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## ON THE NUMBER OF ELLIPTIC CURVES WITH CM OVER LARGE ALGEBRAIC FIELDS (\*)

by Gerhard FREY & Moshe JARDEN (\*\*)

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### Introduction.

The goal of this note is to report on a new phenomena in the theory of large fields.

As usual, we denote the absolute Galois group of  $\mathbb{Q}$  by  $\text{Gal}(\mathbb{Q})$  and equip each of the cartesian powers  $\text{Gal}(\mathbb{Q})^e$  by the normalized Haar measure  $\mu$ . Let  $\tilde{\mathbb{Q}}$  be the algebraic closure of  $\mathbb{Q}$ . For each  $\sigma = (\sigma_1, \dots, \sigma_e)$  let  $\tilde{\mathbb{Q}}(\sigma)$  be the fixed field in  $\tilde{\mathbb{Q}}$  of  $\sigma_1, \dots, \sigma_e$ . The behavior of the fields  $\tilde{\mathbb{Q}}(\sigma)$  becomes regular if we remove sets of measure zero. This is exemplified by the following fundamental result:

**THEOREM A** ([FrJ], Thms. 18.5.6 and 18.6.1). — *The following statements hold for almost all  $\sigma \in \text{Gal}(\mathbb{Q})^e$ :*

(a) *The absolute Galois group of  $\tilde{\mathbb{Q}}(\sigma)$  is isomorphic to the free profinite group  $\hat{F}_e$  on  $e$  generators.*

(b) *The field  $\tilde{\mathbb{Q}}(\sigma)$  is PAC, that is, each absolutely irreducible variety  $V$  defined over  $\tilde{\mathbb{Q}}(\sigma)$  has a  $\tilde{\mathbb{Q}}(\sigma)$ -rational point.*

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Likewise, the following holds for Abelian varieties:

**THEOREM B** ([FyJ]). — *Let  $A$  be an abelian variety over  $\mathbb{Q}$ . Then for almost all  $\sigma \in \text{Gal}(\mathbb{Q})^e$  the rank of  $A(\tilde{\mathbb{Q}}(\sigma))$  is infinite.*

Note that the fields  $\tilde{\mathbb{Q}}(\sigma)$  become smaller as  $e$  increases. Thus, it is expected that in general less arithmetical objects will be defined over  $\tilde{\mathbb{Q}}(\sigma)$  as  $e$  increases. Here are two typical examples:

**THEOREM C** ([JaJ], Main Theorem (a)). — *Let  $A$  be an Abelian variety and  $l$  a prime number. Then for each  $e \geq 1$  and for almost all  $\sigma \in \text{Gal}(K)^e$  the set  $\bigcup_{i=1}^{\infty} A_{l^i}(\tilde{\mathbb{Q}}(\sigma))$  is finite (while  $\bigcup_{i=1}^{\infty} A_{l^i}(\tilde{\mathbb{Q}})$  is infinite, which is the case if  $e = 0$ ).*

Here  $A_n(L) = \{\mathfrak{p} \in A(L) \mid n\mathfrak{p} = 0\}$  for each positive integer  $n$  and each field extension  $L$  of  $K$ .

**THEOREM D** ([Jar] Thms. 8.1 and 8.2). — *The following holds for almost all  $\sigma \in \text{Gal}(\mathbb{Q})^e$ :*

- (a) *If  $e = 1$ , then  $\tilde{\mathbb{Q}}(\sigma)$  contains infinitely many roots of unity.*
- (b) *If  $e \geq 2$ , then  $\tilde{\mathbb{Q}}(\sigma)$  contains only finitely many roots of unity.*

**THEOREM E**. — *Let  $E$  be an elliptic curve over  $\mathbb{Q}$ . Then the following holds for almost all  $\sigma \in \text{Gal}(\mathbb{Q})^e$ :*

- (a) *If  $e = 1$ , then  $E_{\text{tor}}(\tilde{\mathbb{Q}}(\sigma))$  is infinite.*
- (b) *If  $e \geq 2$ , then  $E_{\text{tor}}(\tilde{\mathbb{Q}}(\sigma))$  is finite.*

The arithmetical reason that lies behind the distinction between the cases  $e = 1$  and  $e \geq 2$  in Theorems D and E is that the series  $\sum \frac{1}{l^e}$ , with  $l$  ranges over all prime numbers, diverges for  $e = 1$  and converges for  $e \geq 2$ .

In general, we call a nonnegative integer  $e_0$  a **cut** for the large fields over  $\mathbb{Q}$  if there exists an infinite set  $P$  of arithmetical or geometrical objects defined over  $\tilde{\mathbb{Q}}$  such that for almost all  $\sigma \in \text{Gal}(K)^e$  infinitely many objects of  $P$  are defined over  $\tilde{\mathbb{Q}}(\sigma)$  if  $e < e_0$  and only finitely many objects of  $P$  are defined over  $\tilde{\mathbb{Q}}(\sigma)$  if  $e \geq e_0$ .

Theorem C implies that 1 is a cut for the large fields over  $\mathbb{Q}$ , while Theorems D and E imply that 2 is a cut for the large fields over  $\mathbb{Q}$ .

For a long time 1 and 2 were the only known cuts for large fields over  $\mathbb{Q}$ . The goal of the present note is to prove that also 3 and 4 are cuts for large fields over  $\mathbb{Q}$ . The relevant properties of fields were hidden in the theory of elliptic curves with complex multiplication:

THEOREM F. — *The following holds for almost all  $\sigma \in \text{Gal}(\mathbb{Q})^e$ :*

(a) *If  $e \leq 2$ , then there are infinitely many elliptic curves  $E$  (up to  $\mathbb{C}$ -isomorphism) with CM over  $\tilde{\mathbb{Q}}(\sigma)$  such that  $\text{End}(E) \subseteq \tilde{\mathbb{Q}}(\sigma)$ .*

(b) *If  $e \geq 3$ , then there are only finitely many elliptic curves  $E$  (up to  $\mathbb{C}$ -isomorphism) with CM over  $\tilde{\mathbb{Q}}(\sigma)$  such that  $\text{End}(E) \subseteq \tilde{\mathbb{Q}}(\sigma)$ .*

THEOREM G. — *The following holds for almost all  $\sigma \in \text{Gal}(\mathbb{Q})^e$ :*

(a) *If  $e \leq 3$ , then there are infinitely many elliptic curves  $E$  (up to  $\mathbb{C}$ -isomorphism) with CM over  $\tilde{\mathbb{Q}}(\sigma)$ .*

(b) *If  $e \geq 4$ , then there are only finitely many elliptic curves  $E$  (up to  $\mathbb{C}$ -isomorphism) with CM over  $\tilde{\mathbb{Q}}(\sigma)$ .*

The proofs of Theorems F and G use the standard properties of the  $j$ -function of elliptic curves with CM as in [Shi] and [Lan] and information about the growth of the class number of imaginary quadratic fields:

THEOREM H. — *For each prime number  $p$  let  $h(p)$  be the class number of  $\mathbb{Q}(\sqrt{-p})$ . Then  $\sum \frac{1}{h(p)^2} = \infty$ , where  $p$  ranges on all prime numbers which are congruent to 3 modulo 4.*

The authors are indebted to Ram Murty for kindly supplying the proof of Theorem H.

Finally, we rephrase Theorem F for a family of large fields which are considerably smaller than the fields  $\tilde{\mathbb{Q}}(\sigma)$ . For each  $\sigma \in \text{Gal}(\mathbb{Q})^e$  we denote the maximal Galois extension of  $\mathbb{Q}$  which is contained in  $\tilde{\mathbb{Q}}(\sigma)$  by  $\tilde{\mathbb{Q}}[\sigma]$ . Then the following holds:

THEOREM I. — *The following holds for almost all  $\sigma \in \text{Gal}(\mathbb{Q})^e$ :*

(a) *If  $e \leq 2$ , then there are infinitely many elliptic curves  $E$  (up to  $\mathbb{C}$ -isomorphism) with CM over  $\tilde{\mathbb{Q}}[\sigma]$ .*

(b) *If  $e \geq 3$ , then there are only finitely many elliptic curves  $E$  (up to  $\mathbb{C}$ -isomorphism) with CM over  $\tilde{\mathbb{Q}}[\sigma]$ .*

## 1. On the growth of the class number of imaginary quadratic fields.

For each prime number  $p$  let  $h(p)$  be the class number of  $K_p = \mathbb{Q}(\sqrt{-p})$ . By a theorem of Siegel,  $\log h(p) \sim \log \sqrt{p}$  [Lan], p. 96. Thus, there exists  $\epsilon(p)$  which tends to 0 as  $p \rightarrow \infty$  such that  $\log h(p) = (1 + \epsilon(p)) \log \sqrt{p}$ . It follows that

$$(1) \quad \sum_p \frac{1}{h(p)^2} = \sum_p \frac{1}{p^{1+\epsilon(p)}}.$$

One knows that  $\sum_p \frac{1}{p}$  diverges. Unfortunately, without any additional information about  $\epsilon(p)$  one can not draw from (1) that its left hand side diverges. Still, the sum does diverge, as we prove below:

PROPOSITION 1.1 (Murty). — *With the notation above,*

$$(2) \quad \sum_{p \equiv 3 \pmod{4}} \frac{1}{h(p)^2} = \infty,$$

*Proof.* — Lemma 1.2 below reduces (2) to the proof of the existence of a constant  $c > 0$  such that

$$(3) \quad \sum_{\substack{p \leq x \\ p \equiv 3 \pmod{4}}} \frac{h(p)}{p} \sim \frac{c\sqrt{x}}{\log x}.$$

In order to prove (3) suppose  $p \equiv 3 \pmod{4}$  is a prime number and let  $\chi_p$  be the quadratic character of  $K_p$ . Thus,  $\chi_p(n) = (-1)^{\frac{n-1}{2}} \left(\frac{n}{p}\right)$  if  $p \nmid n$  [BoS], Chap. 3, § 8.2. Let  $l$  be a prime number satisfying  $l \nmid 2p$ . Then  $l$  decomposes in  $K_p$  into two distinct primes if  $\chi_p(l) = 1$  and  $l$  remains prime in  $K_p$  if  $\chi_p(l) = -1$  [BoS], Chap. 3, § 8.2, Thm. 2. Let  $L(s, \chi_p) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$  be the corresponding  $L$ -series. By the Dirichlet class number formula [BoS, Chap. 5, § 4.1],  $h(p)$  is a multiple of  $\sqrt{p}L(1, \chi_p)$  by a constant. Hence, (3) is equivalent to the existence of  $c > 0$  such that

$$(4) \quad \sum_{\substack{p \leq x \\ p \equiv 3 \pmod{4}}} \frac{L(1, \chi_p)}{\sqrt{p}} \sim \frac{c\sqrt{x}}{\log x}$$

Statement (4) is essentially proved in [FoM], pp. 91–93. □

The rest of this section proves the equivalence of (2) and (3).

For each set  $P$  of prime numbers let  $\pi(P, x)$  be the number of  $p \in P$  with  $p \leq x$ . In particular, if  $P$  is the set of all prime numbers, then

$\pi(P, x) = \pi(x)$ . If  $P$  is the set of all prime numbers  $p \equiv a \pmod n$ , we write  $\pi_{a,n}(x)$  for  $\pi(P, x)$ . By the prime number theorem for arithmetical progressions [LaO], Thms. 1.3 and 1.4 applied to the case of  $L = \mathbb{Q}(\zeta_n)$ ,

$$(5) \quad \pi(x) = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right) \quad \text{and} \quad \pi_{a,n}(x) = \frac{1}{\varphi(n)} \cdot \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right),$$

where  $\varphi(n)$  is Euler's totient function.

LEMMA 1.2. — For each prime number  $p$  let  $h(p)$  be a positive real number. Suppose that there exists  $c > 0$  such that

$$(6) \quad \sum_{\substack{p \leq x \\ p \equiv 3 \pmod 4}} \frac{h(p)}{p} \sim \frac{c\sqrt{x}}{\log x}.$$

Then (2) is true.

Proof. — Apply summation by parts:

$$\begin{aligned} \sum_{\substack{p \leq x \\ p \equiv 3 \pmod 4}} \frac{h(p)}{\sqrt{p}} &= \sum_{\substack{p \leq x \\ p \equiv 3 \pmod 4}} \frac{h(p)}{p} \cdot \sqrt{p} \\ &= \sum_{\substack{p \leq x \\ p \equiv 3 \pmod 4}} \frac{h(p)}{p} \cdot \sqrt{x} - \frac{1}{2} \int_2^x \sum_{\substack{p \leq t \\ p \equiv 3 \pmod 4}} \frac{h(p)}{p} \cdot \frac{1}{\sqrt{t}} dt \\ &\sim c \frac{\sqrt{x}}{\log x} \cdot \sqrt{x} - \frac{c}{2} \int_2^x \frac{\sqrt{t}}{\log t} \cdot \frac{1}{\sqrt{t}} dt \quad \text{by (6)} \\ &= c \frac{x}{\log x} - \frac{c}{2} \int_2^x \frac{dt}{\log t} \sim \frac{c}{2} \frac{x}{\log x}. \end{aligned}$$

The latter approximation is a consequence of the formula  $\int_2^x \frac{dt}{\log t} \sim \frac{x}{\log x}$  [Gol], pp. 254–255, Remark (2). Hence, by (5) there exists  $x_0$  such that

$$c\pi_{3,4}(x) \geq \frac{1}{2} \sum_{\substack{p \leq x \\ p \equiv 3 \pmod 4}} \frac{h(p)}{\sqrt{p}}$$

and  $\pi_{3,4}(x) \geq \frac{1}{3}\pi(x)$  for all  $x \geq x_0$ . Let  $P = \{p \equiv 3 \pmod 4 \mid h(p) > 6c\sqrt{p}\}$  and let  $P' = \{p \equiv 3 \pmod 4 \mid h(p) \leq 6c\sqrt{p}\}$ . Then, for all  $x \geq x_0$

$$\pi_{3,4}(x) \geq \frac{1}{2c} \sum_{\substack{p \leq x \\ p \equiv 3 \pmod 4}} \frac{h(p)}{\sqrt{p}} \geq \frac{1}{2c} \sum_{\substack{p \leq x \\ p \in P'}} \frac{h(p)}{\sqrt{p}} \geq 3\pi(P, x).$$

It follows from  $\pi_{3,4}(x) = \pi(P, x) + \pi(P', x)$  that  $\pi(P', x) \geq \frac{2}{3}\pi_{3,4}(x) \geq \frac{2}{9}\pi(x)$  for all  $x \geq x_0$ . It follows from Lemma 1.3 below that

$$\sum_{p \equiv 3 \pmod 4} \frac{1}{h(p)^2} \geq \sum_{p \in P'} \frac{1}{h(p)^2} \geq \frac{1}{36c^2} \sum_{p \in P'} \frac{1}{p} = \infty,$$

as contended. □

LEMMA 1.3. — Let  $Q$  be a set of prime numbers,  $0 < b \leq 1$ , and  $x_0 > 0$  such that  $\pi(Q, x) \geq b\pi(x)$  for all  $x \geq x_0$ . Then  $\sum_{p \in Q} \frac{1}{p} = \infty$ .

*Proof.* — We reduce the statement to the well known fact that  $\sum \frac{1}{p} = \infty$  [LeV], p. 100, Thm. 6-13. To this end make  $b$  smaller and add all prime numbers  $p \leq x_0$  to  $Q$  if necessary, in order to assume that  $x_0 = 1$ . Then write the set of all prime numbers as an ascending sequence,  $p_1 < p_2 < p_3 < \dots$  and define

$$\chi(n) = \begin{cases} 1 & p_n \in Q \\ 0 & p_n \notin Q. \end{cases}$$

Then  $s(n) = \sum_{i=1}^n \chi(i) = \pi(Q, p_n) \geq b\pi(p_n) = bn$ . Therefore, with  $s(0) = 0$ , we have

$$\begin{aligned} \sum_{\substack{i=1 \\ p_i \in Q}}^n \frac{1}{p_i} &= \sum_{i=1}^n \frac{\chi(i)}{p_i} = \sum_{i=1}^n \frac{s(i) - s(i-1)}{p_i} = \sum_{i=1}^n \frac{s(i)}{p_i} - \sum_{i=1}^n \frac{s(i-1)}{p_i} \\ &= \sum_{i=1}^n \frac{s(i)}{p_i} - \sum_{i=1}^{n-1} \frac{s(i)}{p_{i+1}} = \frac{s(n)}{p_n} + \sum_{i=1}^{n-1} s(i) \left( \frac{1}{p_i} - \frac{1}{p_{i+1}} \right) \\ &\geq \frac{bn}{p_n} + b \sum_{i=1}^{n-1} i \left( \frac{1}{p_i} - \frac{1}{p_{i+1}} \right) = \frac{bn}{p_n} + b \sum_{i=1}^{n-1} \frac{i}{p_i} - b \sum_{i=1}^{n-1} \frac{i}{p_{i+1}} \\ &= b \sum_{i=1}^n \frac{i}{p_i} - b \sum_{i=1}^n \frac{i-1}{p_i} = b \sum_{i=1}^n \frac{1}{p_i} \rightarrow \infty \quad \text{as } n \rightarrow \infty \end{aligned}$$

as contended. □

## 2. On the number of elliptic curves with CM over large algebraic fields.

Consider a positive integer  $e$  and choose  $\sigma$  in  $\text{Gal}(\mathbb{Q})^e$  at random. We would like to know whether there are infinitely many elliptic curves  $E$  (up to  $\mathbb{C}$ -isomorphism) with CM which are defined over  $\tilde{\mathbb{Q}}(\sigma)$ . We would also like to know whether there are infinitely many elliptic curves  $E$  (up to  $\mathbb{C}$ -isomorphism) which are defined over  $\tilde{\mathbb{Q}}(\sigma)$  and such that all  $\mathbb{C}$ -endomorphisms of  $E$  are defined over  $\tilde{\mathbb{Q}}(\sigma)$ . Since  $\tilde{\mathbb{Q}}(\sigma)$  becomes smaller as  $e$  increases, we expect to find for each of those questions an  $e_0$  such that the answer to the question is affirmative if and only if  $e \leq e_0$ . Indeed, we prove that  $e_0 = 3$  for the former question and  $e_0 = 2$  for the latter.

These results reflect the distribution of the modular  $j$ -function at **singular values**, that is complex values which correspond to elliptic curves with CM. To be more precise consider an imaginary quadratic field  $K$ , an order  $O$  of  $K$ , and a proper  $O$ -ideal  $\mathfrak{a}$ . Then  $\mathfrak{a}$  is a 2-dimensional lattice which is  $O$ -invertible [Lan], p. 91. Let  $z_1, z_2$  be a basis of  $\mathfrak{a}$  and put  $z = z_1/z_2$ . Then  $j(\mathfrak{a}) = j(z)$  is the absolute invariant of an elliptic curve  $E$  with the analytic presentation  $\mathbb{C}/\mathfrak{a}$  and such that  $\text{End}(E) \cong O$ . Moreover,  $E$  can be chosen to be defined by a Weierstrass equation over  $\mathbb{Q}(j(\mathfrak{a}))$ . The basic properties of  $j(\mathfrak{a})$  are intimately connected to class field theory:

PROPOSITION 2.1 ([Shi], p. 123, Thm. 5.7). — *Let  $K$  be an imaginary quadratic field,  $O$  an order of  $K$ , and  $\mathfrak{a}$  a proper  $O$ -ideal. Then:*

(a)  $K(j(\mathfrak{a}))/K$  is a Galois extension and  $\text{Gal}(K(j(\mathfrak{a}))/K)$  is isomorphic to the group of all classes of proper  $O$ -ideals through the correspondence  $\sigma \mapsto \mathfrak{b}$  such that  $j(\mathfrak{a})^\sigma = j(\mathfrak{b}^{-1}\mathfrak{a})$ .

(b)  $[K(j(\mathfrak{a})) : K] = [\mathbb{Q}(j(\mathfrak{a})) : \mathbb{Q}]$ .

(c) If  $\mathfrak{a}_1, \dots, \mathfrak{a}_n$  are representatives of the classes of proper  $O$ -ideals, then the values  $j(\mathfrak{a}_1), \dots, j(\mathfrak{a}_n)$  form a complete set of conjugates of  $j(\mathfrak{a})$  over  $\mathbb{Q}$ , and over  $K$ .

(d) If  $O$  is the ring of integers of  $K$  (hence,  $\mathfrak{a}$  is a fractional ideal of  $O$  in  $K$ ), then  $K(j(\mathfrak{a}))$  is the maximal unramified abelian extension of  $K$ , and for each fractional ideal  $\mathfrak{b}$  of  $K$  we have  $j(\mathfrak{a})^\sigma = j(\mathfrak{b}^{-1}\mathfrak{a})$  where  $\sigma = \left(\frac{K(j(\mathfrak{a}))/K}{\mathfrak{b}}\right)$  is the Artin symbol.

COROLLARY 2.2. — *Fix an embedding of  $\tilde{\mathbb{Q}}$  in  $\mathbb{C}$ . Then, with the notation of Proposition 2.1, we have:*

(a)  $K(j(\mathfrak{a}))$  is the Galois closure of  $\mathbb{Q}(j(\mathfrak{a}))$  over  $\mathbb{Q}$ .

(b)  $[K(j(\mathfrak{a})) : \mathbb{Q}(j(\mathfrak{a}))] = 2$ .

(c)  $K(j(\mathfrak{a}))/K$  is an abelian extension.

(d) If  $\tau$  is a conjugate of the restriction to  $K(j(\mathfrak{a}))$  of the complex conjugation, then  $\tau^{-1}\alpha\tau = \alpha^{-1}$  for each  $\alpha \in \text{Gal}(K(j(\mathfrak{a}))/K)$ .



*Proof.* — Statement (d) follows from [Lan], p. 134, Remark 2. Statement (c) is a consequence of Proposition 2.1(a). Statements (a) and (b) follow from Proposition 2.1(b,c) and from (d).  $\square$

Denote the set of all squarefree positive integers by  $D$ . For each  $d \in D$  let  $K_d = \mathbb{Q}(\sqrt{-d})$ . Denote the ring of integers and the class number of  $K_d$ , respectively, by  $O_d$  and  $h(d)$ . Choose a nonzero ideal  $\mathfrak{a}_d$  of  $O_d$  and let  $L_d = K_d(j(\mathfrak{a}_d))$ . By Proposition 2.1(a),  $h(d) = [L_d : K_d]$ . Choose also an elliptic curve  $E^{(d)}$  with  $j(\mathfrak{a}_d)$  as its absolute invariant which is defined over  $\mathbb{Q}(j(\mathfrak{a}_d))$  [Lan], p. 300, Thm. 2.

LEMMA 2.3. — *Let  $\Lambda$  be the set of all prime  $l \equiv 3 \pmod{4}$ . Then, the fields  $L_l$ , with  $l \in \Lambda$ , are linearly disjoint over  $\mathbb{Q}$ .*

*Proof.* — Consider a finite set  $\Lambda_0$  of  $\Lambda$  and an element  $l' \in \Lambda \setminus \Lambda_0$ . Let  $L = \prod_{l \in \Lambda_0} L_l$ . By Corollary 2.2(a), each  $L_l$  is Galois over  $\mathbb{Q}$ . Hence, it suffices to prove that  $L \cap L_{l'} = \mathbb{Q}$ . Since, by a theorem of Minkowski, each proper extension of  $\mathbb{Q}$  is ramified [Jan], p. 57, Cor. 11.11, it suffices to prove that no prime number  $p$  is ramified in  $L \cap L_{l'}$ .

Indeed, for each  $l \in \Lambda$  we have  $-l \equiv 1 \pmod{4}$ . Hence, the discriminant of  $K_l/\mathbb{Q}$  is  $-l$  [BoS], § 2.7, p. 132, Thm. 1, so the only prime number which ramifies in  $K_l$  is  $l$ . Since  $L_l/K_l$  is unramified (Proposition 2.1(d)), the only prime number which ramifies in  $L_l$  is  $l$ . In particular,  $l'$  is unramified in each  $L_l$  with  $l \in \Lambda_0$ . Hence,  $l'$  is unramified in  $L$ , so  $l'$  is unramified in  $L \cap L_{l'}$ . If  $p \neq l'$ , then  $p$  is unramified in  $L_{l'}$ , so  $p$  is also unramified in  $L \cap L_{l'}$ . Consequently,  $L \cap L_{l'} = \mathbb{Q}$ , as asserted.  $\square$

The orders of  $K_d$  have the form  $O_{d,c} = \mathbb{Z} + cO_d$ , where  $c$  ranges over all positive integers. For each  $d \in D$  and  $c \in \mathbb{N}$  choose a proper  $O_{d,c}$ -ideal  $\mathfrak{a}_{d,c}$  and let  $L_{d,c} = K_d(j(\mathfrak{a}_{d,c}))$ . By Proposition 2.1(c),  $h(d,c) = [L_{d,c} : K_d]$  is the class number of  $O_{d,c}$ . It is related to  $h(d)$  by the following formula [Lan], p. 95:

$$(1) \quad h(d,c) = h(d) \frac{\psi(d,c)}{(O_d^\times : O_{d,c}^\times)},$$

where

$$(2) \quad \psi(d,c) = c \prod_{p|c} \left( 1 - \left( \frac{K_d}{p} \right) \frac{1}{p} \right),$$

and  $\left( \frac{K_d}{p} \right)$  is 1 if  $p$  decomposes in  $K_d$ ,  $-1$  if  $p$  remains irreducible in  $K_d$ , and 0 if  $p$  ramifies in  $K_d$ .

LEMMA 2.4. — *Let  $L$  be a finite Galois extension of  $\mathbb{Q}$ . Then there are only finitely many elliptic curves  $E$  with CM (up to  $\mathbb{C}$ -isomorphism) which are defined over  $L$  and satisfy  $\text{End}(E) \subseteq L$ .*

*Proof.* — Let  $E$  be an elliptic curve over  $L$  with CM such that  $\text{End}(E) \subseteq L$ . Then  $\text{End}(E) \otimes \mathbb{Q} = K_d$  for some  $d \in D$  [Shi], p. 103, Prop. 4.5. Moreover,  $\text{End}(E)$  is an order of  $O_d$  and there is a unique  $c \in \mathbb{N}$  with  $\text{End}(E) = O_{d,c}$  [Shi], p. 105, Prop. 4.1. In addition,  $E \cong \mathbb{C}/\mathfrak{a}$  for some proper  $O_{d,c}$ -ideal  $\mathfrak{a}$  [Shi], p. 104, Prop. 4.8. In particular  $j(\mathfrak{a})$  is the absolute invariant of  $E$ , so  $K_d(j(\mathfrak{a})) \subseteq L$ . By the comments preceding the lemma,  $[K_d(j(\mathfrak{a})) : \mathbb{Q}] = 2h(d, c)$  and  $h(d, c)$  tends to infinity if  $d$  or  $c$  tend to infinity. Indeed, by the estimates quoted in the proof of the next lemma,  $\log h(d) \sim \log d^{\frac{1}{2}}$  and  $\psi(d, c) \geq \frac{ac}{\log \log c}$  for some  $a > 0$ . Thus, there are only finitely many possibilities for  $(d, c)$ . For each pair  $(d, c) \in D \times \mathbb{N}$  there are only finitely many possibilities (up to  $\mathbb{C}$ -isomorphism) for  $E$ . They correspond to the number  $h(d, c)$  of classes of proper  $O_{d,c}$ -ideals [Shi], p. 105, Prop. 4.10. Consequently, there are only finitely many  $\mathbb{C}$ -isomorphism classes of elliptic curves  $E$  with CM such that  $j(E) \in L$  and  $\text{End}(E) \subseteq L$ . □

LEMMA 2.5. — *Let  $D$  be the set of all squarefree positive integers. Then*

$$(3) \quad \sum_{d \in D} \sum_{c=1}^{\infty} \frac{1}{h(d, c)^3} < \infty.$$

*Proof.* — By (1), it suffices to prove that

$$(4) \quad \sum_{d \in D} \frac{1}{h(d)^3} \sum_{c=1}^{\infty} \frac{(O_d^\times : O_{d,c}^\times)^3}{\psi(d, c)^3} < \infty.$$

There are at most 6 units in  $O_d$  [BoS], § 2.7.3. Hence, the numerator in the inner sum of the right hand side of (4) is bounded. Next consider the Euler totient function:  $\varphi(c) = c \prod_{p|c} (1 - \frac{1}{p})$ . It has an estimate from below:  $\varphi(c) > \frac{ac}{\log \log c}$  for some positive constant  $a$  [Lev], p. 114, Thm. 6-26. For each  $p$ ,  $1 - (\frac{K_d}{p})^{\frac{1}{p}} \geq 1 - \frac{1}{p}$ . Hence,  $\psi(d, c) \geq \varphi(c)$ , so

$$(5) \quad \sum_{c=1}^{\infty} \frac{1}{\psi(d, c)^3} \leq \sum_{c=1}^{\infty} \frac{1}{\varphi(c)^3} \leq \frac{1}{a^3} \sum_{c=1}^{\infty} \frac{(\log \log c)^3}{c^3} < \infty.$$

Finally, by a theorem of Siegel,  $\log h(d) \sim \log d^{\frac{1}{2}}$  [Lan], p. 96. This means that for each  $d \in D$  there exists  $\epsilon(d) > 0$  such that  $h(d) = d^{\epsilon(d)/2}$

and  $\epsilon(d) \rightarrow 1$  as  $d \rightarrow \infty$ . In particular,  $\epsilon(d) > \frac{3}{4}$  for all  $d$  sufficiently large. Hence,  $\frac{3-\epsilon(d)}{2} > \frac{9}{8}$  for almost all  $d$  sufficiently large, so there exists  $b > 0$  such that

$$(6) \quad \sum_{d \in D} \frac{1}{h(d)^3} = \sum_{d \in D} \frac{1}{d^{3-\epsilon(d)/2}} \leq \sum_{d=1}^{\infty} \frac{b}{d^{9/8}} < \infty.$$

We conclude from (5) and (6) that (4) holds. □

The main tool from probability theory we use is the Borel-Cantelli Lemma. We formulate its Galois theoretic version as appears in [FrJ], Theorem 18.5.3:

LEMMA 2.6. — *Let  $L_1, L_2, L_3, \dots$  be finite separable extensions of a field  $K$ . For each  $i \geq 1$  let  $\bar{A}_i$  be a set of left cosets of  $\text{Gal}(L_i)^e$  in  $\text{Gal}(K)^e$  and*

$$A_i = \{\sigma \in \text{Gal}(K)^e \mid \sigma \text{Gal}(L_i)^e \in \bar{A}_i\}.$$

Let  $A$  be the set of all  $\sigma \in \text{Gal}(K)^e$  which belong to infinitely many  $A_i$ 's.

(a) If  $\sum_{i=1}^{\infty} \frac{|\bar{A}_i|}{[L_i:K]^e} < \infty$ , then  $\mu(A) = 0$ .

(b) Suppose  $L_1, L_2, L_3, \dots$  are linearly disjoint over  $K$  and  $\sum_{i=1}^{\infty} \frac{|\bar{A}_i|}{[L_i:K]^e} = \infty$ , then  $\mu(A) = 1$ .

THEOREM 2.7. — *The following holds for almost all  $\sigma \in \text{Gal}(\mathbb{Q})^e$ :*

(a) *If  $e \leq 2$ , then there are infinitely many elliptic curves  $E$  (up to  $\mathbb{C}$ -isomorphism) with CM over  $\tilde{\mathbb{Q}}(\sigma)$  such that  $\text{End}(E) \subseteq \tilde{\mathbb{Q}}(\sigma)$ .*

(b) *If  $e \geq 3$ , then there are only finitely many elliptic curves  $E$  (up to  $\mathbb{C}$ -isomorphism) with CM over  $\tilde{\mathbb{Q}}(\sigma)$  such that  $\text{End}(E) \subseteq \tilde{\mathbb{Q}}(\sigma)$ .*

*Proof of (a).* — Let  $\Lambda$  be the set of all prime numbers  $l \equiv 3 \pmod 4$ . For each  $l$  we have  $[L_l : K_l] = h(l)$  and  $[L_l : \mathbb{Q}_l] = 2h(l)$ . In addition,  $E^{(l)}$  is defined over  $\mathbb{Q}(j(\mathfrak{a}_l))$  and  $\text{End}(E^{(l)}) = O_l$ . Hence, if  $\sigma \in \text{Gal}(L_l)$ , then  $E^{(l)}$  is defined over  $\tilde{\mathbb{Q}}(\sigma)$  and  $\text{End}(E^{(l)}) \subseteq \tilde{\mathbb{Q}}(\sigma)$ . By Proposition 1.1,

$$\sum_{l \in \Lambda} \frac{1}{[L_l : \mathbb{Q}]^e} = \frac{1}{2^e} \sum_{l \in \Lambda} \frac{1}{h(l)^e} \geq \frac{1}{2^2} \sum_{l \in \Lambda} \frac{1}{h(l)^2} = \infty.$$

By Lemma 2.3, the fields  $L_l, l \in \Lambda$ , are linearly disjoint. In particular  $j(\mathfrak{a}_l) \neq j(\mathfrak{a}_{l'})$ , so  $E^{(l)} \not\cong E^{(l')}$  if  $l \neq l'$ . It follows from Borel-Cantelli [FrJ], Lemma 18.5.3(b) that for almost all  $\sigma \in \text{Gal}(\mathbb{Q})^e$  there are infinitely many

primes  $l$  such that  $E^{(l)}$  is defined over  $\tilde{\mathbb{Q}}(\sigma)$  and  $\text{End}(E^{(l)}) \subseteq \tilde{\mathbb{Q}}(\sigma)$ , as desired.

*Proof of (b).* — Let  $\sigma \in \text{Gal}(\mathbb{Q})^e$ . If an elliptic curve  $E$  with CM is defined over  $\tilde{\mathbb{Q}}(\sigma)$  and  $\text{End}(E) \subseteq \tilde{\mathbb{Q}}(\sigma)$ , then there exist  $d \in D$  and a positive integer  $c$  such that  $L_{d,c} \subseteq \tilde{\mathbb{Q}}(\sigma)$ . By Lemma 2.4, for each  $d$  and  $c$  there are only finitely many  $E$ 's (up to a  $\mathbb{C}$ -isomorphism) which are defined together with their endomorphisms over  $L_{d,c}$ . Thus, if there are infinitely many elliptic curves with CM which are defined together with their endomorphisms over  $\tilde{\mathbb{Q}}(\sigma)$ , then  $\sigma$  belongs to infinitely many sets  $\text{Gal}(L_{d,c})^e$ . Since  $[L_{d,c} : \mathbb{Q}] = 2h(d,c)$ , Lemma 2.5 implies that  $\sum_{d \in D} \sum_{c=1}^{\infty} \frac{1}{[L_{d,c} : \mathbb{Q}]^e} \leq \sum_{d \in D} \sum_{e=1}^{\infty} \frac{1}{h(d,c)^3} < \infty$ . Hence, by Borel-Cantelli [Fr.J], Lemma 18.5.3.(a), the measure of those  $\sigma$ 's is 0.  $\square$

If an elliptic curve  $E$  with CM is defined over a field  $K$  and if  $\text{End}(E) \subseteq K$ , then, by Proposition 2.1, all conjugates of  $j_E$  are in  $K(j_E)$ . Therefore, for  $\sigma \in \text{Gal}(\mathbb{Q})^e$ , if we drop the condition that the endomorphisms of the elliptic curves are defined over  $\tilde{\mathbb{Q}}(\sigma)$ , then the probability that there are infinitely many elliptic curves with CM over  $\tilde{\mathbb{Q}}(\sigma)$  increases. This is reflected in the following result:

**THEOREM 2.8.** — *The following holds for almost all  $\sigma \in \text{Gal}(\mathbb{Q})^e$ :*

(a) *If  $e \leq 3$ , then there are infinitely many elliptic curves  $E$  (up to isomorphism) with CM over  $\tilde{\mathbb{Q}}(\sigma)$ .*

(b) *If  $e \geq 4$ , then there are only finitely many elliptic curves  $E$  (up to isomorphism) with CM over  $\tilde{\mathbb{Q}}(\sigma)$ .*

*Proof of (a).* — As in the proof of Theorem 2.7 let  $\Lambda$  be the set of primes  $l \equiv 3 \pmod{4}$ . Consider  $l \in \Lambda$  and let  $K_l, O_l, L_l, \mathfrak{a}_l, E^{(l)}$ , and  $h(l)$  be as above. Let  $\tau$  be a generator of  $\text{Gal}(L_l/\mathbb{Q}(j(\mathfrak{a}_l)))$ . If  $\alpha \in \text{Gal}(L_l/K_l)$ , then  $\tau^\alpha$  generates  $\text{Gal}(L_l/\mathbb{Q}(j(\mathfrak{a}_l))^\alpha)$  and  $(E^{(l)})^\alpha$  is an elliptic curve with CM which is defined over  $\mathbb{Q}(j(\mathfrak{a}_l))^\alpha$ . Thus, if  $\sigma \in \text{Gal}(\mathbb{Q})^e$  and  $\text{res}_{L_l} \sigma_i \in \langle \tau^\alpha \rangle^e$ , then  $(E^{(l)})^\alpha$  is defined over  $\tilde{\mathbb{Q}}(\sigma)$ .

**CLAIM.** —  $\#\{\tau^\alpha \mid \alpha \in \text{Gal}(L_l/K_l)\} = h(l)$ .

Indeed, embed  $L_l$  in  $\mathbb{C}$  and let  $\rho$  be the restriction of the complex conjugation to  $L_l$ . Since  $K_l$  is an imaginary quadratic field,  $\text{res}_{K_l} \rho \neq 1$ , so  $\rho^2 = 1$  and  $\rho \neq 1$ . Since  $l \equiv 3 \pmod{4}$ ,  $h(l)$  is odd [BoS], p. 346, Thm. 4. Thus,  $\rho \in \text{Gal}(L_l/\mathbb{Q}) \setminus \text{Gal}(L_l/K_l)$ . Now assume  $\rho^\alpha = \rho$  for some  $\alpha \in \text{Gal}(L_l/K_l)$ .

By Corollary 2.2(d),  $\rho\alpha\rho = \alpha^{-1}$ , hence  $1 = \rho^2 = \alpha^{-1}\rho\alpha\rho = \alpha^{-2}$ , which implies  $\alpha = 1$  (because  $h(l)$  is odd). It follows that the map  $\alpha \mapsto \rho^\alpha$  from  $\text{Gal}(L_l/K_l)$  into  $\text{Gal}(L_l/\mathbb{Q}) \setminus \text{Gal}(L_l/K_l)$  is injective. Since both sets have the same cardinality, the map is bijective. In particular,  $\tau$  is conjugate to  $\rho$  by an element of  $\text{Gal}(L_l/K_l)$ . Consequently,  $\#\{\tau^\alpha \mid \alpha \in \text{Gal}(L_l/K_l)\} = \#\{\rho^\alpha \mid \alpha \in \text{Gal}(L_l/K_l)\} = [L_l : K_l] = h(l)$ .

Let  $\bar{A}_l = \bigcup_{\alpha \in \text{Gal}(L_l/K_l)} \{1, \tau^\alpha\}^e$ . Each of the sets  $\{1, \tau^\alpha\}^e$  has  $2^e$  elements and the intersection of every two of them contains only one element (by the Claim). Thus,  $|\bar{A}_l| = h(l) \cdot 2^e - (h(l) - 1)$ . Let  $A_l = \{\sigma \in \text{Gal}(\tilde{\mathbb{Q}})^e \mid \text{res}_{L_l} \sigma \in \bar{A}_l\}$ . Then,  $\mu(A_l) = \frac{h(l) \cdot 2^e - (h(l) - 1)}{(2h(l))^e}$ . Since  $e \leq 3$ , Proposition 1.1 implies that

$$\sum_{l \in \Lambda} \mu(A_l) = \sum_{l \in \Lambda} \frac{h(l) \cdot 2^e - (h(l) - 1)}{(2h(l))^e} \geq \frac{2^e - 1}{2^e} \sum_{l \in \Lambda} \frac{1}{h(l)^2} = \infty.$$

By Lemma 2.3, the fields  $L_l$ ,  $l \in \Lambda$  are linearly disjoint. It follows from Borel-Cantelli that for almost all  $\sigma \in \text{Gal}(\mathbb{Q})^e$  there are infinitely many elliptic curves with CM which are defined over  $\tilde{\mathbb{Q}}(\sigma)$ .

*Proof of (b).* — Let  $d$  range over  $D$  and let  $c$  range over all positive integers. For each  $d$  and  $c$  let

$$A(d, c) = \bigcup_{\alpha \in \text{Gal}(L_{d,c}/K_d)} \text{Gal}(\mathbb{Q}(j(\mathbf{a}_{d,c})^\alpha))^e.$$

By Proposition 2.1(b),

$$\mu(A(d, c)) \leq [L_{d,c} : K_d] \left( \frac{1}{[\mathbb{Q}(j(\mathbf{a}_{d,c})) : \mathbb{Q}]} \right)^e = \frac{1}{h(d, c)^{e-1}}.$$

If for  $\sigma \in \text{Gal}(\mathbb{Q})^e$  there are infinitely many elliptic curves with CM which are defined over  $\tilde{\mathbb{Q}}(\sigma)$ , then  $\sigma$  belongs to infinitely many of the sets  $A(d, c)$  (as argued in the proof of Lemma 2.4). Since  $e \geq 4$ , we have by Lemma 2.5 that

$$\mu\left(\bigcup_{d,c} A(d, c)\right) \leq \sum_{d,c} \frac{1}{h(d, c)^{e-1}} \leq \sum_{d,c} \frac{1}{h(d, c)^3} < \infty.$$

We conclude from Borel-Cantelli that almost no  $\sigma \in \text{Gal}(\mathbb{Q})^e$  belongs to infinitely many sets  $A(d, c)$ . Thus, for almost all  $\sigma \in \text{Gal}(\mathbb{Q})^e$ , there are only finitely many elliptic curves with CM (up to a  $\mathbb{C}$ -isomorphism) which are defined over  $\tilde{\mathbb{Q}}(\sigma)$ . □

COROLLARY 2.9. — *The following holds for almost all  $\sigma \in \text{Gal}(\mathbb{Q})^e$ :*

(a) *If  $e \leq 2$ , then there are infinitely many elliptic curves  $E$  (up to  $\mathbb{C}$ -isomorphism) with CM over  $\tilde{\mathbb{Q}}[\sigma]$ .*

(b) *If  $e \geq 3$ , then there are only finitely many elliptic curves  $E$  (up to  $\mathbb{C}$ -isomorphism) with CM over  $\tilde{\mathbb{Q}}[\sigma]$ .*

*Proof.* — First suppose  $e \leq 2$ . By Theorem 2.7(a), for almost all  $\sigma \in \text{Gal}(\mathbb{Q})^e$  there are infinitely many elliptic curves  $E$  with CM over  $\tilde{\mathbb{Q}}(\sigma)$  such that  $\text{End}(E) \subseteq \tilde{\mathbb{Q}}(\sigma)$ . For all such  $\sigma$  and  $E$  let  $K_E$  be the quotient field of  $\text{End}(E)$ . Then  $K_E(j_E)$  is a Galois extension of  $\mathbb{Q}$  which is contained in  $\tilde{\mathbb{Q}}(\sigma)$ . Hence,  $K_E(j_E) \subseteq \tilde{\mathbb{Q}}[\sigma]$ . It follows that an isomorphic copy of  $E$  (over  $\mathbb{C}$ ) is defined over  $\tilde{\mathbb{Q}}[\sigma]$ .

Now suppose  $e \geq 3$ . For each  $\sigma \in \text{Gal}(\mathbb{Q})^e$  let  $\mathcal{E}(\sigma)$  be the set of all elliptic curves  $E$  (up to  $\mathbb{C}$ -isomorphism) which are defined over  $\tilde{\mathbb{Q}}(\sigma)$  such that  $\text{End}(E) \subseteq \tilde{\mathbb{Q}}(\sigma)$ . Let  $S$  be the set of all  $\sigma \in \text{Gal}(\mathbb{Q})^e$  such that  $\mathcal{E}(\sigma)$  is a finite set. By Theorem 2.7(b),  $\mu(S) = 1$ .

Consider  $\sigma \in S$  and let  $E$  be an elliptic curve with CM over  $\tilde{\mathbb{Q}}[\sigma]$ . Then  $j_E \in \tilde{\mathbb{Q}}[\sigma]$ . Hence, the Galois closure of  $\mathbb{Q}(j_E)/\mathbb{Q}$  is contained in  $\tilde{\mathbb{Q}}[\sigma]$ . By Corollary 2.2(a), the latter contains  $\text{End}(E)$ . Hence,  $E \in \mathcal{E}(\sigma)$ . Consequently, there are only finitely many elliptic curves (up to  $\mathbb{C}$ -isomorphism with CM over  $\tilde{\mathbb{Q}}[\sigma]$ ). □

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