*(\hat\rho_{\mathrm{prim}})_*(\mathrm{End}(E_S))$. Over a generic subset of the toric locus, $\hat\rho^* \mathrm{End}_\rho$ is indeed the direct sum of $P_S(\mathcal{O}_S)$ and $P_S(\mathcal{O}_S[s_\rho])$, and so is of rank $2|W|$. As seen in the Sl(2) case, this sum is not necessarily direct when the bundle is not toric. On the other hand, semi-continuity tells us that everywhere End_ρ is subbundle of rank at least $2|W|$.

One thus has a flat bundle End_ρ on Σ, whose lift to A contains the subbundle $P_S(\mathcal{O}_S[s_\rho])$ as well as a trivial subbundle $P_S(\mathcal{O}_S)$. The inclusion of the latter into $\hat\rho^*(\mathrm{End}_\rho)$ induces an isomorphism on sections. The quotient $\mathrm{End}_\rho / \hat\rho_* P_S(\mathcal{O}_S)$ is still flat. The following shows that End_ρ is a good candidate for defining Higgs fields:

5.1. Proposition. — *Let s be an element of the fiber of $\hat\rho_* P_S(\mathcal{O}_S[s_\rho])$ over the origin in Σ, satisfying $s^2 = 1$. There is a meromorphic section of End_ρ over Σ with a single pole over the origin and residue s.*

For E_S toric, this can be done explicitly; if $q \in \Sigma^r$ represents one of the points of S, the others are given by acting by W, so that S is the union of points $q_w, w \in W$. One can write the desired section, in terms of appropriate automorphy factors, as

$$\mathrm{diag}_{w\in W}(x(\rho \cdot q_w, z))s. \tag{5.2}$$

In general, let us consider the bundle End_ρ for a one dimensional family E_t of bundles, with E_t toric for $t \neq 0$; let us suppose that a choice of s has been made for each E_t. Let $\mathrm{End}_\rho[s]$ be the sheaf of meromorphic sections of End_ρ with a simple pole at the origin and residue a multiple of s. One has the exact sequence

$$0 \to \mathrm{End}_\rho \to \mathrm{End}_\rho[s] \to \mathbb{C} \to 0 \tag{5.3}$$

and so the induced sequence:

$$0 \to H^0(\Sigma, \mathrm{End}_\rho) \to H^0(\Sigma, \mathrm{End}_\rho[s]) \to \mathbb{C}. \tag{5.4}$$

One has that $H^0(\Sigma, \mathrm{End}_\rho)$ is of constant dimension $|W|$; generically, for $t \neq 0$, $H^0(\Sigma, \mathrm{End}_\rho[s])$ is of dimension $|W|+1$; semi-continuity then forces this everywhere, and so one has the desired section.

5.3. The moduli of bundles and Higgs fields.

With this definition of End_ρ, we now simply extend the definition of the moduli space to the non-toric case:

5.5. Definition. — *Let $\mathcal{M}$ be the space of pairs (E_S, ϕ), where E_S is a W-invariant degree zero bundle on A, obtained as the Fourier-Mukai transform of a length $|W|$ W-invariant 0-dimensional scheme S on A^*, and ϕ is a section of $(\Omega^1 \otimes \mathrm{End}(E_S))_D$, built as above as a sum $\phi = \phi_0 + \sum_\rho \phi_\rho$ with ϕ_0 holomorphic, and ϕ_ρ a section of $\hat{\rho}^*(\mathrm{End}_\rho \otimes K_\Sigma)$ with, as before, a simple pole over D_ρ with residue in $P_S(\mathcal{O}_S[s_\rho])$. One asks, as before,*

– that ϕ be W-invariant,

– that at the origin, the residues $R_\rho = m_{|\rho|}^{-1} \mathrm{res}_{D_\rho}(\phi)$ generate a representation of the Weyl group inside $\mathrm{End}(E_S)$ under $s_\rho \mapsto R_\rho$; in particular $(m_{|\rho|}^{-1} \mathrm{res}_{D_\rho}(\phi))^2$ be the identity along D_ρ.

– $\phi \wedge \phi = 0$.

We have a natural map $\pi : \mathcal{M} \to A^*/W$, which to a pair (E_S, ϕ) associates the scheme S. The fibers of this map over the toric locus are r-dimensional planes, and one can check, modeling oneself on the explicit $Gl(2)$ case, that the inverse image over the generic point in the walls is a union of two r-dimensional planes. As one degenerates from the toric case to the generic bundles in the wall, the picture is fairly similar to that of a conic degenerating into the union of two lines.

6. The spectral variety; duality.

6.1. A Lagrangian spectral variety.

The fact that $\phi \wedge \phi = 0$ tells us that the components ϕ_i commute, and so can be simultaneously diagonalised. Let $P : T^*A \to A$ denote the cotangent bundle of A; let ζ denote the tautological section of the lift of

T^*A to itself. Away from the poles of ϕ, one can define a quotient sheaf $\mathcal{L}$ over T^*A by

$$(\Omega^1)^* \otimes \mathrm{End}(E_S) \xrightarrow{\phi-\zeta} \mathrm{End}(E_S) \longrightarrow \mathcal{L} \longrightarrow 0. \tag{6.1}$$

If $z_1, \ldots, z_r$ are coordinates on A, we let ϕ_i denote the corresponding components of ϕ, and ζ_i the components of the tautological section. The support $X_{\mathcal{L}}$ of $\mathcal{L}$ is defined by the equations:

$$\det(\phi_i - \zeta_i) = 0. \tag{6.2}$$

6.3. Proposition. — *The variety $X_{\mathcal{L}}$ is a Lagrangian subvariety of T^*A.*

Proof. — It suffices to prove this for generic ϕ. The variety $X_{\mathcal{L}}$ is described, locally, as the graph of a section $\zeta(z)$; showing that the variety is Lagrangian amounts to proving that $d\zeta(z) = 0$. Varying the term $\mathrm{diag}_{w\in W}(p^w)$ in our explicit expression for the Higgs field, one can show that ϕ is generically semi-simple. One has, over the semi-simple locus, the defining relation for an eigenvalue and eigenvector

$$(\phi - \zeta)v = 0.$$

Our bundles, by the way they are constructed, have a flat structure, and so one can take derivatives in a natural way. Taking the exterior derivative of the relation above,

$$(d\phi - d\zeta)v + (\phi - \zeta)dv = 0.$$

Looking at the explicit form of ϕ given above, one has $d\phi = 0$. One can split, over the semi-simple locus, the bundle E_S into $\ker(\phi-\zeta)\oplus\mathrm{Im}(\phi-\zeta)$; projecting $d((\phi - \zeta)v)$ to $\ker(\phi - \zeta)$ gives $d\zeta = 0$. □

Of course, the spectral variety is not closed, as, over the divisor D, the Higgs field has poles. One should construct the spectral variety $X_{\mathcal{L}}$ not as a subvariety of the symplectic variety T^*A, but as a subvariety of a Poisson manifold T^*_DA, a suitable desingularisation of the total space of the sheaf of logarithmic one-forms along D.

The variety $X_{\mathcal{L}}$ is W-invariant, and so one can take a quotient:

$$\begin{array}{ccccc} X_{\mathcal{L}} & \rightarrow & X_{\mathcal{L}}/W & \subset & T^*_DA/W \\ \downarrow & & \downarrow & & \\ A & \rightarrow & \mathbb{P}_W. & & \end{array} \tag{6.4}$$

6.2. Duality.

We now have two families of Lagrangian submanifolds, the first given as by the integrable system in (over a generic set) T^*A^*/W, the second, after quotienting by W, given by the spectral varieties, living in T^*A/W. One could hope that the two are related, so that the spectral varieties for one system are the leaves of the integrable system for the dual root system. This is indeed the case, as the following calculation shows. We start with the formula for the generic Higgs field:

$$\phi(q_w, p^w; \xi) = \mathrm{diag}_{w\in W}(p^w) + \sum_{\rho\in\Delta_+} m_{|\rho|}\rho^\vee \otimes \mathrm{diag}_{w\in W}(x(\rho\cdot q^w, \rho^\vee\cdot\xi))\hat{s}_\rho. \tag{6.5}$$

This is invariant under the action of W: $\phi_{w,w'}(\xi^{v^{-1}}) = v^{-1}(\phi_{vw,vw'}(\xi))$. Here the elements of Weyl group in subscript serve as indices, and elements of the group in superscripts are acting on the elements to which they are attached.

Let us suppose that we are on the spectral curve at (ξ,λ): there is then a vector x_w in the kernel of $\phi-\lambda$: for all w, we have:

$$0 = (p^w - \lambda)x_w + \sum_{\rho\in\Delta_+} m_{|\rho|}\rho^\vee(x(\rho\cdot q^w, \rho^\vee\cdot\xi))x_{\hat{s}_\rho w}. \tag{6.6}$$

Acting by $u\in W$ on the forms, for all u, w we have:

$$0 = (p^{uw} - \lambda^u)x_w + \sum_{\rho\in\Delta_+} m_{|\rho|}(\rho^\vee)^u(x(\rho^u\cdot q^{uw}, (\rho^\vee)^u\cdot\xi^u))x_{\hat{s}_\rho w}. \tag{6.7}$$

Relabeling:

$$0 = (p^{uw} - \lambda^u)x_w + \sum_{\rho\in\Delta_+} m_{|\rho|}\rho^\vee(x(\rho\cdot q^{uw}, \rho^\vee\cdot\xi^u))x_{u\hat{s}_\rho u^{-1}w}. \tag{6.8}$$

Now restrict to $u = w^{-1}$:

$$0 = (p - \lambda^{w^{-1}})x_w + \sum_{\rho\in\Delta_+} m_{|\rho|}\rho^\vee(x(\rho\cdot q, \rho^\vee\cdot\xi^{w^{-1}}))x_{w\hat{s}_\rho}. \tag{6.9}$$

Now note that replacing the forms on the torus by those on the dual torus, and renormalising the constants, gives:

$$0 = (p - \lambda^{w^{-1}})x_w + \sum_{\rho\in\Delta_+} m_{|\rho^\vee|}\rho\cdot(x(\rho\cdot q, \rho^\vee\cdot\xi^{w^{-1}}))x_{w\hat{s}_\rho} \tag{6.10}$$

which is the dual Calogero Moser Higgs field.

Invariantly, we can define, at least over an open set, a branched cover $\widetilde{\mathcal{M}}$ over $\mathcal{M}$ as a pull-back by the quotient map $A^* \to A^*/W$. On the other side, one should blow up $T^*_D A$ in a suitable fashion, so that the dual $\widetilde{\mathcal{M}}^*$ lies inside the blow up. One has the universal spectral variety $\mathbb{S}$ in $\widetilde{\mathcal{M}} \times \widetilde{\mathcal{M}}^*$ as the zero set of

$$\det(\phi(p^w, q^w, \xi) - \lambda)$$

or as the zero set of

$$\det(\phi^*(\lambda^u, \xi^u, q) - p).$$

There is a double fibration:

$$\begin{array}{ccccc} & & \mathbb{S} & & \\ & \swarrow & & \searrow & \\ \widetilde{\mathcal{M}} & & & & \widetilde{\mathcal{M}}^* \\ & H \searrow & & \swarrow H^* & \\ & & U. & & \end{array} \tag{6.11}$$

The fibers over U are Lagrangian, and $H^{-1}(u)/W$, $(H^*)^{-1}(u)/W$ are the Lagrangian (and we hope, Abelian) varieties corresponding to the integrable systems. We hope to return to this question in another paper.

7. Restriction to curves; the toric $\mathbf{GL}(N)$ case.

For $\mathbf{GL}(N)$, the Weyl group is the symmetric group $\mathcal{S}_N$, acting on $\mathfrak{h} = \mathbb{C}^N = \{\mathrm{diag}(a_1, \ldots, a_N) | a_i \in \mathbb{C}\}$ by permutations. Invariant schemes $S \in (A^* - D^*)/W$ are given by an N-tuple of distinct points $q = (q_1, q_2, \ldots, q_N)$ of Σ^*, with the Weyl group again acting by permutations. Each point q of Σ^* gives a degree zero line bundle L_q on Σ. The construction outlined above gives a rank $N!$ bundle on $A = \Sigma^N$, which is a sum over $w \in \mathcal{S}_N$ of line bundles:

$$E = \oplus_{w \in \mathcal{S}_N} (\pi_1^* L_{q_{w^{-1}(1)}} \otimes \cdots \otimes \pi_N^* L_{q_{w^{-1}(N)}}).$$

This bundle is invariant under the diagonal action of permutation of the summands and permutation of coordinates on the base.

Choose $(p^1, \ldots, p^N) \in \mathbb{C}^N$. The Higgs field is given by

$$\begin{aligned} \phi = & \,\mathrm{diag}_{w \in \mathcal{S}_N} \Big(\sum p^{w^{-1}(i)} d\xi_i \Big) \\ & + \sum_{1 \leqslant i < j \leqslant N} (d\xi_i - d\xi_j) \otimes \mathrm{diag}_{w \in \mathcal{S}_N} (x(q^{w^{-1}(i)} - q^{w^{-1}(j)}, \xi_i - \xi_j)) \hat{s}_{i,j}. \end{aligned}$$

There is a method given in [BCS] for turning their universal Lax element into a Lax matrix over an elliptic curve. Translated into our context, this is done as follows

– One chooses a family of co-weights α_j, $j = 1, \ldots, M$ which is Weyl invariant.

– Each co-weight gives an embedding $\hat{\alpha}_j : \Sigma \to A$; one pulls back the bundle E_S and the section ϕ (as a section of the endomorphism bundle and as a form) via each of these embeddings, and sums over the co-weights, to obtain a rank $M|W|$ bundle $\widehat{E}$ and a section $\hat{\phi}$, which are both Weyl invariant, this time under an action which maps each fiber to itself.

– One then restricts to the fixed locus, obtaining a rank M bundle $\widetilde{E} = \widehat{E}^W$ and a section $\tilde{\phi}$ of $\widetilde{E}$ with a pole at the origin. Alternately, if there is just one orbit of co-weights under the action of the Weyl group, one can just pull-back using one co-weight and quotient by the stabiliser of that co-weight.

In general, this gives only an embedding of the Calogero-Moser space into a much bigger phase space. For $\mathbf{GL}(N)$, however, it gives a phase space which coincides exactly with one defined by Krichever. For $\mathbf{GL}(N)$, one chooses co-weights $\alpha_i, i = 1, \ldots, N$ of the form

$$(\alpha_i)_j = \delta_{i,j},$$

which forms a Weyl invariant set, on which the symmetric group acts transitively. The pull-back of the bundle E_S under $\hat{\alpha}_1$ is given by $\oplus_{w\in\mathcal{S}_N} L_{q_{w(1)}}$. The pull-back of the Higgs field by $\hat{\alpha}_1$ to Σ is given by

$$\begin{aligned}(7.1)\qquad \hat{\alpha}_1^*\phi = \mathrm{diag}_{w\in\mathcal{S}_N}\Big(\sum_i p^{w^{-1}(1)} d\xi\Big) \\ + \sum_{1<j\leqslant N} (d\xi) \otimes \mathrm{diag}_{w\in\mathcal{S}_N}(x(q^{w^{-1}(1)} - q^{w^{-1}(j)}, \xi))\hat{s}_{1,j}.\end{aligned}$$

One must look at the invariant part of the pull-back bundle and of $\hat{\alpha}_1^*\phi$ under $W_1 = \mathrm{Stab}(1)$. The pull-back bundle is a sum of line bundles L_w indexed by the elements of W. Choose a basis for one of them, and then act by V to give a basis e_w for each of them. The W_1-invariant subspace is generated by

$$f_i = \sum_{w\in W_1} e_{ws_{1,i}}.$$

One then has:

$$\mathrm{diag}_{w\in\mathcal{S}_N}(p^{w^{-1}(1)})(f_i) = \sum p^i f_i$$

$$
\begin{aligned}
(7.2)\quad &\sum_{1<j} \mathrm{diag}_{w\in\mathcal{S}_N}\, x(q^{w^{-1}(1)} - q^{w^{-1}(j)}, \xi)(f_i)\\
&= \sum_{w\in W_1}\sum_{1<j} x(q^i - q^{s_{1,i}w^{-1}s_{1,j}(1)}, \xi) e_{s_{1,j}ws_{1,i}}\\
&= \sum_{k=1}^{N} x(q^i - q^k, \xi)(\sum_{1<j}(\sum_{\{w\in\mathcal{S}_N \mid s_{1,i}w^{-1}s_{1,j}(1)=k\}} e_{s_{1,j}ws_{1,i}}))\\
&= \sum_{k=1}^{N} x(q^i - q^k, \xi)(\sum_{1<j}(\sum_{\{w\in\mathcal{S}_N \mid w^{-1}(j)=1, w^{-1}(1)=k\}} e_w))\\
&= \sum_{k\neq i} x(q^i - q^k, \xi) f_k.
\end{aligned}
$$

This gives the $N \times N$ matrix:

$$\tilde{\phi}_{i,j} = \delta_{i,j} p^i + (1-\delta_{i,j}) x(q_i - q_j, \xi).$$

For $\mathbf{GL}(N)$, there is the alternate description of the Calogero-Moser system, due to Krichever: one looks at the space of pairs $(\widetilde{E}, \tilde{\phi})$, where $\widetilde{E}$ is a rank N vector bundle over Σ and $\tilde{\phi}$ is a section of $\mathrm{End}(E)$ with a pole at the origin and a residue conjugate to $r_{i,j} = (1-\delta_{i,j})$. The formula above is exactly the explicit expression of these sections, and so we have obtained, for the $\mathbf{GL}(N)$-bundle corresponding to $(q_1, \ldots, q_N)$, the original $\mathbf{GL}(N)$ construction.

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Jacques HURTUBISE,
McGill University
Department of Mathematics
Montréal QC H3A 2K6 (Canada)
and
Université de Montréal
Centre de Recherches Mathématiques
Montréal, QC H3P 3J7 (Canada)
hurtubis@math.mcgill.ca

Thomas NEVINS,
University of Illinois at Urbana-Champaign
Department of Mathematics
Urbana, IL 61801 (USA)
nevins@math.uiuc.edu