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# HAMILTONIAN STABILITY AND SUBANALYTIC GEOMETRY

by Laurent NIEDERMAN

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ABSTRACT. — In the 70's, Nekhorochev proved that for an analytic nearly integrable Hamiltonian system, the action variables of the unperturbed Hamiltonian remain nearly constant over an exponentially long time with respect to the size of the perturbation, provided that the unperturbed Hamiltonian satisfies some generic transversality condition known as *steepness*. Using theorems of real subanalytic geometry, we derive a geometric criterion for steepness: a numerical function  $h$  which is real analytic around a compact set in  $\mathbb{R}^n$  is steep if and only if its restriction to any affine subspace of  $\mathbb{R}^n$  admits only isolated critical points. We also state a necessary condition for exponential stability, which is close to steepness.

Finally, we give methods to compute lower bounds for the steepness indices of an arbitrary steep function.

RÉSUMÉ. — La notion de raideur a été introduite pour étudier la stabilité effective des systèmes Hamiltoniens quasi-intégrables. À l'aide de théorèmes de géométrie sous-analytique, on donne une condition géométrique simple qui est équivalente à la raideur pour une fonction réelle analytique.

## 1. Introduction

### 1.1. Set-up

One of the main problem in Hamiltonian dynamic is the stability of motions in nearly-integrable systems (for example: the  $n$ -body planetary problem). The main tool of investigation is the construction of normal forms (see Giorgilli [7] for an introduction and a survey about these topics).

This yields two kinds of theorems:

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(i) Results of stability over infinite times provided by KAM theory which are valid for solutions with initial conditions in a Cantor set of large measure but no information is given on the other trajectories.

(ii) On the other hand, global results of stability over open sets which are valid only over exponentially long times with respect to the size of the perturbation.

Here, we focus our attention on the integrable Hamiltonians which satisfy the following property:

DEFINITION 1.1 (exponential stability). — Consider an open set  $\mathcal{P} \subset \mathbb{R}^n$ , an analytic integrable Hamiltonian  $h : \mathcal{P} \rightarrow \mathbb{R}$  and action-angle variables  $(I, \varphi) \in \mathcal{P} \times \mathbb{T}^n$  with  $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ .

For an arbitrary  $\rho > 0$ , let  $\mathcal{O}_\rho$  be the space of analytic functions over a complex neighborhood  $\mathcal{P}_\rho \subset \mathbb{C}^{2n}$  of size  $\rho$  around  $\mathcal{P} \times \mathbb{T}^n$  equipped with the supremum norm  $\|\cdot\|_\rho$  over  $\mathcal{P}_\rho$ .

We say that the Hamiltonian  $h$  is exponentially stable if there exists an open set  $\tilde{\mathcal{P}} \subset \mathcal{P}$  and positive constants  $\rho, C_1, C_2, a, b$  and  $\varepsilon_0$  which depend only on  $h$  and  $\tilde{\mathcal{P}}$  such that:

(i)  $h \in \mathcal{O}_\rho$ ;

(ii) for any function  $\mathcal{H} \in \mathcal{O}_\rho$  such that  $\|\mathcal{H} - h\|_\rho = \varepsilon < \varepsilon_0$ , an arbitrary solution  $(I(t), \varphi(t))$  of the Hamiltonian system associated to  $\mathcal{H}$  with an initial action  $I(t_0) \in \tilde{\mathcal{P}}$  is defined over a time  $\exp(C_2/\varepsilon^a)$  and satisfies:

$$\|I(t) - I(t_0)\| \leq C_1 \varepsilon^b \quad \text{for } |t - t_0| \leq \exp(C_2/\varepsilon^a);$$

$a$  and  $b$  are called stability exponents.

Remark 1.2. — Along the same lines, the previous definition can be extended to an integrable Hamiltonian in the Gevrey class.

In the seventies, Nekhorochev [18], [19] introduced the class of steep functions in order to get a sufficient condition for exponential stability.

DEFINITION 1.3 (steepness). — Consider an open set  $\mathcal{P}$  in  $\mathbb{R}^n$ . A real analytic function  $h : \mathcal{P} \rightarrow \mathbb{R}$  is said to be steep at a point  $I \in \mathcal{P}$  along an affine subspace  $\Lambda$  which contains  $I$  if there are constants  $C > 0, \delta > 0$  and  $p > 0$  such that along any continuous curve  $\gamma$  in  $\Lambda$  connecting  $I$  and a point at a distance  $d < \delta$ , the norm of the projection of the gradient  $\nabla h(x)$  onto the direction of  $\Lambda$  is greater than  $Cd^p$  at some point  $\gamma(t_*)$ ;  $(C, \delta)$  and  $p$  are respectively called the steepness coefficients and the steepness index.

Under the previous assumptions, the function  $h$  is said to be steep at the point  $I \in \mathcal{P}$  if  $I$  is not a critical point for  $h$  and if, for every

$k \in \{1, \dots, n-1\}$ , there exists positive constants  $C_k$ ,  $\delta_k$  and  $p_k$  such that  $h$  is steep at  $I$  along any affine subspace of dimension  $k$  containing  $I$  uniformly with respect to the coefficients  $(C_k, \delta_k)$  and the index  $p_k$ .

Finally, a real analytic function  $h$  is steep over a domain  $\Sigma \subset \mathcal{P} \subseteq \mathbb{R}^n$  with the steepness coefficients  $(C_1, \dots, C_{n-1}, \delta_1, \dots, \delta_{n-1})$  and the steepness indices  $(p_1, \dots, p_{n-1})$  if there are no critical points for  $h$  in  $\Sigma$  and  $h$  is steep at any point  $I \in \Sigma$  uniformly with respect to these coefficients and indices.

For instance, convex functions are steep with all the steepness indices equal to *one*. On the other hand,

$$h(x, y) = x^2 - y^2$$

is a typical non steep function but by adding a third order term (e.g.  $y^3$ ) we recover steepness. Moreover, this definition is minimal since a function can be steep along all subspaces of dimension lower than or equal to  $k < n-1$  and not steep for a subspace of dimension  $\ell$  greater than  $k$  (consider the function  $h(x, y, z) = (x^2 - y^2)^2 + z$  at  $(0, 0, 0)$  along all the lines and along the plane  $z = 0$ ).

Actually, these definitions look slightly less restrictive than the initial one given by Nekhorochev. But they retain the key property needed to derive estimates of stability. We will actually prove in Section 3 that they are equivalent to the original one.

In this setting, Nekhorochev proved the following:

**THEOREM 1.4** (see [19], [20]). — *If  $h$  is real analytic, non-degenerate ( $|\nabla^2 h(I)| \neq 0$  for any  $I \in \mathcal{P}$ ) and steep then  $h$  is exponentially stable.*

If  $h$  is quasi-convex, Lochak-Neishtadt [14], [15] and Pöschel [25] have improved the previous estimates with the exponents  $a = b = (2n)^{-1}$ . This result has been generalized to the steep case by Niederman [22] with the values  $a = b = (2np_1 \cdots p_{n-1})^{-1}$  ( $= 1/2n$  if  $h$  is convex).

Recently, Marco and Sauzin [17], following an idea of Herman showed that if  $h$  is quasi-convex and the total Hamiltonian  $\mathcal{H}$  is Gevrey of order  $\alpha$  (i.e.  $\mathcal{H}$  is infinitely differentiable and  $|\partial^k \mathcal{H}| \leq C^k (k!)^\alpha$ ) then the previous estimates are valid with  $a = b = (2n\alpha)^{-1}$ . Indeed these exponential bounds come from the Gevrey character of the normalizing transformations involved in the proof. Moreover, in the same setting ( $h$  quasi-convex and  $\mathcal{H}$  Gevrey of order  $\alpha > 1$ , i.e.  $\mathcal{H}$  non analytic), Marco and Sauzin [17] build examples of nearly integrable systems where an important instability of the actionvariables occurs for arbitrary small perturbations over times

of order  $\exp(1/\varepsilon^{a_*})$  with  $a_* = (2(n-2)\alpha)^{-1}$ . Hence, the times of stability in these estimates are nearly optimal (they are actually optimal in the neighborhood of resonances, see [17]) and the Gevrey character of the perturbation is a close to minimal *regularity* condition for exponential stability.

## 1.2. Statements of the results

Here, we study a minimal *non-degeneracy* condition on the unperturbed Hamiltonian needed to derive exponential stability results and give a *geometric* criterion equivalent to steepness.

Firstly, the absence of critical points and the nondegeneracy condition on the unperturbed Hamiltonian are not necessary to ensure exponential stability results (see [23]). The presence of critical points is overcome along the lines of Nekhorochev's reasonings (see [19, Section 1.9]) by assuming that our integrable Hamiltonian is steep not only along the proper affine subspaces but also on the whole space  $\mathbb{R}^n$  with the coefficients  $(C_n, \delta_n)$  and the index  $p_n$ , the stability exponents are modified accordingly with  $a = (1 + 2np_1 \cdots p_{n-1})^{-1}$  and  $b = a/p_n$ . Following Nekhorochev's terminology, we will say that a function steep along any affine subspaces including  $\mathbb{R}^n$  is *symmetrically steep* (or *S-steep*).

Using tools of real subanalytic geometry (see [3], [4], [16]): the curve selection lemma and the Lojasiewicz's inequalities for continuous subanalytic functions, we prove the following:

**THEOREM 1.5.** — *Let  $h$  be a numerical function which is real analytic on the open set  $\mathcal{P} \subset \mathbb{R}^n$ . Then  $h$  is S-steep (resp. steep) on any compact set  $\Sigma \subset \mathcal{P}$  if and only if its restriction  $h|_\Lambda$  to any affine subspace  $\Lambda \subseteq \mathbb{R}^n$  admits only isolated critical points (resp. if  $f$  has no critical points,  $\nabla h(x) \neq 0$  for all  $x \in \mathcal{P}$ , and if its restriction  $h|_\Lambda$  to any proper affine subspace  $\Lambda \subset \mathbb{R}^n$  admits only isolated critical points).*

Actually, a similar geometric criterion which ensures steepness has already been obtained by Ilyashenko [13]. He showed that a complex-valued holomorphic function on a domain of  $\mathbb{C}^n$  whose restriction to any (complex) affine subspace admits only  $\mathbb{C}$ -isolated critical points is steep on  $\mathbb{C}^n$  (with a generalization of our Definition 1.3 in the complex field). Hence, Ilyashenko considers a stronger property than steepness in the real case, moreover our estimates on the steepness indices are sharper (see Section 5).

We also prove the following necessary condition for exponential stability:

**THEOREM 1.6.** — *Consider an integrable Hamiltonian  $h$  which is real analytic on the open ball  $\mathcal{P} \subset \mathbb{R}^n$  (here it is the action space). If  $h$  is exponentially stable then its restriction to any proper affine subspace whose direction is generated by vectors with integer components admits only isolated critical points.*

This last statement is proved thanks to a result of Nekhorochev [20] about sufficient conditions on an integrable Hamiltonian which ensure the existence of arbitrary small perturbations giving rise to solutions with a drift of the action variables over linear times with respect to the size of the perturbation (“systems with fast drift”). The same problem has been studied in the realm of KAM theory by Michael Herman [11] who exhibited nearly integrable Hamiltonian systems with a dense Cantor set of invariant tori together with orbits which drift away to infinity.

We see that a gap subsists between the sufficient geometric condition for exponential stability given in Theorem 1.5 and the necessary condition derived in Theorem 1.6.

Actually, steepness is only a sufficient condition for exponential stability but the converse is not true. On one hand,

$$h(I_1, I_2) = I_1^2 - I_2^2$$

is not steep and the perturbed Hamiltonian

$$\mathcal{H}(I_1, I_2, \varphi_1, \varphi_2) = h(I_1, I_2) + \varepsilon f(\varphi_1, \varphi_2)$$

with  $f(\varphi_1, \varphi_2) = \sin(\varphi_1 + \varphi_2)$  admits the special solution

$$I(t) = (\varepsilon t, \varepsilon t), \quad \varphi(t) = (-\varepsilon t^2, \varepsilon t^2)$$

hence  $\|I(t) - I(0)\| = \sqrt{2}\varepsilon t$  and we have a drift over polynomial times (even linear times). On the other hand, Guzzo [8] have proved that for a Diophantine number  $\delta$ , the integrable Hamiltonian

$$h(I_1, I_2) = I_1^2 - (\delta I_2)^2$$

is not steep but is exponentially stable (its isotropic directions are the lines directed by  $(1, \pm\delta)$  and not generated by integer vectors). More generally, the Hamiltonian

$$h(I_1, I_2) = I_1^2 - \kappa I_2^2$$

for  $\kappa > 0$  is not steep but it is difficult to determine if it is exponentially stable (for instance when  $\kappa$  is the square of a Liouville number). This problem of stability under a relaxed assumption of steepness have been accurately studied in a recent paper ([23]).

In the context of KAM theory, the usual non-degeneracy condition is the invertibility of the gradient map associated to the unperturbed Hamiltonian. But the minimal condition needed for the existence of invariant tori in the perturbed system is the Rüssmann condition (see [5] or [26]): the image of the gradient map should not be included in a hyperplane. This last property is much easier to check for an arbitrary integrable Hamiltonian. Especially in the  $n$ -body problem, the unperturbed system given by uncoupled Kepler problems is strongly degenerate and Herman showed that the use of Rüssmann's condition is crucial to prove the existence of quasi-periodic planetary motions. A complete proof of this latter result has been given recently by Féjoz [6]. Over exponentially long times, our condition should be useful to prove stability results in the secular planetary problem (see also [21]). In the same way, Benettin, Fasso, Guzzo [2] and Guzzo, Morbidelli [9] have also studied stability properties of problems in celestial mechanics by reducing them to a perturbed steep, non-convex, integrable Hamiltonian system.

## 2. Subanalytic geometry and subanalytic functions

### 2.1. Essential results of subanalytic geometry

In order to get a self-consistent paper, we introduce the theorems which will be used in our proof. The definitions come from [3]:

DEFINITION 2.1. — *Let  $M$  be a real analytic manifold. If  $U$  is an open set in  $M$ , let  $\mathcal{O}(U)$  denote the ring of real analytic functions on  $U$ . A subset  $X \subset M$  is semianalytic if each  $a \in M$  has a neighborhood  $U$  such that*

$$X \cap U = \bigcup_{i=1}^p \bigcap_{j=1}^q X_{ij}$$

where each  $X_{ij}$  is either a set defined as  $\{f_{ij} = 0\}$  or  $\{f_{ij} > 0\}$  for some  $f_{ij} \in \mathcal{O}(U)$  (we say that  $X$  is described by  $\{f_{ij}\}$ ).

DEFINITION 2.2. — *A subset  $X$  of a real analytic manifold  $M$  is subanalytic if each point of  $M$  admits a neighborhood  $U$  such that  $X \cap U$  is a projection of a relatively compact semianalytic set  $A$  (i.e., there is a real analytic manifold  $N$  and a relatively compact semianalytic set  $A \subset M \times N$  such that  $X \cap U = \Pi(A)$  with the canonical projection  $\Pi$  from  $M \times N$  to  $N$ ).*

DEFINITION 2.3. — *Let  $X \subset M$  and let  $N$  be a real analytic manifold. A mapping  $f : X \rightarrow N$  is subanalytic if its graph is subanalytic in  $M \times N$*

THEOREM 2.4. — (i) *The intersection or the union of a finite collection of subanalytic sets is subanalytic.*

(ii) *The closure of a subanalytic set remains subanalytic.*

(iii) *The complement of a subanalytic set is subanalytic.*

(iv) *The image of a relatively compact subanalytic set by a subanalytic mapping remains subanalytic.*

(v) *For a numerical function  $f$  continuous subanalytic over a real analytic manifold  $M$ , the set  $X = \{x \in M \mid f(x) > 0\}$  is a subanalytic set (indeed,  $X$  is the projection of the intersection of the graph of  $f$  with  $\{(x, y) \in M \times \mathbb{R} \mid y > 0\}$ ).*

Two examples of subanalytic functions (see [3, p. 19]):

a) Let  $X$  be a subanalytic set of  $\mathbb{R}^n$ , the distance function

$$\delta_X(x) = \text{Min}_{x' \in \bar{X}} (\|x - x'\|)$$

is continuous subanalytic (while  $\delta_X$  is not analytic even if  $X$  is analytic). For instance, the norm is subanalytic.

b) Let  $M$  and  $N$  be real analytic manifolds and  $X$  (resp.  $K$ ) be subanalytic subsets of  $M$  (resp.  $N$ ), where  $K$  is compact. If  $f : X \times K \rightarrow \mathbb{R}$  is a continuous subanalytic function, then

$$m(x) = \text{Min}_{u \in K} (f(x, u)) \quad \text{and} \quad \mathcal{M}(x) = \text{Max}_{u \in K} (f(x, u))$$

are continuous subanalytic.

The continuity of  $m$  and  $\mathcal{M}$  are proved by abstract nonsense. Moreover, the set

$$\mathcal{A} = \{(x, u, v) \in M \times K \times K \text{ such that } f(x, u) > f(x, v)\}$$

is subanalytic since  $\mathcal{A} = \{(x, u, v) \in M \times K \times K \text{ such that } g(x, u, v) > 0\}$  with the continuous subanalytic function  $g(x, u, v) = f(x, u) - f(x, v)$ . Let  $\Pi$  be the projection  $\Pi : M \times K \times K \rightarrow M \times K$  defined by  $\Pi(x, u, v) = (x, u)$ , then the set  $\Pi(\mathcal{A})$  is subanalytic and also its complement  $\mathcal{B} = M \times K \setminus \Pi(\mathcal{A})$ . Finally,  $(x, u) \in \mathcal{B}$  implies that  $f(x, u) = m(x)$  and the graph of  $m$  is subanalytic as the image of  $\mathcal{B}$  by the mapping  $F(x, u) = (x, f(x, u))$  which admits subanalytic components.

In the same way,  $\mathcal{M}$  is a continuous subanalytic function over  $M$ . □

The first key ingredient for our proofs is the following:

THEOREM 2.5 (Curve selection Lemma, see [12], [16]). — Consider a subanalytic set  $X$  of a real analytic manifold  $M$  and let  $x$  be an accumulation point of  $X$ , then there exists a non-constant analytic arc

$$\gamma : ] - 1, +1[ \longrightarrow M$$

with  $\gamma(0) = x$  and  $\gamma(]0, 1[) \subset X$ .

The second point is that, under the assumptions of Theorem 1.5, the steepness indices can be seen as the Lojasiewicz's exponents of two functions according to the following:

DEFINITION 2.6 (Lojasiewicz's exponent [3], [4], [16]). — (i) Let  $M$  be a real analytic manifold,  $K$  a compact subset of  $M$  and  $f, g$  two vector-valued functions continuous over  $K$ , we set:

$$\mathcal{E}_K(f, g) = \{ \alpha \in \mathbb{R}_+ \text{ such that there exists a constant } C \\ \text{with } \|f(u)\|^\alpha \leq C\|g(u)\|, \forall u \in K \},$$

$$\alpha_K(f, g) = \text{Inf}\{\mathcal{E}_K(f, g)\} \text{ with } \text{Inf}\{\emptyset\} \text{ defined as } +\infty;$$

$\alpha_K(f, g)$  is called the Lojasiewicz's exponent of  $f$  with respect to  $g$  over  $K$ .

(ii) We will also consider the case where  $f$  is defined on a compact subset of  $\mathbb{R}^n$  and admits an isolated zero at  $x$ , then we set:

$$\alpha_x(f) = \text{Inf}\{ \alpha \in \mathbb{R}_+ \text{ such that } \exists C > 0, r > 0 \\ \text{with } \|u - x\|^\alpha \leq C\|f(u)\| \text{ if } \|u - x\| \leq r \}$$

hence  $\alpha_x(f) = \alpha_K(f, \text{dist}(\cdot, x))$ .

With these definitions, we have the important theorem:

THEOREM 2.7 (Lojasiewicz's inequalities, see [3], [4]). — Let  $f$  and  $g$  be two vector-valued continuous subanalytic functions over a compact set  $K$  in a real analytic manifold  $M$  such that their zero sets satisfy  $Z_g \subset Z_f$ , then  $\mathcal{E}_K(f, g)$  is non-empty,  $\alpha_K(f, g) \in \mathbb{Q}$  and  $\alpha_K(f, g) \in \mathcal{E}_K(f, g)$ .

Remark 2.8. — Specifically, if  $g$  is a vector-valued continuous subanalytic function over a compact set  $K \subset \mathbb{R}^n$ , if  $X = Z_g$  and  $f(x) = \text{dist}(x, Z_g)$  then for all  $x \in K$  we get

$$(\text{dist}(x, Z_g))^\alpha \leq C\|g(y)\|.$$

**2.2. Uniform lower estimates for a family of subanalytic germs**

Let  $0 < R$  and  $\bar{B}_R$  be the closed ball of radius  $R$  centered at the origin in  $\mathbb{R}^k$ .

With a real-analytic compact manifold  $K$ , we consider a family of subanalytic germs near the origin given by a positive function  $\mathcal{F} : \bar{B}_R \times K \rightarrow \mathbb{R}^+$  continuous subanalytic over  $\bar{B}_R \times K$  such that, for a given  $y \in K$ , either  $\mathcal{F}(0, y) > 0$  or the origin in  $\bar{B}_R$  is an *isolated* zero of  $\mathcal{F}_y = \mathcal{F}(\cdot, y)$ .

We look for lower estimates on the growth of the function  $\mathcal{F}_y$  which are uniform with respect to  $y$ . In most cases, one cannot expect the existence of constants  $C > 0$  and  $\alpha > 0$  such that  $\mathcal{F}_y(x) \geq C\|x\|^\alpha$  for all  $(x, y) \in \bar{B}_R \times K$ . For instance, the function  $\mathcal{F}(x, y) = |x(x - y)|$  defined over  $[-R, +R] \times [0, 1]$  satisfies our assumptions but the latter inequality is obviously false. On the other hand, one can prove that

$$\text{Max}_{0 \leq \xi \leq r} (\text{Min}_{|x|=\xi} (\mathcal{F}(x, y))) \geq (3 - 2\sqrt{2})r^2$$

for all  $(r, y) \in [0, R] \times [0, 1]$  and we want to generalize this kind of estimates for an arbitrary subanalytic function which satisfy our assumption of isolated zero for a fixed parameter.

PROPOSITION 2.9. — *With the previous notations:*

(i) *The function  $m(r, y) = \text{Min}_{\|x\|=r} (\mathcal{F}(x, y))$  for  $0 \leq r \leq R$  and  $y \in K$  is continuous subanalytic over  $[0, R] \times K$ .*

(ii) *The function  $\mathcal{M}(r, y) = \text{Max}_{t \in [0, r]} (m(t, y))$  is also continuous subanalytic.*

*Proof.* — Consider the continuous subanalytic function  $f(r, y, \theta) = \mathcal{F}(r\theta, y)$  defined over  $[0, R] \times K \times \mathcal{S}_n$  where  $\mathcal{S}_n$  is the unit sphere in  $\mathbb{R}^n$  then

$$m(r, y) = \text{Min}_{\theta \in \mathcal{S}_n} (f(r, y, \theta))$$

is continuous subanalytic over  $[0, R] \times K$  as it was proved in the previous section.

Now, we consider the function  $g(r, y, t) = m(tr, y)$  defined over  $[0, R] \times K \times [0, 1]$  and

$$\mathcal{M}(r, y) = \text{Max}_{0 \leq t \leq 1} (g(r, y, t)) = \text{Max}_{0 \leq u \leq r} (m(u, y))$$

is also continuous subanalytic. This proves the desired claims. □

Gathering all the results of the previous section, we obtain the following:

DEFINITION AND THEOREM 2.10. — *Let  $\mathcal{F}$  be a positive numerical function continuous subanalytic over  $\bar{B}_R \times K$ , we will say that  $\mathcal{F}$  is arc-steep if it satisfies one of the following equivalent properties:*

(i) *For any fixed value of the parameter  $y \in K$ , the function  $\mathcal{F}_y = \mathcal{F}(\cdot, y)$  admits an isolated zero at the origin in  $\bar{B}_R$  or  $\mathcal{F}_y(0) > 0$ .*

(ii) *There exists two constants  $C > 0$  and  $\alpha > 0$  such that the inequality  $\text{Max}_{0 \leq t \leq r}(\text{Min}_{\|x\|=t} \mathcal{F}(x, y)) \geq Cr^\alpha$  is valid for all  $(r, y) \in [0, R] \times K$ .*

(iii) *There exists constants  $C > 0$  and  $\alpha > 0$  such that along any continuous curve  $\gamma$  in  $\bar{B}_R$  connecting the origin to a point at a distance  $d \leq R$  and for any parameter  $y \in K$ , the modulus of the function  $\mathcal{F}_y = \mathcal{F}(\cdot, y)$  is greater than  $Cd^\alpha$  at some point  $\gamma(t_*)$*

*Proof.* — The latter proposition implies that

$$\mathcal{M}(r, y) = \text{Max}_{0 \leq t \leq r}(\text{Min}_{\|x\|=t} \mathcal{F}(x, y))$$

is continuous subanalytic over  $[0, R] \times K$ . Moreover, the zero set  $Z_{\mathcal{M}}$  is included in  $\{0\} \times K$ .

Indeed,  $\mathcal{M}(\cdot, y)$  is a nondecreasing function with respect to  $r$  and  $\mathcal{M}(0, y) > 0$  implies  $\mathcal{M}(r, y) > 0$  for all  $r \in [0, R]$ . Conversely,  $\mathcal{M}(0, y) = 0$  implies that  $\mathcal{F}(0, y) = 0$  and, under our assumptions, such a zero is isolated; hence  $\mathcal{F}(x, y) > 0$  for any  $x$  close to 0 and, by monotonicity, we can ensure that  $\mathcal{M}(r, y) > 0$  for any  $r \in ]0, R]$ .

Now, the function  $\mathcal{N}(r, y) = r$  is continuous subanalytic over  $[0, R] \times K$  and the zero sets of  $\mathcal{M}$  and  $\mathcal{N}$  satisfy  $Z_{\mathcal{M}} \subseteq \{0\} \times K = Z_{\mathcal{N}}$ . Finally, the existence of the Lojasiewicz's exponent (Theorem 2.7) on the compact real analytic manifold  $[0, R] \times K$  yields the constants  $C > 0$  and  $\alpha = \alpha_{[0, R] \times K}(\mathcal{M}, \mathcal{N}) > 0$  such that  $\mathcal{M}(r, y) \geq Cr^\alpha$  for all  $(r, y) \in [0, R] \times K$ , hence (i) implies (ii).

If (ii) is satisfied then, for any parameter  $y \in K$ , an arbitrary continuous curve  $\gamma$  in  $\bar{B}_R$  connecting the origin to a point at a distance  $d \leq R$  cross a sphere of radius  $0 \leq t \leq d$  centered at  $\gamma(0)$  where  $\mathcal{F}(x, y)$  is greater than  $Cd^\alpha$  at any point and the last property (iii) is ensured.

Finally, if the function  $\mathcal{F}_y = \mathcal{F}(\cdot, y)$  admits an accumulation of zeros at the origin for a certain value of the parameter  $y \in K$  then the zero set  $Z_y = \{x \in \bar{B}_R \text{ such that } \mathcal{F}_y(x) = 0\}$  is a real analytic set, hence a subanalytic set, and the curve selection lemma (Theorem 2.5) allows to find a non-constant continuous arc starting at the origin included in  $Z_y$ .

Consequently, the property (iii) cannot be satisfied and, by abstract nonsense, (iii) implies (i).  $\square$

### 3. Proof of the geometric criterion for steepness (Theorem 1.5)

Going back to the initial Definition 1.3 of steepness, we consider a numerical function  $h$  real analytic over an open set  $\mathcal{P} \subset \mathbb{R}^n$ .

For any  $k \in \{1, \dots, n\}$ , the  $k$ -dimensional Grassmannian in  $\mathbb{R}^n$  will be denoted by  $G_k(\mathbb{R}^n)$  and we introduce the numerical function

$$h_k(X, X_0, \Lambda_k) = \|\text{Proj}_{\Lambda_k}(\nabla h(X + X_0))\|$$

defined over the set

$$\{(X, X_0, \Lambda_k) \in \mathbb{R}^n \times \mathcal{P} \times G_k(\mathbb{R}^n) \text{ such that } X \in \Lambda_k \text{ and } X + X_0 \in \mathcal{P}\}.$$

With the previous notations, for  $k \in \{1, \dots, n\}$  a point  $(X, X_0, \Lambda_k) \in \mathbb{R}^n \times \mathcal{P} \times G_k(\mathbb{R}^n)$  is a zero of  $h_k$  if and only if  $X + X_0$  is a critical point for the restriction of  $h$  on the affine subspace  $X_0 + \Lambda_k$  as it can be seen by choosing an orthonormal basis in  $\Lambda_k$ .

Now, we consider a numerical function  $h$  whose restriction to any affine subspace admits only isolated critical points. This property is insured if and only if, for any  $(k, X_0, \Lambda_k) \in \{1, \dots, n\} \times \mathcal{P} \times G_k(\mathbb{R}^n)$ , the function  $h_k(\cdot, X_0, \Lambda_k)$  admits an isolated zero at the origin or  $h_k(0, X_0, \Lambda_k) \neq 0$ .

Let  $\Sigma$  be an arbitrary compact set in  $\mathcal{P}$ , since  $\mathcal{P}$  is open, the function  $h_k(X, X_0, \Lambda_k)$  is defined over the set

$$\{(X, X_0, \Lambda_k) \in \mathbb{R}^n \times \Sigma \times G_k(\mathbb{R}^n) \text{ such that } X \in \overline{B}_R^{(n)} \cap \Lambda_k\}$$

where  $\overline{B}_R^{(n)}$  is the closed ball of radius  $R = \text{dist}(\Sigma, \mathbb{R}^n \setminus \mathcal{P})$  in  $\mathbb{R}^n$  centered at the origin.

Following [28], we consider the Stiefel manifold  $V_k^0(\mathbb{R}^n)$  composed of all orthonormal families in  $\mathbb{R}^n$  of cardinality  $k$  and the  $k$ -dimensional Grassmannian  $G_k(\mathbb{R}^n)$ .

$G_k(\mathbb{R}^n)$  is isomorphic to the quotient  $V_k^0(\mathbb{R}^n)/(\mathcal{O}(k) \times \mathcal{O}(n - k))$  where the latter component is the stabilizer of a subspace of dimension  $k$  under the action of the orthonormal group  $\mathcal{O}(n)$ .

Hence, around any subspace in  $\mathbb{R}^n$ , there exists a local section of  $V_k^0(\mathbb{R}^n)$  over  $G_k(\mathbb{R}^n)$ . Moreover, since all the previous manifolds are real analytic, these sections can be real analytic. Actually, around any subspace  $\Lambda_k \in G_k(\mathbb{R}^n)$ , we can choose a compact neighborhood  $\overline{\Omega}_k \subset G_k(\mathbb{R}^n)$  and a real analytic map  $\mathcal{T} : \overline{\Omega}_k \rightarrow \mathbb{R}^{nk}$  such that  $\mathcal{T}(\Lambda) = (\mathcal{T}_1(\Lambda), \dots, \mathcal{T}_k(\Lambda))$  is an orthonormal basis of  $\Lambda$  for any  $\Lambda \in \overline{\Omega}_k$ .

Consequently, we can introduce the function

$$\mathcal{H}_k(x, X_0, \Lambda_k) = h_k \left( \sum_{j=1}^n x_j \mathcal{T}_j(\Lambda_k), X_0, \Lambda_k \right)$$

and

$$\mathcal{H}_k(x, X_0, \Lambda_k) = \left( \sum_{\ell=1}^k \left\langle \nabla h \left( X_0 + \sum_{j=1}^k x_j \mathcal{T}_j(\Lambda_k) \right) \middle| \mathcal{T}_\ell(\Lambda_k) \right\rangle^2 \right)^{\frac{1}{2}}$$

is continuous subanalytic over  $\bar{B}_R \times \Sigma \times \bar{\Omega}_k$  where  $\bar{B}_R$  is the closed ball of radius  $R$  in  $\mathbb{R}^k$  centered at the origin.

Finally, the union of these neighborhoods  $\bar{\Omega}_k$  around each point of  $G_k(\mathbb{R}^n)$  recover the whole  $k$ -dimensional Grassmannian and the compactness of  $G_k(\mathbb{R}^n)$  implies the existence of a finite collection of compact sets  $\bar{\Omega}_k^{(\ell)} \subset G_k(\mathbb{R}^n)$  for  $\ell \in \{1, \dots, m_k\}$  such that  $G_k(\mathbb{R}^n)$  is equal to the union  $\bigcup_{\ell=1}^{m_k} \bar{\Omega}_k^{(\ell)}$ . The corresponding sections of  $V_k^0(\mathbb{R}^n)$  over  $\bar{\Omega}_k^{(\ell)} \subset G_k(\mathbb{R}^n)$  will be denoted  $\mathcal{T}^{(\ell)} = (\mathcal{T}_1^{(\ell)}, \dots, \mathcal{T}_k^{(\ell)})$  for  $\ell \in \{1, \dots, m_k\}$ .

Now,  $h$  satisfies our assumption of isolated critical points if and only if, for any  $k \in \{1, \dots, n\}$  and any  $\ell \in \{1, \dots, m_k\}$ , the function

$$\mathcal{H}_k^{(\ell)}(x, X_0, \Lambda_k) = h_k \left( \sum_{j=1}^n x_j \mathcal{T}_j^{(\ell)}(\Lambda_k), X_0, \Lambda_k \right)$$

is arc-steep over  $\bar{B}_R \times \Sigma \times \bar{\Omega}_k^{(\ell)}$ .

Indeed,  $\mathcal{H}_k^{(\ell)}(0, X_0, \Lambda_k) \neq 0$  if and only if the restriction of  $h$  on the affine subspace  $X_0 + \Lambda_k$  admits a critical point at  $X_0$ . This is an isolated critical point according to our assumption and  $\mathcal{H}_k^{(\ell)}(0, X_0, \Lambda_k) \neq 0$  for any  $x$  close enough to the origin.

Conversely, if  $\mathcal{H}_k^{(\ell)}$  is arc-steep for any  $k \in \{1, \dots, n\}$  and any  $\ell \in \{1, \dots, m_k\}$ , then we consider a point  $X_0 \in \Sigma$  which is a critical point for the restriction of  $h$  to the affine subspace of  $X_0 + \Lambda_k$  where  $\Lambda_k \in G_k(\mathbb{R}^n)$ . There exists  $\ell \in \{1, \dots, m_k\}$  such that  $\Lambda_k \in \bar{\Omega}_k^{(\ell)}$  and  $\mathcal{H}_k^{(\ell)}(0, X_0, \Lambda_k) = 0$ , hence  $\mathcal{H}_k^{(\ell)}(0, X_0, \Lambda_k) \neq 0$  for any  $x$  close enough to the origin and  $X_0$  is an isolated critical point for the restriction of  $h$  on the affine subspace  $X_0 + \Lambda_k$ .

With the Theorem 2.10, the function  $h$  satisfies our assumption of isolated critical point if and only if, for any  $k \in \{1, \dots, n\}$  and any  $\ell \in \{1, \dots, m_k\}$ , there exists constants  $C_k^{(\ell)}$  and  $\alpha_k^{(\ell)}$  such that, along any continuous curve  $\gamma(t)$  in  $\bar{B}_R$  connecting the origin to a point at a distance  $d \leq R = \text{dist}(\Sigma, \mathbb{R}^n \setminus \mathcal{P})$  and any  $(X_0, \Lambda_k) \in \Sigma \times \bar{\Omega}_k^{(\ell)}$ , one can find  $t_*$  such

that

$$\mathcal{H}_k^{(\ell)}(\gamma(t_*), X_0, \Lambda_k) \geq C_k^{(\ell)} d^{\alpha_k^{(\ell)}}.$$

With  $C_k = \text{Min}_\ell(C_k^{(\ell)})$  and  $p_k = \text{Max}_\ell(\alpha_k^{(\ell)})$ , for any curve  $\Gamma$  in  $\overline{B}_R^{(n)} \cap \Lambda_k$  where  $\Gamma(0) = 0$  and  $\|\Gamma(1)\| = d \leq R$ , there exists  $t_*$  such that

$$\|\text{Proj}_{\Lambda_k}(\nabla h(X_0 + \Gamma(t_*)))\|$$

is bigger than  $C_k d^{p_k}$ . Since  $X_0 + \Lambda_k$  for all  $(X_0, \Lambda_k) \in \Sigma \times \overline{\Omega}_k^{(\ell)}$  describes all the affine subspaces which intersect  $\Sigma$ , we see that  $h$  is steep on  $\Sigma$  with the steepness coefficients  $(C_1, \dots, C_n)$  and the steepness indices  $(p_1, \dots, p_n)$  for curves of length shorter than  $R = \text{dist}(\Sigma, \mathbb{R}^n \setminus \mathcal{P})$ , which is exactly the required equivalence. □

Actually, we have proved the following:

**THEOREM 3.1.** — *Let  $f$  be a numerical function real analytic over an open set  $\mathcal{P} \subset \mathbb{R}^n$ . For a compact set  $\Sigma \subset \mathcal{P}$  and  $k \in \{1, \dots, n\}$ , we introduce the function  $\mathcal{M}_k^{(\Sigma)}$  which measure the steepness of  $f$  along the set  $\text{Graff}_\Sigma(k, n)$  of all affine subspaces of dimension  $k$  which intersect  $\Sigma$ . For any  $\tilde{\Lambda}_k \in \text{Graff}_\Sigma(k, n)$  of direction  $\Lambda_k$  and any  $d \leq R = \text{dist}(\Sigma, \mathbb{R}^n \setminus \mathcal{P})$ , we define*

$$\mathcal{M}_k^{(\Sigma)}(d, X_0, \tilde{\Lambda}_k) = \text{Max}_{0 \leq \xi \leq d} (\text{Min}_{X \in \mathcal{S}_\xi(X_0) \cap \tilde{\Lambda}_k} \|\text{Proj}_{\Lambda_k}(\nabla h(X_0 + X))\|)$$

where  $\mathcal{S}_\xi(X_0)$  is the sphere of radius  $\xi$  centered at  $X_0$  in  $\mathbb{R}^n$ . Equivalently, if  $\Lambda_k \in \overline{\Omega}_k^{(\ell)}$  for some  $\ell \in \{1, \dots, m_k\}$ , we have

$$\mathcal{M}_k^{(\Sigma)}(d, X_0, \tilde{\Lambda}_k) = \text{Max}_{0 \leq \xi \leq d} (\text{Min}_{\|x\|=\xi} (\mathcal{H}_k^{(\ell)}(x, X_0, \Lambda_k))).$$

The function  $h$  is steep over  $\Sigma$  according to our Definition 1.3 if and only if, for any  $k \in \{1, \dots, n\}$ , there exists constants  $C_k > 0$  and  $p_k > 0$  such that

$$\mathcal{M}_k^{(\Sigma)}(d, X_0, \tilde{\Lambda}_k) \geq C_k d^{p_k} \text{ for all } (d, X_0, \tilde{\Lambda}_k) \in [0, R] \times \Sigma \times \text{Graff}_\Sigma(k, n).$$

This latter inequality corresponds to the original definition of steepness in Nekhorochev’s work [19].

### 4. Proof of our necessary condition for exponential stability (Theorem 1.6)

We prove this theorem by abstract nonsense; Nekhorochev [20, Section 4], considered the following class of functions:

DEFINITION 4.1. — Let  $\mathcal{F}$  be the class of functions  $f : \mathcal{P} \rightarrow \mathbb{R}$  real analytic over a domain  $\mathcal{P} \subseteq \mathbb{R}^n$  such that there exist an affine subspace  $\tilde{\Lambda}$  whose direction  $\Lambda$  is generated by vectors with integer components and a regular curve  $\gamma_f : [0, 1] \rightarrow \tilde{\Lambda} \cap \mathcal{P}$  where

$$\text{Proj}_{\Lambda}(\nabla f(\gamma_f(t))) = 0 \quad \text{for all } t \in [0, 1].$$

In this setting, we have:

THEOREM 4.2 (Systems with fast drift [20]). — For any Hamiltonian  $h \in \mathcal{F}$  (defined above) and any  $\varepsilon > 0$ , there exists a nearly-integrable Hamiltonian system deriving from  $\mathcal{H}(I, \varphi) = h(I) + \varepsilon f(I, \varphi)$  in the action-angle variables  $(I, \varphi) \in \mathcal{P} \times \mathbb{T}^n$  which admits a solution  $(I(t), \varphi(t))$  defined over  $[0, 1/\varepsilon]$  such that  $I(0) = \gamma_h(0)$  and  $I(1/\varepsilon) = \gamma_h(1)$ .

Hence, we have a drift along a curve with a length independent of  $\varepsilon$  over a linear time  $1/\varepsilon$ .

Remark 4.3. — This is the strongest possible drift with a perturbation of magnitude  $\varepsilon$ .

Proof of Theorem 1.6. — Here, we consider an integrable Hamiltonian  $h$  with an affine subspace  $\tilde{\Lambda}$  whose direction  $\Lambda$  is generated by vectors with integer components such that the zero set  $Z_g \cap \tilde{\Lambda}$  of the real analytic function  $g = \text{Proj}_{\Lambda}(\nabla h(x))$  admits an accumulation point. As in the proof of the Theorem 2.10, this zero set is real analytic and we can find a non trivial analytic arc  $\gamma_h$  in  $\tilde{\Lambda}$  composed of critical points. Hence,  $\gamma_h$  satisfies Nekhorochev's conditions for systems with fast drift (Definition 4.1). Consequently, there exists an arbitrary small analytic perturbation of  $h$  where the action variables drift over linear times along  $\gamma_h$ .  $\square$

## 5. Explicit computation of lower bounds for the steepness indices

In the study of the dynamic of nearly-integrable Hamiltonian systems, the most important result given by Nekhorochev's estimates is the order of magnitude of the stability time which depend almost only of the steepness indices through the expression of the exponents  $a = (1 + 2np_1 \cdots p_{n-1})^{-1}$  and  $b = a/p_n$  (see [23]).

We still consider a numerical function  $h$  real analytic over an open set  $\mathcal{P} \subset \mathbb{R}^n$  whose restriction to any affine subspace admits only isolated critical points. Along the lines of the previous reasonings,  $h$  is steep on any compact set  $\Sigma \subset \mathcal{P}$  with the steepness indices given by  $p_k = \text{Max}_{\ell \in \{0, \dots, m_k\}} (\alpha_k^{(\ell)})$

where  $\alpha_k^{(\ell)} = \alpha_{\mathcal{K}_{k,\ell}^{(\Sigma)}}(\mathcal{M}_k^{(\Sigma)}, \mathcal{N}_k)$  is the Lojasiewicz's exponent of the function  $\mathcal{M}_k^{(\Sigma)}$  defined in the Theorem 3.1 with respect to  $\mathcal{N}_k(r, X_0, \Lambda_k) = r$  defined over the compact

$$\mathcal{K}_{k,\ell}^{(\Sigma)} = [0, R] \times \Sigma \times \overline{\Omega}_k^{(\ell)} \subset \mathbb{R}^+ \times \mathcal{P} \times G_k(\mathbb{R}^n)$$

for  $R = \text{dist}(\Sigma, \mathbb{R}^n \setminus \mathcal{P})$ .

The explicit computation of the Lojasiewicz's exponent in a general setting is a difficult question (see [27]). Indeed, we look for the sharpest lower estimates on the growth of a real analytic function with respect to the distance to the zero set  $Z_f = \{x \text{ such that } f(x) = 0\}$ .

In Ilyashenko's study [13], the problem of steepness of holomorphic functions is tackled with the following property. Consider an holomorphic function on an open set in  $\mathbb{C}^n$  which admits a finite number of zeros, then  $\|f(z)\| \geq C \text{dist}(z, Z_f)^p$  where  $p$  is the number of zeros counted with their multiplicity (or Milnor number, see [1]) which is the number of zeros obtained by bifurcation ( $\#\{x \text{ such that } \|f(x)\| = \varepsilon\}$  for  $\varepsilon$  close to 0). For instance, the function  $f(z_1, z_2) = (z_1^2, z_2^3)$  admits only one zero over  $\mathbb{C}^2$  with the multiplicity  $p_f(0, 0) = 6$  while the Lojasiewicz's exponent is  $\alpha_f(0, 0) = 3$ , hence the previous lemma gives only  $\|f(z_1, z_2)\| \geq C\|(z_1, z_2)\|^6$  in the vicinity of  $(0, 0)$  instead of  $\|f(z_1, z_2)\| \geq C\|(z_1, z_2)\|^3$  which is the best lower bound. The same phenomena occurs in a general setting (see [24]), hence the use of multiplicity instead of Lojasiewicz's exponents yields only rough upper estimates on the steepness indices.

In the sequel, we give a method to compute sharp lower estimates on the steepness indices based on the following

**THEOREM 5.1.** — *Consider an integrable Hamiltonian  $h$  which satisfies the assumptions of our Main Theorem 1.5. For  $\Lambda$  an affine subspace in  $\mathbb{R}^n$ , we consider the zero set*

$$Z_\Lambda = \{x \in \Lambda \text{ such that } \text{Proj}_{\vec{\Lambda}}(\nabla h(x)) = 0\}$$

where  $\vec{\Lambda}$  is the direction of  $\Lambda$ , hence  $Z_\Lambda$  is composed of isolated points.

Then, the steepness index of order  $k$  satisfies

$$p_k \geq \text{Sup}_{\Lambda \in \text{Graff}_\Sigma(k,n)} (\text{Sup}_{x \in Z_\Lambda} (\alpha_x(f_\Lambda)))$$

where  $\alpha_x(f_\Lambda)$  is the Lojasiewicz's exponent at the isolated zero  $x \in Z_\Lambda$  of the restriction  $f_\Lambda$  on the affine subspace  $\Lambda$  of the function  $f(u) = \|\text{Proj}_{\vec{\Lambda}}(\nabla h(u))\|$ .

The point of this refinement lies in the fact that our estimates on the steepness indices can be obtained as the maximum of a family of Lojasiewicz's exponents at an isolated zero of a real-analytic function and the latter quantity can be computed along the lines of the following theorem of Gwozdziwicz [10].

Let  $g$  be a numerical function real analytic on an open set  $\mathcal{P}$  in  $\mathbb{R}^k$ , with  $(\vec{e}_1, \dots, \vec{e}_k)$  the canonical basis of  $\mathbb{R}^k$ , we consider the set (called polar curve):

$$P^{(j)} = \nabla g^{-1}(\mathbb{R}\vec{e}_j) = \{u \in \mathcal{V} \text{ such that } \nabla g(u) = \lambda \vec{e}_j \text{ for } \lambda \in \mathbb{R}\}.$$

For  $x \in \mathcal{P}$  an isolated zero of  $g$  and  $j \in \{1, \dots, k\}$ , we define the partial exponent:

$$\alpha_x^{(j)}(g) = \text{Inf} \{ \alpha \in \mathbb{R}_+ \text{ such that } \exists C > 0, R > 0 \\ \text{with } \|u - x\|^\alpha \leq C|g(u)| \text{ if } \|u - x\| \leq R \text{ and } u \in P^{(j)} \}$$

then Gwozdziwicz's Theorem asserts that the Lojasiewicz's exponent at  $x$  is given by  $\alpha_x(g) = \text{Max}_{1 \leq j \leq k}(\alpha_x^{(j)}(g))$ .

Summarizing, in order to compute the Lojasiewicz's exponent of a real analytic function at an isolated zero, one has only to estimate the growth of the function along one of the polar curves which is usually an analytic set of dimension one.

Here, for an affine subspace  $\Lambda \subset \mathbb{R}^n$ , we apply the previous result to the real analytic function  $g_\Lambda(u) = (f_\Lambda(u))^2 = \|\text{Proj}_{\bar{\Lambda}}(\nabla h(u))\|^2$  and  $\alpha_x(g_\Lambda) = 2\alpha_x(f_\Lambda)$  for any isolated zero  $x$  of  $f_\Lambda$ , hence

$$p_k \geq \frac{1}{2} \text{Sup}_{\Lambda \in \text{Graff}_\Sigma(k,n)} (\text{Sup}_{x \in Z_\Lambda} (\alpha_x(g_\Lambda))).$$

*Proof of Theorem 5.1.* — We first need the following:

LEMMA 5.2 (Lojasiewicz's exponent for a family of subanalytic functions). — *Consider a numerical subanalytic function  $f$  defined on a product set  $M = [0, R] \times K$  for some  $R > 0$  and a real analytic compact manifold  $K$  with the zero set  $Z_f \subset \{0\} \times K$ , consequently if  $g(x, r) = r$  over  $M$  then  $Z_f \subset Z_g$ .*

We denote by  $f_x$  and  $g_x$  the restrictions of  $f$  and  $g$  on the fibers

$$M_x = [0, R] \times \{x\}$$

for an arbitrary  $x \in K$  and  $\alpha = \alpha_M(f, g)$ ,  $\alpha_x = \alpha_{M_x}(f_x, g_x)$  are the Lojasiewicz's exponents of these functions on their domain of definition, then  $\alpha \geq \text{Sup}_{x \in K}(\alpha_x)$ .

*Remark 5.3.* — The equality  $\alpha = \text{Sup}_{x \in K}(\alpha_x)$  seems natural and should be true in many cases, this question is worthwhile considering.

*Proof.* — Since each fiber  $M_x = [0, R] \times \{x\}$  is included in  $M$ , by definition we have  $\alpha \geq \alpha_x$  for any  $x \in K$  and  $\alpha \geq \tilde{\alpha} = \text{Sup}_{x \in K}(\alpha_x)$ .  $\square$

Consider a compact set  $\Sigma \subset \mathcal{P}$ , we define the function  $\mathcal{M}_k^{(\Sigma)}$  as in the Theorem 3.1 and the previous lemma allows to compute a lower bound on the Lojasiewicz’s exponent  $\alpha_k^{(\ell)} = \alpha_{\mathcal{K}_{k,\ell}^{(\Sigma)}}(\mathcal{M}_k^{(\Sigma)}, \mathcal{N}_k)$  over the compact  $\mathcal{K}_{k,\ell}^{(\Sigma)} = [0, R] \times \Sigma \times \overline{\Omega}_k^{(\ell)}$ . Hence

$$\alpha_k^{(\ell)} \geq \text{Sup}_{(X_0, \Lambda_k) \in \Sigma \times \overline{\Omega}_k^{(\ell)}}(\alpha_{k, (X_0, \Lambda_k)}^{(\ell)})$$

where  $\alpha_{k, (X_0, \Lambda_k)}^{(\ell)}$  is the Lojasiewicz’s exponent of the function  $\mathcal{M}_k^{(\Sigma)}(\cdot, X_0, \Lambda_k)$  with respect to  $\mathcal{N}_k(\cdot, X_0, \Lambda_k)$  defined over the fiber  $[0, R] \times \{(X_0, \Lambda_k)\}$ .

Following the reasonings and the notations of Section 3, under our assumption of isolated zero, the function  $\mathcal{H}_k^{(\ell)}$  is arc-steep and  $\mathcal{M}_k^{(\Sigma)}(\cdot, X_0, \Lambda_k)$  admits at most one zero located at the origin.

If  $\mathcal{M}_k^{(\Sigma)}(0, X_0, \Lambda_k) > 0$  then  $\alpha_{k, (X_0, \Lambda_k)}^{(\ell)} \leq 1$  since  $\mathcal{M}_k^{(\Sigma)}(r, X_0, \Lambda_k) > 0$  for  $0 \leq r \leq R$ , otherwise  $\mathcal{M}_k^{(\Sigma)}(0, X_0, \Lambda_k) = 0$  if and only if  $\mathcal{H}_k^{(\ell)}(0, X_0, \Lambda_k) = 0$  which implies that  $\mathcal{H}_k^{(\ell)}(x, X_0, \Lambda_k) \neq 0$  for  $x$  close enough to 0.

In this latter case,  $\text{m}_k^{(\Sigma)}(r, X_0, \Lambda_k) = \text{Min}_{\|x\|=r}(\mathcal{H}_k^{(\ell)}(x, X_0, \Lambda_k))$  is a non-decreasing function with respect to  $r$  and  $\mathcal{M}_k^{(\Sigma)}(r, X_0, \Lambda_k) = \text{m}_k^{(\Sigma)}(r, X_0, \Lambda_k)$  for  $r$  small enough and we easily see that the Lojasiewicz’s exponents  $\alpha_0(\text{m}_k^{(\Sigma)}(\cdot, X_0, \Lambda_k)) = \alpha_x(f_\Lambda) \geq 1$  if  $\Lambda = X_0 + \Lambda_k$  and  $x = X_0 \in Z_\Lambda$ , hence

$$p_k \geq \text{Sup}_{\ell \in \{1, \dots, m_k\}}(\text{Sup}_{\Lambda_k \in \overline{\Omega}_k^{(\ell)}}(\text{Sup}_{X_0 \in Z_\Lambda}(\alpha_0(\text{m}_k^{(\Sigma)}(r, X_0, \Lambda_k))))$$

and

$$\begin{aligned} \text{Graff}_\Sigma(k, n) &= \{X_0 + \Lambda_k \text{ for } (X_0, \Lambda_k) \in \Sigma \times \text{G}_k(\mathbb{R}^n)\} \\ &= \bigcup_{\ell=1}^{m_k} \{X_0 + \Lambda_k \text{ for } (X_0, \Lambda_k) \in \Sigma \times \overline{\Omega}_k^{(\ell)}\} \end{aligned}$$

implies our claim.  $\square$

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