ANNALES

DE

L'INSTITUT FOURIER

Barbara DRINOVEC DRNOVŠEK

On proper discs in complex manifolds

Tome 57, n° 5 (2007), p. 1521-1535.

<http://aif.cedram.org/item?id=AIF_2007__57_5_1521_0>

Ann. Inst. Fourier, Grenoble
57, 5 (2007) 1521-1535

ON PROPER DISCS IN COMPLEX MANIFOLDS

by Barbara DRINOVEC DRNOVŠEK

ABSTRACT. — Let X be a complex manifold of dimension at least 2 which has an exhaustion function whose Levi form has at each point at least 2 strictly positive eigenvalues. We construct proper holomorphic discs in X through any given point and in any given direction.

RÉSUMÉ. — Soit X une variété analytique complexe de dimension au moins 2 qui possède une fonction d'exhaustion telle que sa forme de Levi possède au moins 2 valeurs propres strictement positives en tout point de X. On construit les disques holomorphes dans X par n'importe quel point donné et dans n'importe quelle direction donnée.

1. Introduction and the results

Denote by $\triangle$ the open unit disc in $\mathbb{C}$. Our main result is the following

THEOREM 1.1. — *Let X be a complex manifold of dimension $n \geqslant 2$ which has an exhaustion function whose Levi form has at each point at least 2 positive eigenvalues. Given $p \in X$ and a vector v tangent to X at p, there is a proper holomorphic map $f\colon \triangle \to X$ such that $f(0) = p$ and $f'(0) = \lambda v$ for some $\lambda > 0$.*

Note that in the case $n = 2$ the manifold X having the property in the theorem is Stein. It is known that if X is a Stein manifold of dimension at least 2, then for each point p in X and for each vector v tangent to X at p, there is a proper holomorphic map $f\colon \triangle \to X$ such that $f(0) = p$ and $f'(0) = \lambda v$ for some $\lambda > 0$ [6, 4]. Therefore, our result is new in the case $n > 2$.

In the theory of q-convex manifolds the manifolds with the above property are called $(n-1)$-complete manifolds. For general theory of q-convex

Keywords: Complex manifolds, proper holomorphic discs.
Math. classification: 32H35, 32C25.

manifolds we refer to [7, 9]. The survey [1] gives a list of examples (with references) and some open questions concerning q-convexity.

The conclusion of the theorem is not valid in general for complex manifolds of dimension at least 2 which have an exhaustion function whose Levi form has at each point at least 1 positive eigenvalue. Indeed, in [4] the authors constructed for every $n \geqslant 2$ a smoothly bounded domain $\Omega \subset\subset \mathbb{C}^n$ and a point $p \in \Omega$ such that there is no proper holomorphic map $f\colon \triangle \to \Omega$ with $p \in f(\triangle)$. Since every noncompact complex manifold of dimension $n \geqslant 2$ has an exhaustion function whose Levi form has at each point at least 1 positive eigenvalue [8] we get the desired example for which the conclusion of Theorem 1.1 fails.

A. Dor [2] proved that there exists a bounded domain Ω in $\mathbb{C}^n$ $(n \geqslant 2)$ such that there is no proper holomorphic mapping from the unit disc to Ω.

We prove the following result.

Theorem 1.2. — *Let X be a complex manifold of dimension $n \geqslant 2$ and assume that $\rho\colon X \to \mathbb{R}$ is an exhaustion function whose Levi form has at least 2 positive eigenvalues at each point of the set $\{\rho > M\}$ for some $M \in \mathbb{R}$. Let d be a complete metric on X which induces the manifold topology. Given $\varepsilon > 0$, $0 < r < 1$, and a continuous map $f\colon \overline{\triangle} \to X$ such that $\rho(f(\zeta)) > M$ $(\zeta \in b\triangle)$ there is a proper holomorphic map $g\colon \triangle \to X$ such that*

(i) $d(g(\zeta), f(\zeta)) < \varepsilon$ *for* $|\zeta| < r$,
(ii) $g(0) = f(0)$,
(iii) $g'(0) = \lambda f'(0)$ *for some* $\lambda > 0$.

Since there are small holomorphic discs through any given point in any given direction on each complex manifold, Theorem 1.2 easily implies Theorem 1.1.

If $\dim X = 2$ then the function ρ is strictly plurisubharmonic in the set $\{\rho > M\}$ and in this case theorem was proved in [6],[5, Theorem 1.1]. So we only need to treat the case $\dim X \geqslant 3$.

In the proof of Theorem 1.2 we shall push the boundary of a given analytic disc outside a given sublevel set of ρ. Since our manifold does not necessarily lie in Euclidean space, we are not able to do this by adding a suitable polynomial map as it was done in [6]. Instead we use convex bumps. At the first step we push the boundary outside the given sublevel set union one bump. At the next step the boundary lies outside the sublevel set union two bumps. These bumps are constructed in such a way that they fill the space between two level sets of ρ. In a finite number of steps the boundary of the disc lies outside the bigger sublevel set.

In section 2 we prove that we can push the boundary of a given holomorphic disc along a continuous family of small holomorphic discs attached to the boundary; these small discs are not constant only in a fixed coordinate neighborhood. To do this we first solve approximately a Riemann-Hilbert boundary value problem [6, Lemma 5.1], [4] to get a holomorphic map from the part of the open unit disc which was initially mapped into the fixed coordinate neighborhood and then we obtain the new holomorphic disc as a solution of a nonlinear Cousin problem (due to J.-P. Rosay [12]).

In section 3 we construct convex bumps which provide the continuous family of holomorphic discs, and for this family we use the result from section 2. We prove Theorem 1.2 in section 4.

2. The main lemma

We will need the following elementary lemma

LEMMA 2.1. — *Let M be a metric space with distance function d. Let U_1, U_2 be open sets in $\mathbb{C}$ such that $\overline{\triangle} \subset U_1 \cup U_2$ and assume that $f_k\colon \overline{U_k \cap \triangle} \to M$ $(k = 1, 2)$ are continuous maps. Further, assume that for some $\varepsilon > 0$ there is a continuous map $g_0\colon \triangle \to M$ such that*

$$d(g_0(\zeta), f_k(\zeta)) < \varepsilon \ (\zeta \in U_k \cap \triangle, k = 1, 2).$$

Then there is R, $0 < R < 1$, so close to 1 that

$$d(g_0(R\zeta), f_k(\zeta)) < 5\varepsilon \ (\zeta \in U_k \cap \overline{\triangle}, k = 1, 2).$$

Proof. — It is easy to see that $(U_1 \setminus U_2) \cap \triangle \subset\subset U_1$ and $(U_2 \setminus U_1) \cap \triangle \subset\subset U_2$. Therefore there is R, $0 < R < 1$, so close to 1 that

$$R((U_1 \setminus U_2) \cap \triangle) \subset U_1 \text{ and } R((U_2 \setminus U_1) \cap \triangle) \subset U_2. \tag{2.1}$$

Since f_k is uniformly continuous on $\overline{U_k \cap \triangle}$ $(k = 1, 2)$ by increasing R, $R < 1$, we obtain

$$\text{if } \zeta, \eta \in \overline{U_k \cap \triangle},\ |\zeta - \eta| \leqslant 1 - R \text{ then } d(f_k(\zeta), f_k(\eta)) < \varepsilon \ (k = 1, 2). \tag{2.2}$$

For $\zeta \in (U_1 \setminus U_2) \cap \triangle$ using (2.1) and (2.2) we get

$$d(g_0(R\zeta), f_1(\zeta)) \leqslant d(g_0(R\zeta), f_1(R\zeta)) + d(f_1(R\zeta), f_1(\zeta)) < 2\varepsilon,$$

and similarly for $\zeta \in (U_2 \setminus U_1) \cap \triangle$. Take $\zeta \in U_1 \cap U_2 \cap \triangle$. If $R\zeta \in U_1$ then as above we get $d(g_0(R\zeta), f_1(\zeta)) < 2\varepsilon$. If $R\zeta \notin U_1$ then $R\zeta \in U_2$, and there is t, $R < t < 1$, such that $t\zeta \in U_1 \cap U_2$, and we have $d(g_0(R\zeta), f_1(\zeta)) \leqslant$

$d(g_0(R\zeta), f_2(R\zeta)) + d(f_2(R\zeta), f_2(t\zeta)) + d(f_2(t\zeta), g_0(t\zeta)) + d(g_0(t\zeta), f_1(t\zeta)) + d(f_1(t\zeta), f_1(\zeta)) < 5\varepsilon$. Similarly $d(g_0(R\zeta), f_2(\zeta)) < 5\varepsilon$. This completes the proof. □

The following lemma, which holds for any complex manifold of dimension at least two, is the main tool in the inductive construction of a proper holomorphic disc. The proof depends on the solution of a nonlinear Cousin problem due to J.-P. Rosay [12].

Lemma 2.2. — *Let X be a complex manifold of dimension $n \geqslant 2$ endowed with a Riemannian metric, which induces the distance function d on X. Assume that $\Omega \subset\subset X$ is an open coordinate neighborhood and $\Omega_0 \subset\subset \Omega$. Let f be a holomorphic map from a neighborhood of $\overline{\triangle}$ to X. Let K be a compact subset of X and r, $0 < r < 1$, such that $K \cap \Omega = \emptyset$ and $K \cap f(\overline{\triangle} \setminus r\triangle) = \emptyset$. Assume that $H\colon b\triangle \times \overline{\triangle} \to X$ is a continuous map with the following properties:*

(i) *for each $\zeta \in b\triangle$ the map $\eta \mapsto H(\zeta, \eta)$ is holomorphic on $\triangle$,*
(ii) $H(\zeta, 0) = f(\zeta)$ $(\zeta \in b\triangle)$,
(iii) *if for some $\zeta \in b\triangle$ we have $f(\zeta) \notin \Omega_0$ then $H(\zeta, \eta) = f(\zeta)$ $(\eta \in \overline{\triangle})$,*
(iv) *if for some $\zeta \in b\triangle$ we have $f(\zeta) \in \Omega_0$ then $H(\zeta, \eta) \in \Omega$ $(\eta \in \overline{\triangle})$.*

Given $\varepsilon > 0$, there is a continuous map $g\colon \overline{\triangle} \to X$, holomorphic on $\triangle$, such that

(i') $d(g(\zeta), H(\zeta, b\triangle)) < \varepsilon$ $(\zeta \in b\triangle)$,
(ii') $d(g(\zeta), f(\zeta)) < \varepsilon$ $(\zeta \in r\overline{\triangle})$,
(iii') $g(\overline{\triangle} \setminus r\triangle) \cap K = \emptyset$,
(iv') $g(0) = f(0)$,
(v') *$g'(0) = Rf'(0)$ for some R, $r < R < 1$.*

Proof. — By assumption there is $\rho > 1$ such that f is holomorphic on $\rho\triangle$. Choose ρ', $1 < \rho' < \rho$. Define the map $\tilde{f}\colon \rho\triangle \to X \times \mathbb{C}$ by $\tilde{f}(\zeta) = (f(\zeta), \zeta)$. The map $\tilde{f}$ is a holomorphic embedding (not proper), so there is an open neighborhood $\tilde{\Omega}_1$ of $\tilde{f}(\rho'\triangle)$ in $X \times \mathbb{C}$ and a biholomorphic map $\tilde{\Phi}_1$ from $\tilde{\Omega}_1$ onto a bounded open subset of $\mathbb{C}^{n+1}$ (see [14], [11], [13, Lemma 1.1]). With no loss of generality one can in addition assume that the derivative of $\tilde{\Phi}_1$ at 0 is the identity map. Choose a compact neighborhood $\tilde{K}_1$ of $\tilde{f}(\overline{\triangle})$ in $\tilde{\Omega}_1$.

Choose a biholomorphic map Φ_2 from Ω to an open subset of $\mathbb{C}^n$ and an open set Ω_2 such that $\Omega_0 \subset\subset \Omega_2 \subset\subset \Omega$, and $H(f^{-1}(\overline{\Omega}_0), \overline{\triangle}) \subset \Omega_2$. Let $\tilde{\Omega}_2 = \Omega \times 3\triangle$ and $\tilde{K}_2 = \overline{\Omega}_2 \times 2\overline{\triangle}$. Let $\tilde{\Phi}_2(z, \zeta) = (\Phi_2(z), \zeta)$ for $(z, \zeta) \in \tilde{\Omega}_2$ and note that $\tilde{\Phi}_2$ maps $\tilde{\Omega}_2$ biholomorphically into $\mathbb{C}^{n+1}$.

By decreasing $\varepsilon > 0$ if necessary we may assume that

$$d(K, f(\overline{\triangle} \setminus r\triangle)) > \varepsilon \text{ and } d(K, \Omega_2) > \varepsilon. \tag{2.3}$$

Denote by $\mathbb{B}$ the open unit ball in $\mathbb{C}^n$ and by $\Pi\colon X \times \mathbb{C} \to X$ the canonical projection to the first factor. There is $\alpha > 0$ so small that

$$\Phi_2(H(f^{-1}(\overline{\Omega}_0), \overline{\triangle})) + \alpha\mathbb{B} \subset \Phi_2(\Omega_2), \tag{2.4}$$

$$\begin{cases} \text{if } z \in \tilde{\Phi}_1(\tilde{K}_1),\ z' \in \mathbb{C}^{n+1},\ |z - z'| < \alpha, \\ \text{then } d(\Pi(\tilde{\Phi}_1^{-1}(z)), \Pi(\tilde{\Phi}_1^{-1}(z'))) < \frac{\varepsilon}{6}, \end{cases} \tag{2.5}$$

$$\begin{cases} \text{if } z \in \tilde{\Phi}_2(\tilde{K}_2),\ z' \in \mathbb{C}^{n+1},\ |z - z'| < \alpha, \\ \text{then } d(\Pi(\tilde{\Phi}_2^{-1}(z)), \Pi(\tilde{\Phi}_2^{-1}(z'))) < \frac{\varepsilon}{6}. \end{cases} \tag{2.6}$$

By slightly enlarging Ω_0 we may assume that either $f(b\triangle) \subset \overline{\Omega}_0$ or the set $f^{-1}(\overline{\Omega}_0) \cap b\triangle$ is at most finite union of disjoint closed arcs. Consider first the second case: denote these arcs by $\{I_j\}_{j\in\mathcal{J}}$ where $\mathcal{J}$ is finite and where I_j are pairwise disjoint. For each $j \in \mathcal{J}$ one can find a smooth simple closed curve $\Gamma_j \subset \overline{\triangle} \setminus r\overline{\triangle}$ such that $\Gamma_j \cap b\triangle$ is a neighborhood of I_j in $b\triangle$ and Γ_j are pairwise disjoint. Each Γ_j bounds a domain $D_j \subset \triangle \setminus r\overline{\triangle}$, which is conformally equivalent to the unit disc. Since $f(I_j) \subset \Omega_0$ one can choose Γ_j in such a way that, in addition to the above, we have $f(\overline{D}_j) \subset \Omega_2$ $(j \in \mathcal{J})$. Choose a homeomorphic map h_j from $\overline{\triangle}$ to $\overline{D}_j$, which is holomorphic on $\triangle$, and let $V_j = h_j(\{th_j^{-1}(\zeta); \zeta \in I_j, t \in [0,1]\})$. One can choose an open neighborhood W_j of V_j in $\mathbb{C}$ such that $W_j \cap \triangle \subset D_j$. Denote by U_1 the set $\rho'\triangle \setminus \overline{\cup_j V_j}$ and by U_2 the union $\cup_j W_j$.

If $f(b\triangle) \subset \overline{\Omega}_0$ then let $I_1 = b\triangle$ and choose r_0, $r < r_0 < 1$, such that $f(\overline{\triangle} \setminus r_0\triangle) \subset \Omega_2$. Let $U_1 = \frac{1+r_0}{2}\triangle$ and let $U_2 = W_1 = \rho'\triangle \setminus \overline{r_0\triangle}$.

Now we are in the situation of Section 5 in [12]: The sets U_1 and U_2 are open in $\mathbb{C}$ and satisfy $\overline{\triangle} \subset U_1 \cup U_2$. Let $\omega_{12}' = \tilde{\Phi}_1(\tilde{\Omega}_1 \cap \tilde{\Omega}_2)$ and let ω_{12} be the image under $\tilde{\Phi}_1$ of a neighborhood of $\tilde{K}_1 \cap \tilde{K}_2$ such that $\omega_{12} \subset\subset \omega_{12}'$. By [12, Proposition 1'] there is a $\delta > 0$ with the following property: if u_k is a holomorphic map from $U_k \cap \triangle$ into $\mathbb{C}^{n+1}$ $(k \in \{1,2\})$ such that $u_1(U_1 \cap U_2 \cap \triangle) \subset \omega_{12}$, and $|u_2(\zeta) - (\tilde{\Phi}_2 \circ \tilde{\Phi}_1^{-1} \circ u_1)(\zeta)| \leqslant \delta$ $(\zeta \in U_1 \cap U_2 \cap \triangle)$, then there are holomorphic maps v_k from $U_k \cap \triangle$ into $\mathbb{C}^{n+1}$ such that $|v_k(\zeta)| \leqslant \alpha$ $(\zeta \in U_k \cap \triangle)$ for $k = 1, 2$ and $u_2 + v_2 = \tilde{\Phi}_2 \circ \tilde{\Phi}_1^{-1} \circ (u_1 + v_1)$ on $U_1 \cap U_2 \cap \triangle$. Moreover, one can impose $v_1(0) = 0$. By a minor change in the proof of [12, Proposition 1'] one can further impose $v_1'(0) = 0$. For the sake of completeness we provide the details. We will adapt the same notations as in the proof of [12, Proposition 1']. We only need to change the solution operator T which solves the standard additive Cousin problem. This continuous linear operator associates to a bounded holomorphic map

$\alpha_{12} \in (H^\infty(U_1 \cap U_2 \cap \triangle))^{n+1}$ holomorphic maps $T_j(\alpha_{12}) \in (H^\infty(U_j \cap \triangle))^{n+1}$ $(j = 1, 2)$ such that $\alpha_{12} = T_1(\alpha_{12}) - T_2(\alpha_{12})$. We need that in addition to the above, it satisfies $T_1(\alpha_{12})(0) = 0$ and $T_1(\alpha_{12})'(0) = 0$. This property of operator T implies that $v_1'(0) = 0$. Following the proof of [10, Theorem 1.4.5] the solution of the additive Cousin problem is reduced to solving $\overline{\partial}$-equation as follows. We can choose a cut off function φ such that $\operatorname{supp}\varphi \subset\subset U_1$ and φ equals 1 on $(U_1 \setminus U_2) \cap \overline{\triangle}$. The solution of the additive Cousin problem is of the form

$$T_1(\alpha_{12}) = (1 - \varphi)\alpha_{12} + u\ ,\ T_2(\alpha_{12}) = -\varphi\alpha_{12} + u\ , \tag{2.7}$$

where u is a solution of $\overline{\partial}$-equation to assure that the maps $T_j(\alpha_{12})$ are holomorphic on $U_j \cap \triangle$ $(j = 1, 2)$.

Instead of solving (2.7) we will solve the following

$$\begin{aligned} T_1(\alpha_{12})(\zeta) &= (1 - \varphi(\zeta))\alpha_{12}(\zeta) + \zeta^2 u(\zeta) \quad (\zeta \in U_1 \cap \triangle),\\ T_2(\alpha_{12})(\zeta) &= -\varphi(\zeta)\alpha_{12}(\zeta) + \zeta^2 u(\zeta) \qquad\quad (\zeta \in U_2 \cap \triangle). \end{aligned}$$

Therefore, if the map u satisfies

$$\frac{\partial u}{\partial \overline{\zeta}}(\zeta) = \frac{\alpha_{12}(\zeta)}{\zeta^2}\frac{\partial \varphi}{\partial \overline{\zeta}}(\zeta) \quad (\zeta \in \triangle),$$

then the maps $T_j(\alpha_{12})$ $(j = 1, 2)$ are holomorphic. By the properties of φ one can solve the above $\overline{\partial}$-equation with estimates to prove that T is continuous and linear.

Now we proceed with the proof of Lemma 2.2. Let $u_1(\zeta) = \tilde{\Phi}_1(f(\zeta), \zeta)$ $(\zeta \in U_1 \cap \overline{\triangle})$. In the next paragraph we shall define map u_2.

First consider the case that $f^{-1}(\overline{\Omega}_0) \cap b\triangle$ consists of finitely many closed arcs denoted by I_j. For each $j \in \mathcal{J}$ define the map $H_j \colon \Gamma_j \times \overline{\triangle} \to X$ by $H_j(\zeta, \eta) = H(\zeta, \eta)$ $(\zeta \in I_j, \eta \in \overline{\triangle})$ and $H_j(\zeta, \eta) = f(\zeta)$ $(\zeta \in \Gamma_j \setminus I_j, \eta \in \overline{\triangle})$. Property (iii) implies that H_j is continuous, and by (i), H_j is holomorphic in the second variable. By (2.4), and as $f(\overline{D}_j) \subset \Omega_2$, there is an α_j, $0 < \alpha_j < \alpha$, so small that

$$\Phi_2(H_j(\Gamma_j, \overline{\triangle})) + \alpha_j\mathbb{B} \subset \Phi_2(\Omega_2). \tag{2.8}$$

Let $\gamma > 0$. Using [6, Lemma 5.1] for the map $G_j(\zeta, \eta) = \Phi_2(H_j(h_j(\zeta), \eta)) - \Phi_2(f(h_j(\zeta)))$ $((\zeta, \eta) \in b\triangle \times \overline{\triangle})$ we get the polynomial map P_j from $\mathbb{C}$ into $\mathbb{C}^n$ with the following properties

$$\begin{gathered} P_j(\zeta) \in G_j(\zeta, b\triangle) + \gamma\mathbb{B},\ (\zeta \in b\triangle),\\ P_j(t\zeta) \in G_j(\zeta, \overline{\triangle}) + \gamma\mathbb{B},\ (\zeta \in b\triangle, 0 \leqslant t \leqslant 1). \end{gathered}$$

If $\gamma > 0$ is small enough then the map $p_j\colon \overline{D}_j \to \mathbb{C}^n$, defined by $p_j(\zeta) = P_j(h_j^{-1}(\zeta)) + \Phi_2(f(\zeta))$ $(\zeta \in \overline{D}_j)$, is continuous, holomorphic on D_j, and satisfies the following

$$p_j(\zeta) \in \Phi_2(H_j(\zeta, b\triangle)) + \alpha\mathbb{B}\ (\zeta \in \Gamma_j), \tag{2.9}$$

$$p_j(\zeta) \in \Phi_2(H_j(\Gamma_j, \overline{\triangle})) + \alpha_j\mathbb{B}\ (\zeta \in \overline{D}_j), \tag{2.10}$$

$$|p_j(\zeta) - \Phi_2(f(\zeta))| < \delta\ (\zeta \in D_j \setminus V_j). \tag{2.11}$$

Note that by (2.8) and (2.10), we get $p_j(\overline{D}_j) \subset \Phi_2(\Omega_2)$.

Define the map $u_2\colon U_2 \cap \overline{\triangle} \to \mathbb{C}^{n+1}$ by $u_2(\zeta) = (p_j(\zeta), \zeta)$ $(\zeta \in \overline{W_j \cap \triangle})$.

In the simpler case when $f(b\triangle) \subset \overline{\Omega}_0$ we solve approximately the Riemann-Hilbert problem for the map $G_1(\zeta,\eta) = \Phi_2(H(\zeta,\eta)) - \Phi_2(f(\zeta))$ $((\zeta,\eta) \in b\triangle \times \overline{\triangle})$ such that the solution polynomial map P_1 as above satisfies

$$|P_1(\zeta)| < \min\{\delta, \alpha\} \text{ for } |\zeta| < \frac{1+r_0}{2}. \tag{2.12}$$

Similarly as above we define $p_1(\zeta) = P_1(\zeta) + \Phi_2(f(\zeta))$ $(\zeta \in U_2 \cap \overline{\triangle})$ and $u_2(\zeta) = (p_1(\zeta), \zeta)$ $(\zeta \in U_2 \cap \overline{\triangle})$.

Property (2.11) (or (2.12)) implies that $|u_2(\zeta) - (\tilde{\Phi}_2 \circ \tilde{\Phi}_1^{-1} \circ u_1)(\zeta)| \leqslant \delta$ $(\zeta \in U_1 \cap U_2 \cap \triangle)$. Therefore by the above there exist holomorphic maps $v_k\colon U_k \cap \triangle \to \mathbb{C}^{n+1}$ $(k = 1, 2)$ such that

$$|v_k(\zeta)| < \alpha\ (\zeta \in U_k \cap \triangle,\ k = 1, 2),$$
$$u_2 + v_2 = \tilde{\Phi}_2 \circ \tilde{\Phi}_1^{-1} \circ (u_1 + v_1) \text{ on } U_1 \cap U_2 \cap \triangle$$
$$v_1(0) = 0, \quad v_1'(0) = 0.$$

This implies that one can define

$$g_0(\zeta) = \Pi(\tilde{\Phi}_k^{-1}((u_k + v_k)(\zeta)))\ (\zeta \in U_k \cap \triangle, k = 1, 2).$$

The map g_0 is holomorphic on $\triangle$, $g_0(0) = f(0)$, $g_0'(0) = f'(0)$, and by (2.5) and (2.6), we have

$$d(g_0(\zeta), f(\zeta)) < \tfrac{\varepsilon}{6}\ (\zeta \in U_1 \cap \triangle), \tag{2.13}$$

$$d(g_0(\zeta), \Phi_2^{-1}(p_j(\zeta))) < \tfrac{\varepsilon}{6}\ (\zeta \in W_j \cap \triangle,\ j \in \mathcal{J}). \tag{2.14}$$

By Lemma 2.1, there is R, $r < R < 1$, so close to 1 that the map $g(\zeta) = g_0(R\zeta)$ $(\zeta \in \overline{\triangle})$ satisfies the following

$$d(g(\zeta), f(\zeta)) < \tfrac{5\varepsilon}{6}\ (\zeta \in U_1 \cap \overline{\triangle}), \tag{2.15}$$

$$d(g(\zeta), \Phi_2^{-1}(p_j(\zeta))) < \tfrac{5\varepsilon}{6}\ (\zeta \in W_j \cap \overline{\triangle},\ j \in \mathcal{J}). \tag{2.16}$$

The map g is continuous on $\overline{\triangle}$ and holomorphic on $\triangle$. To prove (i') choose $\zeta \in b\triangle \cap U_1$. Since $f(b\triangle \cap U_1) \cap \Omega_0 = \emptyset$, properties (iii) and (2.15) imply that $d(g(\zeta), H(\zeta, b\triangle)) = d(g(\zeta), f(\zeta)) < \varepsilon$. For $\zeta \in U_2 \cap b\triangle$ it holds that $\zeta \in W_j$ for some $j \in \mathcal{J}$ and by (2.16), (2.9) and (2.6) we get

$$d(g(\zeta), H(\zeta, b\triangle)) \leqslant d(g(\zeta), \Phi_2^{-1}(p_j(\zeta))) + d(\Phi_2^{-1}(p_j(\zeta)), H(\zeta, b\triangle)) < \varepsilon.$$

So (i') holds. Since $r\overline{\triangle} \subset U_1$ property (2.15) implies (ii'). Take $\zeta \in \overline{\triangle} \setminus r\triangle$. If $\zeta \in U_1$, then (2.15) and (2.3) imply that $g(\zeta) \notin K$. If $\zeta \in U_2$, then $\zeta \in W_j$ for some $j \in \mathcal{J}$ and since $p_j(\zeta) \in \Phi_2(\Omega_2)$ properties (2.16) and (2.3) imply that $g(\zeta) \notin K$. Thus we proved (iii'). It is easy to see that $g(0) = f(0)$ and $g'(0) = Rf'(0)$, which proves (iv') and (v'). This proves the lemma. □

3. Convex bumps

We will construct convex bumps introduced by Grauert (see [9]) in order to provide a continuous family of analytic discs, which is needed to use Lemma 2.2 in the inductive proof of Theorem 1.2.

We denote by $d_{1,2}$ the partial differential with respect to the first two complex coordinates on $\mathbb{C}^n$.

Let X be a n-dimensional complex manifold and let $A, B \subset X$ be relatively compact open sets in X. We say that *B is a 2-bump on A* if there exist an open set $\Omega \subset X$ containing $\overline{B}$, a biholomorphic map Φ from Ω onto a convex subset ω in $\mathbb{C}^n$, and smooth real functions $\rho_B \leqslant \rho_A$ on ω such that

$$\Phi(A \cap \Omega) = \{z \in \omega;\ \rho_A(z) < 0\},\ \ \Phi((A \cup B) \cap \Omega) = \{z \in \omega;\ \rho_B(z) < 0\},$$

and the functions ρ_A and ρ_B are strictly plurisubharmonic with respect to the first two coordinates.

We say that *B is a convex 2-bump on A* if, in addition to the above, ρ_A and ρ_B are strictly convex in z_1, z_2 (with respect to the underlying real coordinates), and $d_{1,2}(t\rho_A + (1-t)\rho_B)$ is non degenerate on ω for each $t \in [0, 1]$.

Lemma 3.1. — *Let X be a complex manifold of dimension $n \geqslant 2$, equipped with some metric d. Let $A, B \subset X$ such that B is a convex 2-bump on A and let K be a compact subset of A. Assume that $f\colon \overline{\triangle} \to X$ is a continuous map, holomorphic on $\triangle$, such that $f(b\triangle) \cap \overline{A} = \emptyset$, and r, $0 < r < 1$, such that $f(\overline{\triangle} \setminus r\triangle) \cap K = \emptyset$. Given $\varepsilon > 0$ there is a continuous map $g\colon \overline{\triangle} \to X$, holomorphic on $\triangle$, with the following properties*

(i) $g(b\triangle) \cap (\overline{A \cup B}) = \emptyset$,
(ii) $d(g(\zeta), f(\zeta)) < \varepsilon$ $(|\zeta| \leqslant r)$,
(iii) $g(\overline{\triangle} \setminus r\triangle) \cap K = \emptyset$,

(iv) $g(0) = f(0)$,
(v) $g'(0) = Rf'(0)$ *for some* R, $r < R < 1$.

Proof. — We may assume that f is holomorphic in a neighborhood of $\overline{\triangle}$. Since B is a convex 2-bump on A there are a biholomorphic map $\Phi\colon \Omega \to \mathbb{C}^n$ onto a convex subset ω of $\mathbb{C}^n$, and smooth functions $\rho_A, \rho_B\colon \omega \to \mathbb{R}$ such that

$$\Phi(A \cap \Omega) = \{z \in \omega;\ \rho_A(z) < 0\},\ \ \Phi((A \cup B) \cap \Omega) = \{z \in \omega;\ \rho_B(z) < 0\},$$

and the functions ρ_A and ρ_B are strictly convex with respect to the first two coordinates, and $d_{1,2}(t\rho_A + (1-t)\rho_B)$ is non degenerate on ω for each $t \in [0,1]$.

We can choose $\lambda > 0$ so small that $\Phi(f(b\triangle)\cap\Omega)\cap\{z \in \omega; \rho_A(z) \leqslant \lambda\} = \emptyset$, the set $\omega_0 = \{z \in \omega; \rho_A(z) > \lambda, \rho_B(z) \leqslant \lambda\}$ is relatively compact in ω, and $b\triangle \cap f^{-1}(\Phi^{-1}(\{z \in \omega;\ \rho_B(z) \leqslant \lambda\}))$ is either $b\triangle =: I_1$ or a union of finitely many closed arcs which we denote by I_j.

Choose a point $q \in \omega_0$ and write $q = (q_1, q_2, q'')$. There is exactly one μ, $0 \leqslant \mu \leqslant 1$, such that $\mu(\rho_A(q) - \lambda) + (1-\mu)(\rho_B(q) - \lambda) = 0$. The function $\mu(\rho_A - \lambda) + (1-\mu)(\rho_B - \lambda)$ is defined on ω and it is strictly convex in the first two coordinates. Denote by $M_{\mu,q''}$ the set $\{(z_1, z_2, q'') \in \omega; \mu(\rho_A(z_1, z_2, q'') - \lambda) + (1-\mu)(\rho_B(z_1, z_2, q'') - \lambda) = 0\}$. Note that $M_{\mu,q''}$ is a real submanifold of dimension 3 in $\mathbb{C}^2 \times \{q''\}$. Denote by $T_q M_{\mu,q''}$ its real tangent space at q. The intersection $E_q = T_q M_{\mu,q''} \cap \imath T_q M_{\mu,q''}$ is a complex line. By strict convexity the intersection of $\{q\}+E_q$ with $\{z \in \Omega; \rho_B(z) \leqslant \lambda\}$ is a bounded connected convex subset of $\{q\}+E_q$ therefore it is conformally equivalent to the unit disc. If we vary q smoothly these convex sets vary smoothly. The set $\Phi(f(I_j))$ is contained in ω_0 for each j. Therefore with a proof similar to the proof of [6, Lemma 4.1] we obtain a continuous map $H_j\colon I_j \times \overline{\triangle} \to \mathbb{C}^n$ such that

(a) for each $\zeta \in I_j$ the map $H_j(\zeta, \eta)$ is holomorphic in η,
(b) $H_j(\zeta, 0) = \Phi(f(\zeta))$ $(\zeta \in I_j)$,
(c) $H_j(I_j, \overline{\triangle}) \subset \omega$,
(d) $\rho_B(H_j(\zeta, \eta)) = \lambda$ $(\zeta \in I_j, \eta \in b\triangle)$,
(e) if $\rho_B(\Phi(f(\zeta))) = \lambda$ then $H_j(\zeta, \eta) = \Phi(f(\zeta))$ $(\zeta \in I_j, \eta \in \overline{\triangle})$.

We define a map $H\colon b\triangle \times \triangle \to X$ by

$$H(\zeta, \eta) = \begin{cases} \Phi^{-1}(H_j(\zeta, \eta)), & \zeta \in I_j, \\ f(\zeta), & \zeta \in b\triangle \setminus \cup_j I_j. \end{cases}$$

The map H is continuous by construction and it satisfies the following

(a') for each $\zeta \in b\triangle$ the map $H(\zeta, \eta)$ is holomorphic in η,

(b') $H(\zeta,0)=f(\zeta)$ $(\zeta\in b\triangle)$,
(c') if for some $\zeta\in b\triangle$ we have $f(\zeta)\notin\Phi^{-1}(\omega_0)$ then $H(\zeta,\eta)=f(\zeta)$ $(\eta\in\overline{\triangle})$,
(d') if for some $\zeta\in b\triangle$ we have $f(\zeta)\in\Phi^{-1}(\omega_0)$ then $H(\zeta,\overline{\triangle})\subset\Phi^{-1}(\omega)$ and $H(\zeta,\eta)\in\Phi^{-1}(\{z\in\omega;\rho_B(z)=\lambda,\rho_A(z)\geqslant\lambda\})$ $(\eta\in b\triangle)$.

Take ε_0, $0<\varepsilon_0<\varepsilon$, so small that

$$d\left(\Phi^{-1}\left(\{z\in\omega;\rho_B(z)=\lambda,\rho_A(z)\geqslant\lambda\}\right),\Phi^{-1}(\{z\in\omega;\rho_B(z)\leqslant 0\})\right)>\varepsilon_0. \tag{3.1}$$

Now we use Lemma 2.2 to get the map g such that

(i') $d(g(\zeta),H(\zeta,b\triangle))<\varepsilon_0$ $(\zeta\in b\triangle)$,
(ii') $d(g(\zeta),f(\zeta))<\varepsilon_0$ $(\zeta\in r\overline{\triangle})$,
(iii') $g(\overline{\triangle}\smallsetminus r\triangle)\cap K=\emptyset$,
(iv') $g(0)=f(0)$,
(v') $g'(0)=Rf'(0)$ for some R, $r<R<1$.

Properties (ii)–(v) follow from (ii')–(v'). By (3.1), (d') and (i') we get (i). □

4. Proof of Theorem 1.2

As we have already explained in the introduction we only need to treat the case $\dim X\geqslant 3$.

LEMMA 4.1. — *Let X be a complex manifold of dimension $n\geqslant 3$. Let $\Omega\subset\subset X$ and let $\rho\colon\Omega\to\mathbb{R}$ be a smooth function such that $\{z\in\Omega;\,a\leqslant\rho(z)\leqslant b\}\subset\subset\Omega$ and such that the Levi form of ρ has at each point at least 2 positive eigenvalues. Assume that ρ has at most one critical point in $\{z\in\Omega;\,a\leqslant\rho(z)\leqslant b\}$ and, if q is a critical point of ρ, then further assume that $a<\rho(q)<b$, and that q is a non-degenerate critical point. Let K be a compact subset of X such that $K\cap\Omega=\emptyset$. Assume that $f\colon\overline{\triangle}\to X$ is a continuous map, holomorphic on $\triangle$, such that $f(b\triangle)\subset\{z\in\Omega;\rho(z)>a\}$ and choose r, $0<r<1$, such that $f(\overline{\triangle}\smallsetminus r\triangle)\cap K=\emptyset$. Given $\varepsilon>0$ there exists a continuous map $g\colon\overline{\triangle}\to X$, holomorphic on $\triangle$, with the following properties*

(i) $g(b\triangle)\subset\{z\in\Omega;\rho(z)>b\}$,
(ii) $d(g(\zeta),f(\zeta))<\varepsilon$ $(|\zeta|\leqslant r)$,
(iii) $g(\overline{\triangle}\smallsetminus r\triangle)\cap K=\emptyset$,
(iv) $g(0)=f(0)$,
(v) $g'(0)=\lambda f'(0)$ *for some* λ, $r<\lambda<1$.

Proof. — Note that, if ρ is a smooth function defined on a complex manifold X, whose Levi form has at least 2 positive eigenvalues at some point $w \in X$, then there are holomorphic coordinates near w such that ρ is strictly plurisubharmonic in z_1,z_2. Moreover, if w is a regular point of ρ, then Narasimhan's lemma on local convexification implies that, in local holomorphic coordinates, ρ can be made strictly convex in z_1,z_2. Both conditions are stable under small perturbations. If ρ does not have local minima in Ω, then by [9, Lemma 12.3] we get finitely many domains $\{z \in \Omega; \rho(z) < a\} = A_0 \subset A_1 \subset \cdots \subset A_m = \{z \in \Omega; \rho(z) < b\}$ in X such that for every $k = 0, 1, \ldots, m-1$ we have $A_{k+1} = A_k \cup B_k$, where B_k is a 2-bump on A_k. Moreover, for any open covering $\{U_i\}$ of Ω we can in addition insure that each set B_k is contained in some U_i.

If ρ does not have any critical points in the set U_i for some i then we can further achieve using Narasimhan's lemma on local convexification that for every $k = 0, 1, \ldots, m-1$ for which B_k lies in U_i it holds that B_k is a convex 2-bump on A_k.

Therefore, if ρ does not have any critical points in the set $\{z \in \Omega; a \leqslant \rho(z) \leqslant b\}$, then we can insure that B_k is a convex 2-bump on A_k for every $k = 0, 1, \ldots, m-1$. In this case we obtain the map g by using Lemma 3.1 m times, where each time we push the boundary of the disc to the complement of the set A_j.

Now assume that ρ has exactly one critical point in the set $\{z \in \Omega; a < \rho(z) < b\}$. Since by assumption this critical point is non-degenerate we can achieve that in addition it does not lie on $f(b\triangle)$. Denote this critical point by q. If q is a local minimum then the boundary of the disc already lies above the critical level set; we cannot approach q by the non-critical procedure so we can continue in the same way as if there were no critical points.

Therefore from now on we may assume that q is not a local minimum. We shall push the boundary of the disc to the higher level sets of ρ and we will keep the boundary away from the critical point. Here we need that the complex dimension of the manifold is at least 3. In the local coordinates around the critical point we choose complex 2 dimensional subspaces L_1, L_2 and L_3 and we move the boundary of the disc in these directions. Away from the critical point we use convex 2-bumps. The details are as follows. We can choose a biholomorphic change of coordinates Φ from a neighborhood of q to a neighborhood ω of 0 in $\mathbb{C}^n$ such that $\Phi(q) = 0$ and $\rho \circ \Phi^{-1}|\omega$ is strictly plurisubharmonic in the first two coordinates. Denote by L the complex 2-dimensional subspace generated by z_1, z_2. Let $\omega_1 \subset\subset \omega$ be a neighborhood of 0. Then for every small perturbation L' of L and for each $z \in \omega_1$ the map $\rho \circ \Phi^{-1}$ is strictly plurisubharmonic on $(z + L') \cap \omega_1$. Therefore one can choose three complex linear subspaces L_1, L_2 and L_3 in $\mathbb{C}^n$ such that $L_1 \cap L_2 \cap L_3 = \{0\}$ and for each j, $1 \leqslant j \leqslant 3$, and for $z \in \omega_1$, the map $\rho \circ \Phi^{-1}$ is strictly plurisubharmonic on $(z + L_j) \cap \omega_1$. There is a $\delta > 0$ so small that $\{z \in \mathbb{C}^n; \mathrm{dist}(z, L_j) < \delta, 1 \leqslant j \leqslant 3\} \subset\subset \omega_1$. Denote this set by ω_0. By taking smaller δ if necessary we may assume that $\Phi(f(b\triangle)) \cap \omega_0 = \emptyset$.

For each $w \in \{z \in \Omega; a \leqslant \rho(z) \leqslant b\} \setminus \Phi^{-1}(\omega_1)$ we can choose a coordinate neighborhood Ω_w, a biholomorphic map $\Phi_w \colon \Omega_w \to \Phi_w(\Omega_w) \subset \mathbb{C}^n$, where $\Phi_w(\Omega_w)$ is convex such that $\rho \circ \Phi_w^{-1}$ is strictly convex in the first two coordinates. We can further assume that $\Omega_w \subset\subset \Omega \setminus \overline{\Phi^{-1}(\omega_0)}$.

For each $w \in \Phi^{-1}(\omega_1 \setminus \omega_0)$ there is j, $1 \leqslant j \leqslant 3$, such that $\mathrm{dist}(\Phi(w), L_j) \geqslant \delta$. By the above $\rho \circ \Phi^{-1}$ is strictly plurisubharmonic on $(\Phi(w) + L_j) \cap \omega_1$. Hence there is a biholomorphic change of coordinates on $\Phi(w) + L_j$ near $\Phi(w)$ such that in the new coordinates $\rho \circ \Phi^{-1}$ is strictly convex on $\Phi(w) + L_j$ near $\Phi(w)$. Since strict convexity is preserved by small perturbations it follows that there are new coordinates near $\Phi(w)$ in $\mathbb{C}^n$ and a neighborhood Ω_w of w in X such that $\rho \circ \Phi^{-1}$ is strictly convex in the new coordinates on $(\Phi(z) + L_j) \cap \Phi(\Omega_w)$ for each $z \in \Omega_w$. By construction the small tangent discs corresponding to $\Phi(z)$ along which we lift in Lemma 3.1 lie in $\Phi(z) + L_j$. Denote by Ω_q the set $\Phi^{-1}(\omega_0)$. Note that $\{\Omega_w\}$ is an open covering of $\{z \in \Omega; a \leqslant \rho(z) \leqslant b\}$. By the above there are a finite number of domains $\{z \in \Omega; \rho(z) < a\} = A_0 \subset A_1 \subset \cdots \subset A_m = \{z \in \Omega; \rho(z) < b\}$ such that for each k, $0 \leqslant k \leqslant m-1$, we have $A_{k+1} = A_k \cup B_k$, where B_k is a 2-bump on A_k, and there is a set Ω_w such that $B_k \subset \Omega_w$ and if $w \neq q$ then B_k is a convex 2-bump on A_k.

We construct the map g inductively. At each step we construct a continuous map $f_k \colon \overline{\triangle} \to X$, holomorphic on $\triangle$, with the following properties

(a) $f_k(\overline{\triangle} \setminus r\triangle) \cap K = \emptyset$,
(b) $f_k(b\triangle) \cap (\overline{\Omega_q \cup A_k}) = \emptyset$,
(c) $d(f_k(\zeta), f(\zeta)) \leqslant \frac{k}{2m}\varepsilon$ $(|\zeta| \leqslant r)$,
(d) $f_k(0) = f(0)$,
(e) $f_k'(0) = R_k f'(0)$ for some R_k, $\sqrt[m]{r} < R_k < 1$.

Let $f_0 = f$ and note that f_0 satisfies all the properties. Assume that we have already constructed the map f_k with the properties (a)-(e) for some k, $0 \leqslant k \leqslant m-1$. If $B_k \subset \Omega_q$ then we put $f_{k+1} = f_k$. In this case the map f_{k+1} obviously satisfies (a), (c), (d) and (e). The property (b) follows from the fact that $B_k \subset \Omega_q$ and that the map f_k satisfies (b). Otherwise, if $B_k \subset \Omega_w$, $w \neq q$, then we use Lemma 3.1 to get the map f_{k+1}. The fact that $f_{k+1}(b\triangle)$ misses $\overline{\Omega}_q$ follows from the properties of the covering; if Ω_w misses $\Phi^{-1}(\overline{\omega}_0)$ and if the perturbation constants are small enough then obviously $f_{k+1}(b\triangle)$ misses $\overline{\Omega}_q$. Otherwise, there is j, $1 \leqslant j \leqslant 3$, such that $\operatorname{dist}(\Phi(w), L_j) \geqslant \delta$ and there is a biholomorphic change of coordinates such that in the new coordinates $\rho \circ \Phi^{-1}$ is strictly convex on $\Phi(z) + L_j$ for $z \in \Omega_w$. Since the boundary of the disc f_k does not intersect $\overline{\Omega}_q$, at each point $\zeta \in b\triangle$ such that $f_k(\zeta) \in \Omega_w$ we have $\operatorname{dist}(\Phi(f_k(\zeta)), L_j) > \delta$ and the small tangent disc, along which we lift the boundary of the disc f_k, lies in $\Phi(f_k(\zeta)) + L_j$. Therefore, if the perturbation constants are small enough, it holds that $\operatorname{dist}(\Phi(f_{k+1}(\zeta)), L_j) > \delta$ $(\zeta \in b\triangle)$. This proves (b). The properties (a), (c), (d) and (e) are easily satisfied. The construction is finished. The map $g = f_m$ has all the required properties and the proof is complete. □

Proof of Theorem 1.2. — By Morse theory ([9, Observation 4.15] and [9, Proposition 0.5]) we get an exhaustion function ρ of class $\mathcal{C}^\infty$ without degenerate critical points and M' such that the Levi form of ρ has at each point of $\{\rho > M'\}$ at least 2 positive eigenvalues and such that for $\zeta \in b\triangle$ it holds that $\rho(f(\zeta)) > M'$. We may additionally assume that there is only one critical point on each critical level set.

Choose an increasing sequence a_j of regular values of ρ, converging to ∞, and such that $\rho(f(\zeta)) > a_1$ $(\zeta \in b\triangle)$ and for each $j \in \mathbb{N}$ there is at most one critical value on (a_j, a_{j+1}). Choose a decreasing sequence $\varepsilon_j > 0$ such that

$$\text{(4.1)}\quad \text{if } z \in X,\ \rho(z) \leqslant a_j,\ w \in X,\ d(z,w) \leqslant \varepsilon_j \text{ then } |\rho(z) - \rho(w)| < 1.$$

Using Lemma 4.1 one can construct inductively a sequence of continuous maps $f_n \colon \overline{\triangle} \to X$, holomorphic on $\triangle$, an increasing sequence r_n of positive numbers converging to 1, such that $\sum_{n=1}^{\infty}(1-r_n)$ converges, and a sequence R_n, $r_n < R_n < 1$, such that for each n,

(a) $\rho(f_n(\zeta)) > a_n$ $(\zeta \in b\triangle)$,
(b) $\rho(f_n(\zeta)) > a_{n-1} - 1$ $(r_n \leqslant |\zeta| \leqslant 1)$,
(c) $d(f_n(\zeta), f_{n-1}(\zeta)) < \frac{\varepsilon_n}{2^n}$ $(|\zeta| \leqslant r_n)$,
(d) $f_n(0) = f(0)$,
(e) $f_n'(0) = \prod_{j=1}^{n} R_j f'(0)$.

By (c) the sequence f_n converges uniformly on compacta on $\triangle$ and the limit map g is holomorphic on $\triangle$. Take $n \in \mathbb{N}$ and fix ζ, $r_n \leqslant |\zeta| \leqslant r_{n+1}$. Then we get by (c) that

$$\begin{aligned} d(f_n(\zeta), g(\zeta)) &\leqslant d(f_n(\zeta), f_{n+1}(\zeta)) + d(f_{n+1}(\zeta), f_{n+2}(\zeta)) + \cdots \\ &\leqslant \frac{\varepsilon_{n+1}}{2^{n+1}} + \frac{\varepsilon_{n+2}}{2^{n+2}} + \cdots < \varepsilon_n. \end{aligned}$$

Therefore, if $\rho(g(\zeta)) \leqslant a_n$, then this together with (b) and (4.1) implies that $\rho(g(\zeta)) > a_{n-1} - 2$. Since $\lim_{n\to\infty} r_n = 1$ and since $\lim_{n\to\infty} a_n = \infty$, it follows that g is a proper map. By (d) we obtain that $g(0) = f(0)$. Since $\sum_{n=1}^{\infty}(1 - r_n)$ converges, $\sum_{n=1}^{\infty}(1 - R_n)$ converges, and using [15, Theorem 15.5] we get that the infinite product $\prod_{j=1}^{\infty} R_j$ converges to $\lambda > 0$. Therefore $g'(0) = \lambda f'(0)$. This completes the proof. □

Added in the final revision. — In the subsequent paper [3] written by Franc Forstnerič and the author the conclusion of Theorem 1.2 is extended to complex spaces with singularities.

Acknowledgments. — The author wishes to thank F. Forstnerič for useful discussions while working on this paper. She would also like to thank J. Globevnik, M. Slapar, S. Strle and J. Winkelmann for helpful remarks.

This research has been supported in part by the Ministry of Education, Science and Sport of Slovenia through research program Analysis and Geometry, Contract No. P1-0291 and research project No. J1-6173-0101-04.

BIBLIOGRAPHY

[1] M. Colţoiu, "q-convexity. A survey", in *Complex analysis and geometry (Trento, 1995)* (Harlow), Pitman Res. Notes Math. Ser., vol. 366, Longman, 1997, p. 83-93.

[2] A. Dor, "A domain in $\mathbf{C}^m$ not containing any proper image of the unit disc", *Math. Z.* **222** (1996), p. 615-625.

[3] B. Drinovec-Drnovšek & F. Forstnerič, "Holomorphic curves in complex spaces", to appear in Duke Math. J.

[4] F. Forstnerič & J. Globevnik, "Discs in pseudoconvex domains", *Comment. Math. Helv.* **67** (1992), p. 129-145.

[5] ———, "Proper holomorphic discs in $\mathbb{C}^2$", *Math. Res. Lett.* **8** (2001), p. 257-274.

[6] J. Globevnik, "Discs in Stein manifolds", *Indiana Univ. Math. J.* **49** (2000), p. 553-574.

[7] H. Grauert, "Theory of q-convexity and q-concavity", in *Several complex variables, VII*, Encyclopaedia Math. Sci., vol. 74, Springer, Berlin, 1994, p. 259-284.

[8] R. E. Greene & H. H. Wu, "Embedding of open Riemannian manifolds by harmonic functions", *Ann. Inst. Fourier (Grenoble)* **25** (1975), p. 215-235.

[9] G. M. Henkin & J. Leiterer, *Andreotti-Grauert theory by integral formulas*, Progress in Mathematics, vol. 74, Birkhäuser Boston Inc., Boston, MA, 1988, 270 pages.

[10] L. Hörmander, *An introduction to complex analysis in several variables*, revised ed., North-Holland Publishing Co., Amsterdam, 1973, North-Holland Mathematical Library, Vol. 7, x+213 pages.

[11] F. Lárusson & R. Sigurdsson, "Plurisubharmonic functions and analytic discs on manifolds", *J. Reine Angew. Math.* **501** (1998), p. 1-39.

[12] J.-P. Rosay, "Approximation of non-holomorphic maps, and Poletsky theory of discs", *J. Korean Math. Soc.* **40** (2003), p. 423-434.

[13] ———, "Poletsky theory of disks on holomorphic manifolds", *Indiana Univ. Math. J.* **52** (2003), p. 157-169.

[14] H. L. Royden, "The extension of regular holomorphic maps", *Proc. Amer. Math. Soc.* **43** (1974), p. 306-310.

[15] W. Rudin, *Real and complex analysis*, third ed., McGraw-Hill Book Co., New York, 1987, xiv+416 pages.

Manuscrit reçu le 21 octobre 2005,
accepté le 13 mars 2006.

Barbara DRINOVEC DRNOVŠEK
University of Ljubljana
Institute of Mathematics, Physics and Mechanics
Jadranska 19
SI-1000 Ljubljana (Slovenia)
Barbara.Drinovec@fmf.uni-lj.si