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COMPARISON OF THE REFINED ANALYTIC AND THE BURGHELEA-HALLER TORSIONS

by Maxim BRAVERMAN & Thomas KAPPELER (*)

Abstract. — The refined analytic torsion associated to a flat vector bundle over a closed odd-dimensional manifold canonically defines a quadratic form τ on the determinant line of the cohomology. Both τ and the Burghelea-Haller torsion are refinements of the Ray-Singer torsion. We show that whenever the Burghelea-Haller torsion is defined it is equal to $\pm\tau$. As an application we obtain new results about the Burghelea-Haller torsion. In particular, we prove a weak version of the Burghelea-Haller conjecture relating their torsion with the square of the Farber-Turaev combinatorial torsion.

Résumé. — La torsion analytique raffinée, associée à un fibré vectoriel plat sur une variété fermée et orientée de dimension impaire, définit d'une manière canonique une forme quadratique τ sur le déterminant de la cohomologie. La torsion introduite par Burghelea et Haller et la forme quadratique τ sont des concepts raffinés de la torsion analytique de Ray-Singer. On démontre que dans le cas où la torsion de Burghelea-Haller est définie, elle est identique à $\pm\tau$. Comme application, on obtient des résultats nouveaux pour la torsion de Burghelea-Haller. En particulier, on démontre une version faible de la conjecture de Burghelea-Haller concernant leur torsion et le carré de la torsion combinatoire de Farber-Turaev.

1. Introduction

1.1. The refined analytic torsion

Let M be a closed oriented manifold of odd dimension $d = 2r-1$ and let E be a complex vector bundle over M endowed with a flat connection ∇. In a series of papers [4, 6, 7], we defined and studied the non-zero element

$$\rho_{\mathrm{an}} = \rho_{\mathrm{an}}(\nabla) \in \mathrm{Det}\big(H^\bullet(M,E)\big)$$

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of the determinant line $\mathrm{Det}\big(H^\bullet(M,E)\big)$ of the cohomology $H^\bullet(M,E)$ of M with coefficients in E. This element, called the *refined analytic torsion*, can be viewed as an analogue of the refinement of the Reidemeister torsion due to Turaev [24, 25, 26] and, in a more general context, to Farber-Turaev [15, 16]. The refined analytic torsion carries information about the Ray-Singer metric and about the η-invariant of the odd signature operator associated to ∇ and a Riemannian metric on M. In particular, if ∇ is a hermitian connection, then the Ray-Singer norm of $\rho_{\mathrm{an}}(\nabla)$ is equal to 1. One of the main properties of the refined analytic torsion is that it depends holomorphically on ∇. Using this property we computed the ratio between the refined analytic torsion and the Farber-Turaev torsion up to a factor, which is locally constant on the space of flat connections and is equal to one on every connected component which contains a Hermitian connection, *cf.* Th. 14.5 of [4] and Th. 5.11 of [6]. This result extends the classical Cheeger-Müller theorem about the equality between the Ray-Singer and the Reidemeister torsions [23, 13, 21, 22, 2].

1.2. Quadratic form associated with ρ_{an}

We define the quadratic form $\tau = \tau_\nabla$ on the determinant line $\mathrm{Det}\big(H^\bullet(M,E)\big)$ by setting

$$\tau(\rho_{\mathrm{an}}) = e^{-2\pi i\left(\eta(\nabla)-\mathrm{rank}\,E\cdot\eta_{\mathrm{trivial}}\right)}, \tag{1.1}$$

where $\eta(\nabla)$ stands for the η-invariant of the restriction to the even forms of the odd signature operator, associated to the flat vector bundle (E,∇) and a Riemannian metric on M (*cf.* Definition 2.2), and η_{trivial} is the η-invariant of the trivial line bundle over M.

Properties of ρ_{an}, such as its metric independence or its analyticity established in [4, 7, 6] easily translate into corresponding properties of τ_∇ — see Subsection 1.5.

Remark 1.1. — The difference $\eta(\nabla) - \mathrm{rank}\,E\cdot\eta_{\mathrm{trivial}}$ in (1.1) is called the ρ-invariant of (E,∇) and its reduction modulo $\mathbb{Z}$ is independent of the Riemannian metric.

In the subsequent work [3] we show that τ_∇ can be defined directly, without going through the construction of ρ_{an}.

1.3. The Burghelea-Haller complex Ray-Singer torsion

On a different line of thoughts, Burghelea and Haller [10, 12] have introduced a refinement of the *square* of the Ray-Singer torsion for a closed manifold of arbitrary dimension, *provided that the complex vector bundle E admits a non-degenerate complex valued symmetric bilinear form b.* They defined a complex valued quadratic form

$$\tau^{\mathrm{BH}} = \tau^{\mathrm{BH}}_{b,\nabla} \tag{1.2}$$

on the determinant line $\mathrm{Det}\big(H^\bullet(M,E)\big)$, which depends holomorphically on the flat connection ∇ and is closely related to (the square of) the Ray-Singer torsion. Burghelea and Haller then defined a complex valued quadratic form, referred to as *complex Ray-Singer torsion*. In the case of a closed manifold M of odd dimension it is given by

$$\tau^{\mathrm{BH}}_{b,\alpha,\nabla} := e^{-2\int_M \omega_{\nabla,b}\wedge\alpha}\cdot\tau^{\mathrm{BH}}_{b,\nabla}, \tag{1.3}$$

where $\alpha \in \Omega^{d-1}(M)$ is an arbitrary closed $(d-1)$-form and $\omega_{\nabla,b} \in \Omega^1(M)$ is the Kamber-Tondeur form, *cf.* [12, §2] — see the discussion at the end of Section 5 of [12] for the reasons to introduce this extra factor. Burghelea and Haller conjectured that, for a suitable choice of α, the form $\tau^{\mathrm{BH}}_{b,\alpha,\nabla}$ is roughly speaking equal to the square of the Farber-Turaev torsion, *cf.* [12, Conjecture 5.1] and Theorem 1.3 below.

Note that τ^{BH} seems not to be related to the η-invariant, whereas the refined analytic torsion is closely related to it. In fact, our study of ρ_{an} leads to new results about η, *cf.* [4, Th. 14.10, 14.12] and [6, Prop. 6.2, Cor. 6.4].

1.4. The comparison theorem

The main result of this paper is the following theorem establishing a relationship between the refined analytic torsion and the Burghelea-Haller quadratic form.

THEOREM 1.2. — *Suppose M is a closed oriented manifold of odd dimension $d = 2r-1$ and let E be a complex vector bundle over M endowed with a flat connection ∇. Assume that there exists a symmetric bilinear form b on E so that the quadratic form* (1.2) *on* $\mathrm{Det}\big(H^\bullet(M,E)\big)$ *is defined. Then $\tau^{\mathrm{BH}}_{b,\nabla} = \pm\tau_\nabla$, i.e.,*

$$\tau^{\mathrm{BH}}_{b,\nabla}\big(\rho_{\mathrm{an}}(\nabla)\big) = \pm e^{-2\pi i\left(\eta(\nabla)-\mathrm{rank}\,E\cdot\eta_{\mathrm{trivial}}\right)}. \tag{1.4}$$

The proof is given in Section 4.

Theorem 1.2 implies that for manifolds of odd dimension, the inconvenient assumption of the existence of a non-degenerate complex valued symmetric bilinear form b for the definition of the Burghelea-Haller torsion can be avoided, by defining the quadratic form via the refined analytic torsion as in (1.1).

The relation between $\rho_{\rm an}$ and τ (and, hence, when there exists a quadratic form b, with $\tau^{\rm BH}$) takes an especially simple form, when the bundle (E,∇) is acyclic, *i.e.*, when $H^\bullet(M,E)=0$. Then the determinant line bundle $\operatorname{Det}\big(H^\bullet(M,E)\big)$ is canonically isomorphic to $\mathbb{C}$ and both, τ and $\rho_{\rm an}$, can be viewed as non-zero complex numbers and (1.1) takes the form

$$\tau_\nabla = \Big(\rho_{\rm an}(\nabla)\cdot e^{\pi i(\eta(\nabla)-\operatorname{rank}E\cdot\eta_{\rm trivial})}\Big)^{-2}. \tag{1.5}$$

In general, τ_∇ (and, hence, $\tau^{\rm BH}_\nabla$) does not admit a square root which is holomorphic in ∇, *cf.* Remark 5.12 and the discussion after it in [12]. In particular, the product $\rho_{\rm an}\cdot e^{\pi i(\eta(\nabla)-\operatorname{rank}E\cdot\eta_{\rm trivial})}$ is not a holomorphic function of ∇, since $e^{\pi i(\eta(\nabla)-\operatorname{rank}E\cdot\eta_{\rm trivial})}$ is not even continuous in ∇. Thus the refined analytic torsion can be viewed as a modified version of the inverse square root of τ_∇, which is holomorphic.

1.5. Properties of the quadratic forms τ and $\tau^{\rm BH}$

As an application of our previous papers [4, 7, 6] we obtain various results about the quadratic form τ, some of them generalizing known properties of the Burghelea-Haller torsion $\tau^{\rm BH}$. In particular, we show that τ is independent of the choice of the Riemannian metric. As an application of Theorem 1.2 one sees that $\tau^{\rm BH}_{b,\alpha,\nabla}$ is invariant under the deformation of the non-degenerate bilinear form b (*cf.* Theorem 5.1) — a result, which was first proven by Burghelea and Haller [12, Th. 4.2]. We also slightly improve this result, *cf.* Theorem 5.2.

Next we discuss our main application of Theorem 1.2.

1.6. Comparison between the Farber-Turaev and the Burghelea-Haller torsions

In [12], Burghelea and Haller made a conjecture relating the quadratic form (1.3) with the refinement of the combinatorial torsion introduced by

Turaev [24, 25, 26] and, in a more general context, by Farber and Turaev [15, 16], *cf.* [12, Conjecture 5.1]. Recall that the Turaev torsion depends on the Euler structure ε and a choice of a cohomological orientation, i.e, an orientation $\mathfrak{o}$ of the determinant line of the cohomology $H^\bullet(M,\mathbb{R})$ of M. The set of Euler structures $\mathrm{Eul}(M)$, introduced by Turaev, is an affine version of the integer homology $H_1(M,\mathbb{Z})$ of M. It has several equivalent descriptions [24, 25, 8, 11]. For our purposes, it is convenient to adopt the definition from Section 6 of [25], where an Euler structure is defined as an equivalence class of nowhere vanishing vector fields on M — see [25, §5] for the description of the equivalence relation. The definition of the Turaev torsion was reformulated by Farber and Turaev [15, 16]. The Farber-Turaev torsion, depending on ε, $\mathfrak{o}$, and ∇, is an element of the determinant line $\mathrm{Det}\,\big(H^\bullet(M,E)\big)$, which we denote by $\rho_{\varepsilon,\mathfrak{o}}(\nabla)$.

Though Burghelea and Haller stated their conjecture for manifolds of arbitrary dimensions, we restrict our formulation to the odd dimensional case. Suppose M is a closed oriented odd dimensional manifold. Let $\varepsilon\in\mathrm{Eul}(M)$ be an Euler structure on M represented by a non-vanishing vector field X. Fix a Riemannian metric g^M on M and let $\Psi(g^M)\in\Omega^{d-1}(TM\backslash\{0\})$ denote the Mathai-Quillen form, [20, §7], [2, pp. 40-44]. Set

$$\alpha_\varepsilon=\alpha_\varepsilon(g^M):=X^*\Psi(g^M)\in\Omega^{d-1}(M).$$

This is a closed differential form, whose cohomology class $[\alpha_\varepsilon]\in H^{d-1}(M,\mathbb{R})$ is closely related to the integer cohomology class, introduced by Turaev [25, §5.3] and called *the characteristic class* $c(\varepsilon)\in H_1(M,\mathbb{Z})$ *associated to an Euler structure* ε. More precisely, let $\mathrm{PD}:H_1(M,\mathbb{Z})\to H^{d-1}(M,\mathbb{Z})$ denote the Poincaré isomorphism. For $h\in H_1(M,\mathbb{Z})$ we denote by $\mathrm{PD}'(h)$ the image of $\mathrm{PD}(h)$ in $H^{d-1}(M,\mathbb{R})$. Then

$$\mathrm{PD}'\,\big(c([X])\big)=-2[\alpha_\varepsilon]=-2[X^*\Psi(g^M)],\tag{1.6}$$

and, hence,

$$2\int_M\omega_{\nabla,b}\wedge\alpha_\varepsilon=-\langle[\omega_{\nabla,b}],c(\varepsilon)\rangle,\tag{1.7}$$

where $\omega_{\nabla,b}\in\Omega^1(M)$ is the Kamber-Tondeur form, *cf.* (1.3).

Note that (1.6) implies that $2\alpha_\varepsilon$ represents an integer class in $H^{d-1}(M,\mathbb{R})$.

The following result is the original Burghelea-Haller conjecture [12]. It was proven independently by Burghelea-Haller and Su-Zhang after the first version of our paper was posted to the archive — *cf.* Subsection 1.8.

THEOREM 1.3. — *Assume that (E,∇) is a flat vector bundle over M which admits a non-degenerate symmetric bilinear form b. Then*

$$\tau^{\mathrm{BH}}_{b,\alpha_\varepsilon,\nabla}\big(\rho_{\varepsilon,\mathfrak{o}}(\nabla)\big)=1, \tag{1.8}$$

or, equivalently,

$$\tau^{\mathrm{BH}}_{b,\nabla}\big(\rho_{\varepsilon,\mathfrak{o}}(\nabla)\big)=e^{2\int_M\omega_{\nabla,b}\wedge\alpha_\varepsilon}. \tag{1.9}$$

1.7. A generalization of the Burghelea-Haller conjecture

Following Farber [14], we denote by Arg_∇ the unique cohomology class $\mathrm{Arg}_\nabla\in H^1(M,\mathbb{C}/\mathbb{Z})$ such that for every closed curve γ in M we have

$$\det\big(\mathrm{Mon}_\nabla(\gamma)\big)=\exp\big(2\pi i\langle\mathrm{Arg}_\nabla,[\gamma]\rangle\big), \tag{1.10}$$

where $\mathrm{Mon}_\nabla(\gamma)$ denotes the monodromy of the flat connection ∇ along the curve γ and $\langle\cdot,\cdot\rangle$ denotes the natural pairing $H^1(M,\mathbb{C}/\mathbb{Z})\times H_1(M,\mathbb{Z})\to\mathbb{C}/\mathbb{Z}$.

By Lemma 2.2 of [12] we get

$$e^{-\langle[\omega_{\nabla,b}],c(\varepsilon)\rangle}=\pm\det\mathrm{Mon}_\nabla\big(c(\varepsilon)\big)=\pm e^{2\pi i\langle\mathrm{Arg}_\nabla,c(\varepsilon)\rangle}. \tag{1.11}$$

(Note that $\mathrm{Mon}_\nabla(\gamma)$ is equal to the *inverse* of what is denoted by $\mathrm{hol}^E_x(\gamma)$ in [12]).

Combining (1.7), (1.10) and (1.11) we obtain

$$e^{2\int_M\omega_{\nabla,b}\wedge\alpha_\varepsilon}=\pm e^{2\pi i\langle\mathrm{Arg}_\nabla,c(\varepsilon)\rangle}.$$

Thus, up to sign, the Burghelea-Haller conjecture (1.9) can be rewritten as

$$\tau^{\mathrm{BH}}_{b,\nabla}\big(\rho_{\varepsilon,\mathfrak{o}}(\nabla)\big)=\pm e^{2\pi i\langle\mathrm{Arg}_\nabla,c(\varepsilon)\rangle}. \tag{1.12}$$

In view of Theorem 1.2 we make the following stronger conjecture involving τ_∇ instead of $\tau^{\mathrm{BH}}_{b,\alpha_\varepsilon,\nabla}$, and, hence, meaningful also in the situation, when the bundle E does *not* admit a non-degenerate symmetric bilinear form.

CONJECTURE 1.4. — *Assume that (E,∇) is a flat vector bundle over M. Then*

$$\tau_\nabla\big(\rho_{\varepsilon,\mathfrak{o}}(\nabla)\big)=e^{2\pi i\langle\mathrm{Arg}_\nabla,c(\varepsilon)\rangle}, \tag{1.13}$$

or, equivalently,

$$e^{\pi i\left(\eta(\nabla)-\mathrm{rank}\,E\cdot\eta_{\mathrm{trivial}}\right)}\cdot\rho_{\mathrm{an}}(\nabla)=\pm e^{-\pi i\langle\mathrm{Arg}_\nabla,c(\varepsilon)\rangle}\cdot\rho_{\varepsilon,\mathfrak{o}}(\nabla). \tag{1.14}$$

Clearly Conjecture 1.4 implies (1.8) up to sign.

Remark 1.5. — By construction, the left hand side of (1.14) is independent of the Euler structure ε and the cohomological orientation $\mathfrak{o}$, while the right hand side of (1.14) is independent of the Riemannian metric g^M. Note that the fact that $e^{\pi i\left(\eta(\nabla)-\operatorname{rank} E\cdot\eta_{\text{trivial}}\right)}\cdot\rho_{\text{an}}(\nabla)$ is independent of g^M up to sign follows immediately from Lemma 9.2 of [7], while the fact that $e^{-\pi i\langle \operatorname{Arg}_\nabla, c(\varepsilon)\rangle}\cdot\rho_{\varepsilon,\mathfrak{o}}(\nabla)$ is independent of ε and independent of $\mathfrak{o}$ up to sign is explained on page 212 of [16].

In Theorem 5.1 of [6] we computed the ratio of the refined analytic and the Farber-Turaev torsions. Using this result and Theorem 1.2 we establish the following weak version of Conjecture 1.4 (and, hence, of (1.8)).

THEOREM 1.6.

(i) *Under the same assumptions as in Conjecture 1.4, for each connected component $\mathcal{C}$ of the set* $\operatorname{Flat}(E)$ *of flat connections on E there exists a constant $R_{\mathcal{C}}$ with $|R_{\mathcal{C}}|=1$, such that*

$$\tau_\nabla\big(\rho_{\varepsilon,\mathfrak{o}}(\nabla)\big)=R_{\mathcal{C}}\cdot e^{2\pi i\langle \operatorname{Arg}_\nabla, c(\varepsilon)\rangle},\quad \textit{for all}\quad \nabla\in\mathcal{C}. \tag{1.15}$$

(ii) *If the connected component $\mathcal{C}$ contains an acyclic Hermitian connection then $R_{\mathcal{C}}=1$, i.e.,*

$$\tau_\nabla\big(\rho_{\varepsilon,\mathfrak{o}}(\nabla)\big)=e^{2\pi i\langle \operatorname{Arg}_\nabla, c(\varepsilon)\rangle},\quad \textit{for all}\quad \nabla\in\mathcal{C}. \tag{1.16}$$

The proof is given in Subsection 5.2.

Remark 1.7.

(i) The second part of Theorem 1.6 is due to Rung-Tzung Huang, who also proved it in the case when $\mathcal{C}$ contains a Hermitian connection which is not necessarily acyclic, [18].

(ii) It was brought to our attention by Stefan Haller that one can modify the arguments of our proofs of Theorem 1.2 and of [6, Th. 5.1] so that they can be applied directly to the Burghelea-Haller torsion. It might lead to a proof of an analogue of Theorem 1.6 for $\tau^{\text{BH}}_{\nabla,b}$ on an even dimensional manifold.

1.8. Added in proofs

When the first version of our paper was posted in the archive Theorem 1.3 was still a conjecture. Since then a lot of progress has been made. First, Huang [18] showed that if the connected component $\mathcal{C}\subset\operatorname{Flat}(E)$ contains a Hermitian connection, then the constant $R_{\mathcal{C}}$ of Theorem 1.6

is equal to 1. Part of his result is now incorporated in item (ii) of our Theorem 1.8. Later Burghelea and Haller (D. Burghelea and S. Haller, *Complex valued Ray-Singer torsion II*, `arXiv:math.DG/0610875`) proved the equality (1.8) up to sign. Independently and at the same time Su and Zhang (G. Su and W. Zhang, *A Cheeger-Mueller theorem for symmetric bilinear torsions*, `arXiv:math.DG/0610577`) proved Theorem 1.3 in full generality. Both proofs used methods completely different from ours. In fact, Burghelea-Haller, following [9], and Su-Zhang, following [2], study a Witten-type deformation of the non-self adjoint Laplacian (3.3) and adopt all arguments of these papers to the new situation. In contrast, our Theorem 1.6 provides a "low-tech" approach to the Burghelea-Haller conjecture and, more generally, to Conjecture 1.4. On the other side, it would be interesting to see if the methods of Burghelea-Haller and Su-Zhang can be used to prove Conjecture 1.4.

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2. The refined analytic torsion

In this section we recall the definition of the refined analytic torsion from [7]. The refined analytic torsion is constructed in 3 steps: first, we define the notion of refined torsion of a finite dimensional complex endowed with a chirality operator, *cf.* Definition 2.1. Then we fix a Riemannian metric g^M on M and consider the odd signature operator $\mathcal{B} = \mathcal{B}(\nabla, g^M)$ associated to a flat vector bundle (E, ∇), *cf.* Definition 2.2. Using the *graded determinant* of $\mathcal{B}$ and the definition of the refined torsion of a finite dimensional complex with a chirality operator we construct an element $\rho = \rho(\nabla, g^M)$ in the determinant line of the cohomology, *cf.* (2.14). The element ρ is almost the refined analytic torsion. However, it might depend on the Riemannian metric g^M (though it does not if $\dim M \equiv 1 (\mathrm{mod}\, 4)$ or if $\mathrm{rank}(E)$ is divisible by 4). Finally we "correct" ρ by multiplying it by an explicit factor, the metric anomaly of ρ, to obtain a diffeomorphism invariant $\rho_{\mathrm{an}}(\nabla)$ of the triple (M, E, ∇), *cf.* Definition 2.6.

2.1. The determinant line of a complex

Given a complex vector space V of dimension $\dim V = n$, the *determinant line* of V is the line $\mathrm{Det}(V) := \Lambda^n V$, where $\Lambda^n V$ denotes the n-th exterior power of V. By definition, we set $\mathrm{Det}(0) := \mathbb{C}$. Further, we denote by $\mathrm{Det}(V)^{-1}$ the dual line of $\mathrm{Det}(V)$. Let

$$(C^\bullet, \partial):\quad 0 \to C^0 \xrightarrow{\ \partial\ } C^1 \xrightarrow{\ \partial\ } \cdots \xrightarrow{\ \partial\ } C^d \to 0 \tag{2.1}$$

be a complex of finite dimensional complex vector spaces. We call the integer d the *length* of the complex $(C^\bullet, \partial)$ and denote by $H^\bullet(\partial) = \bigoplus_{i=0}^d H^i(\partial)$ the cohomology of $(C^\bullet, \partial)$. Set

$$\mathrm{Det}(C^\bullet) := \bigotimes_{j=0}^{d} \mathrm{Det}(C^j)^{(-1)^j}, \quad \mathrm{Det}(H^\bullet(\partial)) := \bigotimes_{j=0}^{d} \mathrm{Det}(H^j(\partial))^{(-1)^j}. \tag{2.2}$$

The lines $\mathrm{Det}(C^\bullet)$ and $\mathrm{Det}(H^\bullet(\partial))$ are referred to as the *determinant line of the complex* $C^\bullet$ and the *determinant line of its cohomology*, respectively. There is a canonical isomorphism

$$\phi_{C^\bullet} = \phi_{(C^\bullet,\partial)} : \mathrm{Det}(C^\bullet) \longrightarrow \mathrm{Det}(H^\bullet(\partial)), \tag{2.3}$$

cf. for example, §2.4 of [7].

2.2. The refined torsion of a finite dimensional complex with a chirality operator

Let $d = 2r - 1$ be an odd integer and let $(C^\bullet, \partial)$ be a length d complex of finite dimensional complex vector spaces. A *chirality operator* is an involution $\Gamma : C^\bullet \to C^\bullet$ such that $\Gamma(C^j) = C^{d-j}$, $j = 0, \dots, d$. For $c_j \in \mathrm{Det}(C^j)$ ($j = 0, \dots, d$) we denote by $\Gamma c_j \in \mathrm{Det}(C^{d-j})$ the image of c_j under the isomorphism $\mathrm{Det}(C^j) \to \mathrm{Det}(C^{d-j})$ induced by Γ. Fix non-zero elements $c_j \in \mathrm{Det}(C^j)$, $j = 0, \dots, r-1$ and denote by c_j^{-1} the unique element of $\mathrm{Det}(C^j)^{-1}$ such that $c_j^{-1}(c_j) = 1$. Consider the element

$$c_\Gamma := (-1)^{\mathcal{R}(C^\bullet)} \cdot c_0 \otimes c_1^{-1} \otimes \cdots \otimes c_{r-1}^{(-1)^{r-1}} \otimes (\Gamma c_{r-1})^{(-1)^r} \otimes (\Gamma c_{r-2})^{(-1)^{r-1}} \otimes \cdots \otimes (\Gamma c_0)^{-1} \tag{2.4}$$

of $\mathrm{Det}(C^\bullet)$, where

$$\mathcal{R}(C^\bullet) := \frac{1}{2} \sum_{j=0}^{r-1} \dim C^j \cdot \big(\dim C^j + (-1)^{r+j} \big). \tag{2.5}$$

It follows from the definition of c_j^{-1} that c_Γ is independent of the choice of c_j $(j=0,\dots,r-1)$.

DEFINITION 2.1. — *The* refined torsion *of the pair* $(C^\bullet,\Gamma)$ *is the element*

$$\rho_\Gamma = \rho_{C^\bullet,\Gamma} := \phi_{C^\bullet}(c_\Gamma) \in \operatorname{Det}\big(H^\bullet(\partial)\big), \tag{2.6}$$

where $\phi_{C^\bullet}$ *is the canonical map* (2.3).

2.3. The odd signature operator

Let M be a smooth closed oriented manifold of odd dimension $d=2r-1$ and let (E,∇) be a flat vector bundle over M. We denote by $\Omega^k(M,E)$ the space of smooth differential forms on M of degree k with values in E and by

$$\nabla : \Omega^\bullet(M,E) \longrightarrow \Omega^{\bullet+1}(M,E)$$

the covariant differential induced by the flat connection on E. Fix a Riemannian metric g^M on M and let $* : \Omega^\bullet(M,E) \to \Omega^{d-\bullet}(M,E)$ denote the Hodge $*$-operator. Define the *chirality operator* $\Gamma = \Gamma(g^M) : \Omega^\bullet(M,E) \to \Omega^\bullet(M,E)$ by the formula

$$\Gamma\omega := i^r(-1)^{\frac{k(k+1)}{2}} * \omega, \quad \omega \in \Omega^k(M,E), \tag{2.7}$$

with r given as above by $r=\frac{d+1}{2}$. The numerical factor in (2.7) has been chosen so that $\Gamma^2=1$, *cf.* Proposition 3.58 of [1].

DEFINITION 2.2. — *The* odd signature operator *is the operator*

$$\mathcal{B} = \mathcal{B}(\nabla, g^M) := \Gamma\nabla + \nabla\Gamma : \Omega^\bullet(M,E) \longrightarrow \Omega^\bullet(M,E). \tag{2.8}$$

We denote by $\mathcal{B}_k$ *the restriction of* $\mathcal{B}$ *to the space* $\Omega^k(M,E)$.

2.4. The graded determinant of the odd signature operator

Note that for each $k=0,\dots,d$, the operator $\mathcal{B}^2$ maps $\Omega^k(M,E)$ into itself. Suppose $\mathcal{I}$ is an interval of the form $[0,\lambda]$, $(\lambda,\mu]$, or (λ,∞) $(\mu>\lambda\geqslant 0)$. Denote by $\Pi_{\mathcal{B}^2,\mathcal{I}}$ the spectral projection of $\mathcal{B}^2$ corresponding to the set of eigenvalues, whose absolute values lie in $\mathcal{I}$. Set

$$\Omega^\bullet_{\mathcal{I}}(M,E) := \Pi_{\mathcal{B}^2,\mathcal{I}}\big(\Omega^\bullet(M,E)\big) \subset \Omega^\bullet(M,E).$$

If the interval $\mathcal{I}$ is bounded, then, *cf.* Section 6.10 of [7], the space $\Omega^\bullet_{\mathcal{I}}(M,E)$ is finite dimensional.

For each $k=0,\dots,d$, set

$$\begin{aligned}\Omega^k_{+,\mathcal{I}}(M,E) &:= \operatorname{Ker}(\nabla\Gamma)\cap\Omega^k_{\mathcal{I}}(M,E) = \big(\Gamma(\operatorname{Ker}\nabla)\big)\cap\Omega^k_{\mathcal{I}}(M,E);\\ \Omega^k_{-,\mathcal{I}}(M,E) &:= \operatorname{Ker}(\Gamma\nabla)\cap\Omega^k_{\mathcal{I}}(M,E) = \operatorname{Ker}\nabla\cap\Omega^k_{\mathcal{I}}(M,E).\end{aligned} \tag{2.9}$$

Then

$$\Omega^k_{\mathcal{I}}(M,E) = \Omega^k_{+,\mathcal{I}}(M,E)\oplus\Omega^k_{-,\mathcal{I}}(M,E) \quad \text{if} \quad 0\notin\mathcal{I}. \tag{2.10}$$

We consider the decomposition (2.10) as a *grading* [(1)] of the space $\Omega^\bullet_{\mathcal{I}}(M,E)$, and refer to $\Omega^k_{+,\mathcal{I}}(M,E)$ and $\Omega^k_{-,\mathcal{I}}(M,E)$ as the positive and negative subspaces of $\Omega^k_{\mathcal{I}}(M,E)$.

Set

$$\Omega^{\text{even}}_{\pm,\mathcal{I}}(M,E) = \bigoplus_{p=0}^{r-1}\Omega^{2p}_{\pm,\mathcal{I}}(M,E)$$

and let $\mathcal{B}^{\mathcal{I}}$ and $\mathcal{B}^{\mathcal{I}}_{\text{even}}$ denote the restrictions of $\mathcal{B}$ to the subspaces $\Omega^\bullet_{\mathcal{I}}(M,E)$ and $\Omega^{\text{even}}_{\mathcal{I}}(M,E)$ respectively. Then $\mathcal{B}^{\mathcal{I}}_{\text{even}}$ maps $\Omega^{\text{even}}_{\pm,\mathcal{I}}(M,E)$ to itself. Let $\mathcal{B}^{\pm,\mathcal{I}}_{\text{even}}$ denote the restriction of $\mathcal{B}^{\mathcal{I}}_{\text{even}}$ to the space $\Omega^{\text{even}}_{\pm,\mathcal{I}}(M,E)$. Clearly, the operators $\mathcal{B}^{\pm,\mathcal{I}}_{\text{even}}$ are bijective whenever $0\notin\mathcal{I}$.

Definition 2.3. — *Suppose $0\notin\mathcal{I}$. The* graded determinant *of the operator $\mathcal{B}^{\mathcal{I}}_{\text{even}}$ is defined by*

$$\operatorname{Det}_{\text{gr},\theta}(\mathcal{B}^{\mathcal{I}}_{\text{even}}) := \frac{\operatorname{Det}_\theta(\mathcal{B}^{+,\mathcal{I}}_{\text{even}})}{\operatorname{Det}_\theta(-\mathcal{B}^{-,\mathcal{I}}_{\text{even}})} \in \mathbb{C}\backslash\{0\}, \tag{2.11}$$

where $\operatorname{Det}_\theta$ denotes the ζ-regularized determinant associated to the Agmon angle $\theta\in(-\pi,0)$, cf. for example, §6 of [7].

It follows from formula (6.17) of [7] that (2.11) is independent of the choice of $\theta\in(-\pi,0)$.

2.5. The canonical element of the determinant line

Since the covariant differentiation ∇ commutes with $\mathcal{B}$, the subspace $\Omega^\bullet_{\mathcal{I}}(M,E)$ is a subcomplex of the twisted de Rham complex $(\Omega^\bullet(M,E),\nabla)$. Clearly, for each $\lambda\geqslant 0$, the complex $\Omega^\bullet_{(\lambda,\infty)}(M,E)$ is acyclic. Since

$$\Omega^\bullet(M,E) = \Omega^\bullet_{[0,\lambda]}(M,E)\oplus\Omega^\bullet_{(\lambda,\infty)}(M,E), \tag{2.12}$$

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(1) Note, that our grading is opposite to the one considered in [9, §2].

the cohomology $H^\bullet_{[0,\lambda]}(M,E)$ of the complex $\Omega^\bullet_{[0,\lambda]}(M,E)$ is naturally isomorphic to the cohomology $H^\bullet(M,E)$. Let $\Gamma_{\mathcal{I}}$ denote the restriction of Γ to $\Omega^\bullet_{\mathcal{I}}(M,E)$. For each $\lambda \geqslant 0$, let

$$\rho_{\Gamma_{[0,\lambda]}} = \rho_{\Gamma_{[0,\lambda]}}(\nabla, g^M) \in \operatorname{Det}(H^\bullet_{[0,\lambda]}(M,E)) \tag{2.13}$$

denote the refined torsion of the finite dimensional complex $(\Omega^\bullet_{[0,\lambda]}(M,E),\nabla)$ corresponding to the chirality operator $\Gamma_{[0,\lambda]}$, *cf.* Definition 2.1. We view $\rho_{\Gamma_{[0,\lambda]}}$ as an element of $\operatorname{Det}(H^\bullet(M,E))$ via the canonical isomorphism between $H^\bullet_{[0,\lambda]}(M,E)$ and $H^\bullet(M,E)$.

It is shown in Proposition 7.8 of [7] that the nonzero element

$$\rho(\nabla) = \rho(\nabla, g^M) := \operatorname{Det}_{\mathrm{gr},\theta}(\mathcal{B}^{(\lambda,\infty)}_{\mathrm{even}}) \cdot \rho_{\Gamma_{[0,\lambda]}} \in \operatorname{Det}(H^\bullet(M,E)) \tag{2.14}$$

is independent of the choice of $\lambda \geqslant 0$. Further, $\rho(\nabla)$ is independent of the choice of the Agmon angle $\theta \in (-\pi, 0)$ of $\mathcal{B}_{\mathrm{even}}$. However, in general, $\rho(\nabla)$ might depend on the Riemannian metric g^M (it is independent of g^M if $\dim M \equiv 3 (\mathrm{mod}\, 4)$). The refined analytic torsion, *cf.* Definition 2.6, is a slight modification of $\rho(\nabla)$, which is independent of g^M.

2.6. The η-invariant

First, we recall the definition of the η-function of a non-self-adjoint elliptic operator D, *cf.* [17]. Let $C^\infty(M,E) \to C^\infty(M,E)$ be an elliptic differential operator of order $m \geqslant 1$ whose leading symbol is self-adjoint with respect to some given Hermitian metric on E. Assume that θ is an Agmon angle for D (*cf.* for example, Definition 3.3 of [4]). Let $\Pi_>$ (resp. $\Pi_<$) be the spectral projection whose image contains the span of all generalized eigenvectors of D corresponding to eigenvalues λ with $\operatorname{Re}\lambda > 0$ (resp. with $\operatorname{Re}\lambda < 0$) and whose kernel contains the span of all generalized eigenvectors of D corresponding to eigenvalues λ with $\operatorname{Re}\lambda \leqslant 0$ (resp. with $\operatorname{Re}\lambda \geqslant 0$). For all complex s with $\operatorname{Re} s < -d/m$, we define the η-function of D by the formula

$$\eta_\theta(s, D) = \zeta_\theta(s, \Pi_>, D) - \zeta_\theta(s, \Pi_<, -D), \tag{2.15}$$

where $\zeta_\theta(s,\Pi_>,D) := \operatorname{Tr}(\Pi_> D^s)$ and, similarly, $\zeta_\theta(s,\Pi_<,D) := \operatorname{Tr}(\Pi_< D^s)$. Note that, by the above definition, the purely imaginary eigenvalues of D do not contribute to $\eta_\theta(s,D)$.

It was shown by Gilkey, [17], that $\eta_\theta(s,D)$ has a meromorphic extension to the whole complex plane $\mathbb{C}$ with isolated simple poles, and that it is regular at 0. Moreover, the number $\eta_\theta(0,D)$ is independent of the Agmon angle θ.

Since the leading symbol of D is self-adjoint, the angles $\pm\pi/2$ are principal angles for D. Hence, there are at most finitely many eigenvalues of D on the imaginary axis. Let $m_+(D)$ (resp., $m_-(D)$) denote the number of eigenvalues of D, counted with their algebraic multiplicities, on the positive (resp., negative) part of the imaginary axis. Let $m_0(D)$ denote the algebraic multiplicity of 0 as an eigenvalue of D.

Definition 2.4. — *The η-invariant $\eta(D)$ of D is defined by the formula*

$$\eta(D) = \frac{\eta_\theta(0,D) + m_+(D) - m_-(D) + m_0(D)}{2}. \tag{2.16}$$

As $\eta_\theta(0,D)$ is independent of the choice of the Agmon angle θ for D, *cf.* [17], so is $\eta(D)$.

Remark 2.5. — Note that our definition of $\eta(D)$ is slightly different from the one proposed by Gilkey in [17]. In fact, in our notation, Gilkey's η-invariant is given by $\eta(D) + m_-(D)$. Hence, reduced modulo integers, the two definitions coincide. However, the number $e^{i\pi\eta(D)}$ will be multiplied by $(-1)^{m_-(D)}$ if we replace one definition by the other. In this sense, Definition 2.4 can be viewed as a *sign refinement* of the definition given in [17].

Let ∇ be a flat connection on a complex vector bundle $E \to M$. Fix a Riemannian metric g^M on M and denote by

$$\eta(\nabla) = \eta\big(\mathcal{B}_{\mathrm{even}}(\nabla, g^M)\big) \tag{2.17}$$

the η-invariant of the restriction $\mathcal{B}_{\mathrm{even}}(\nabla, g^M)$ of the odd signature operator $\mathcal{B}(\nabla, g^M)$ to $\Omega^{\mathrm{even}}(M,E)$.

2.7. The refined analytic torsion

Let $\eta_{\mathrm{trivial}} = \eta_{\mathrm{trivial}}(g^M)$ denote the η-invariant of the operator $\mathcal{B}_{\mathrm{trivial}} = \Gamma d + d\Gamma : \Omega^\bullet(M) \to \Omega^\bullet(M)$. In other words, η_{trivial} is the η-invariant corresponding to the trivial line bundle $M \times \mathbb{C} \to M$ over M.

Definition 2.6. — *Let (E,∇) be a flat vector bundle on M. The* refined analytic torsion *is the element*

$$\begin{aligned} \rho_{\mathrm{an}} &= \rho_{\mathrm{an}}(\nabla) \\ &:= \rho(\nabla, g^M)\cdot \exp\Big(i\pi\cdot \operatorname{rank} E\cdot \eta_{\mathrm{trivial}}(g^M)\Big) \in \operatorname{Det}(H^\bullet(M,E)), \end{aligned} \tag{2.18}$$

where g^M is any Riemannian metric on M and $\rho(\nabla, g^M) \in \mathrm{Det}(H^\bullet(M,E))$ is defined by (2.14).

It is shown in Theorem 9.6 of [7] that $\rho_{\mathrm{an}}(\nabla)$ is independent of g^M.

Remark 2.7. — In [4, 7, 6] we introduced an alternative version of the refined analytic torsion. Consider an oriented manifold N whose oriented boundary is the disjoint union of two copies of M. Instead of the exponential factor in (2.18) we used the term

$$\exp\Big(\frac{i\pi\cdot\mathrm{rank}\,E}{2}\int_N L(p,g^M)\Big),$$

where $L(p,g^M)$ is the Hirzebruch L-polynomial in the Pontrjagin forms of any Riemannian metric on N which near M is the product of g^M and the standard metric on the half-line. The advantage of this definition is that the latter factor is simpler to calculate than $e^{i\pi\eta_{\mathrm{trivial}}}$. In addition, if $\dim M \equiv 3 (\mathrm{mod}\, 4)$, then $\int_M L(p,g^M) = 0$ and, hence, the refined analytic torsion then coincides with $\rho(\nabla, g^M)$. However, in general, this version of the refined analytic torsion depends on the choice of N (though only up to a multiplication by $i^{k\cdot\mathrm{rank}(E)}$ ($k\in\mathbb{Z}$)). For this paper, however, the definition (2.18) of the refined analytic torsion is slightly more convenient.

2.8. Relationship with the η-invariant

To simplify the notation set

(2.19)

$$T_\lambda = T_\lambda(\nabla, g^M, \theta) = \prod_{j=0}^{d}\Big(\mathrm{Det}_{2\theta}\Big[\big((\Gamma\nabla)^2 + (\nabla\Gamma)^2\big)\big|_{\Omega^j_{(\lambda,\infty)}(M,E)}\Big]\Big)^{(-1)^{j+1}j}$$

where $\theta \in (-\pi/2, 0)$ and both, θ and $\theta+\pi$, are Agmon angles for $\mathcal{B}_{\mathrm{even}}$ (hence, 2θ is an Agmon angle for $\mathcal{B}^2_{\mathrm{even}}$). We shall use the following proposition, *cf.* [7, Prop. 8.1]:

Proposition 2.8. — *Let ∇ be a flat connection on a vector bundle E over a closed Riemannian manifold (M, g^M) of odd dimension $d = 2r-1$. Assume $\theta \in (-\pi/2, 0)$ is such that both θ and $\theta+\pi$ are Agmon angles for the odd signature operator $\mathcal{B} = \mathcal{B}(\nabla, g^M)$. Then, for every $\lambda \geqslant 0$,*

$$\Big(\mathrm{Det}_{\mathrm{gr},2\theta}(\mathcal{B}^{(\lambda,\infty)}_{\mathrm{even}})\Big)^2 = T_\lambda \cdot e^{-2\pi i\eta(\nabla, g^M)}. \tag{2.20}$$

Note that Proposition 8.1 of [7] gives a similar formula for the logarithm of $\mathrm{Det}_{\mathrm{gr},2\theta}(\mathcal{B}_{\mathrm{even}}^{(\lambda,\infty)})$, thus providing a sign refined version of (2.20). In the present paper we won't need this refinement.

Proof. — Set

$$\eta_\lambda = \eta_\lambda(\nabla, g^M) := \eta(\mathcal{B}_{\mathrm{even}}^{(\lambda,\infty)}). \tag{2.21}$$

From Proposition 8.1 and equality (10.20) of [7] we obtain

$$\mathrm{Det}_{\mathrm{gr},2\theta}(\mathcal{B}_{\mathrm{even}}^{(\lambda,\infty)})^2 = T_\lambda \cdot e^{-2\pi i \eta_\lambda} \cdot e^{-i\pi \dim \Omega_{[0,\lambda]}^{\mathrm{even}}(M,E)}. \tag{2.22}$$

The operator $\mathcal{B}_{\mathrm{even}}^{[0,\lambda]}$ acts on the finite dimensional vector space $\Omega_{[0,\lambda]}^{\mathrm{even}}(M,E)$. Hence, $2\eta\big(\mathcal{B}_{\mathrm{even}}^{[0,\lambda]}\big) \in \mathbb{Z}$ and

$$2\eta\big(\mathcal{B}_{\mathrm{even}}^{[0,\lambda]}\big) \equiv \dim \Omega_{[0,\lambda]}^{\mathrm{even}}(M,E) \mod 2. \tag{2.23}$$

Since $\eta_\lambda = \eta\big(\mathcal{B}_{\mathrm{even}}\big) - \eta\big(\mathcal{B}_{\mathrm{even}}^{[0,\lambda]}\big)$, we obtain from (2.23) that

$$e^{-i\pi\left(2\eta_\lambda + \dim \Omega_{[0,\lambda]}^{\bullet}(M,E)\right)} = e^{-2i\pi\eta\left(\mathcal{B}_{\mathrm{even}}\right)}.$$

The equality (2.20) follows now from (2.22). □

3. The Burghelea-Haller quadratic form

In this section we recall the construction of the quadratic form on the determinant line $\mathrm{Det}\big(H^\bullet(M,E)\big)$ due to Burghelea and Haller, [12]. Throughout the section we assume that the vector bundle $E \to M$ admits a non-degenerate symmetric bilinear form b. Such a form, required for the construction of τ, might not exist on E, but there always exists an integer N such that on the direct sum $E^N = E \oplus \cdots \oplus E$ of N copies of E such a form exists, *cf.* Remark 4.6 of [12].

3.1. A quadratic form on the determinant line of the cohomology of a finite dimensional complex

Consider the complex (2.1) and assume that each vector space C^j ($j = 0,\dots,d$) is endowed with a non-degenerate symmetric bilinear form $b_j : C^j \times C^j \to \mathbb{C}$. Set $b = \oplus b_j$. Then b_j induces a bilinear form on the determinant line $\mathrm{Det}(C^j)$ and, hence, one obtains a bilinear form on the determinant line $\mathrm{Det}(C^\bullet)$. Using the isomorphism (2.3) we thus obtain a bilinear form on $\mathrm{Det}(H^\bullet(\partial))$. This bilinear form induces a quadratic form on $\mathrm{Det}(H^\bullet(\partial))$, which we denote by $\tau_{C^\bullet,b}$.

The following lemma establishes a relationship between $\tau_{C^\bullet,b}$ and the construction of Subsection 2.2 and is an immediate consequence of the definitions.

Lemma 3.1. — *Suppose that d is odd and that the complex $(C^\bullet,\partial)$ is endowed with a chirality operator Γ, cf. Subsection 2.2. Assume further that Γ preserves the bilinear form b, i.e., $b(\Gamma x,\Gamma y)=b(x,y)$, for all $x,y\in C^\bullet$. Then*

$$\tau_{C^\bullet,b}(\rho_\Gamma)=1 \tag{3.1}$$

where ρ_Γ is given by (2.6).

3.2. Determinant of the generalized Laplacian

Assume now that M is a compact oriented manifold and E is a flat vector bundle over M endowed with a non-degenerate symmetric bilinear form b. Then b together with the Riemannian metric g^M on M define a bilinear form

$$\mathfrak{b}:\Omega^\bullet(M,E)\times\Omega^\bullet(M,E)\to\mathbb{C} \tag{3.2}$$

in a natural way.

Let $\nabla:\Omega^\bullet(M,E)\to\Omega^{\bullet+1}(M,E)$ denote the flat connection on E and let $\nabla^\#:\Omega^\bullet(M,E)\to\Omega^{\bullet-1}(M,E)$ denote the formal transpose of ∇ with respect to $\mathfrak{b}$. Following Burghelea and Haller we define a (generalized) Laplacian

$$\Delta=\Delta_{g^M,b}:=\nabla^\#\nabla+\nabla\nabla^\#. \tag{3.3}$$

Given a Hermitian metric on E, Δ is not self-adjoint, but has a self-adjoint positive definite leading symbol, which is the same as the leading symbol of the usual Laplacian. In particular, Δ has a discrete spectrum, *cf.* [12, §4].

Suppose $\mathcal{I}$ is an interval of the form $[0,\lambda]$ or (λ,∞) and let $\Pi_{\Delta_k,\mathcal{I}}$ be the spectral projection of Δ corresponding to $\mathcal{I}$. Set

$$\widehat{\Omega}^k_{\mathcal{I}}(M,E):=\Pi_{\Delta_k,\mathcal{I}}\big(\Omega^k(M,E)\big)\subset\widehat{\Omega}^k(M,E),\quad k=0,\dots,d.$$

For each $\lambda\geqslant 0$, the space $\widehat{\Omega}^\bullet_{[0,\lambda]}(M,E)$ is a finite dimensional subcomplex of the de Rham complex $(\Omega^\bullet(M,E),\nabla)$, whose cohomology is isomorphic to $H^\bullet(M,E)$. Thus, according to Subsection 3.1, the bilinear form (3.2) restricted to $\widehat{\Omega}^\bullet(M,E)$ defines a quadratic form on the determinant line $\mathrm{Det}(H^\bullet(M,E))$, which we denote by $\tau_{[0,\lambda]}=\tau_{b,\nabla,[0,\lambda]}$.

Let $\Delta_k^{\mathcal{I}}$ denote the restriction of Δ_k to $\widehat{\Omega}^k_{\mathcal{I}}(M,E)$. Since the leading symbol of Δ is positive definite the ζ-regularized determinant $\operatorname{Det}'_\theta(\Delta_k^{\mathcal{I}})$ does not depend on the choice of the Agmon angle θ. Set

$$\tau_{b,\nabla,(\lambda,\infty)} := \prod_{j=0}^{d} \big(\operatorname{Det}'_\theta(\Delta_j^{(\lambda,\infty)})\big)^{(-1)^j j} \in \mathbb{C}\backslash\{0\}. \tag{3.4}$$

Note that both, $\tau_{b,\nabla,[0,\lambda]}$ and $\tau_{b,\nabla,(\lambda,\infty)}$, depend on the choice of the Riemannian metric g^M.

Definition 3.2. — *The Burghelea-Haller quadratic form* $\tau^{\mathrm{BH}}_{b,\nabla}$ *on* $\operatorname{Det}\big(H^\bullet(M,E)\big)$ *is defined by the formula*

$$\tau^{\mathrm{BH}} = \tau^{\mathrm{BH}}_{b,\nabla} := \tau_{b,\nabla,[0,\lambda]} \cdot \tau_{b,\nabla,(\lambda,\infty)}. \tag{3.5}$$

It is easy to see, *cf.* [12, Prop. 4.7], that (3.5) is independent of the choice of $\lambda \geqslant 0$. Theorem 4.2 of [12] states that τ^{BH} is independent of g^M and locally constant in b. Since we are not going to use this result in the proof of Theorem 1.4, the latter theorem provides a new proof of Theorem 4.2 of [12] in the case when the dimension of M is odd, *cf.* Subsection 5.1.

4. Proof of the comparison theorem

In this section we prove Theorem 1.4 adopting the arguments which we used in Section 11 of [7] to compute the Ray-Singer norm of the refined analytic torsion.

4.1. The dual connection

Suppose M is a closed oriented manifold of odd dimension $d = 2r-1$. Let $E \to M$ be a complex vector bundle over M and let ∇ be a flat connection on E. Assume that there exists a non-degenerate bilinear form b on E. The *dual connection* ∇' to ∇ with respect to the form b is defined by the formula

$$db(u,v) = b(\nabla u, v) + b(u, \nabla' v), \quad u,v \in C^\infty(M,E).$$

We denote by E' the flat vector bundle (E,∇').

4.2. Choices of the metric and the spectral cut

Till the end of this section we fix a Riemannian metric g^M on M and set $\mathcal{B} = \mathcal{B}(\nabla, g^M)$ and $\mathcal{B}' = \mathcal{B}(\nabla', g^M)$. We also fix $\theta \in (-\pi/2, 0)$ such that both θ and $\theta+\pi$ are Agmon angles for the odd signature operator $\mathcal{B}$. Recall that for an operator A we denote by $A^{\#}$ its formal transpose with respect to the bilinear form (3.2) defined by g^M and b. One easily checks that

$$\nabla^{\#} = \Gamma\nabla'\Gamma, \quad (\nabla')^{\#} = \Gamma\nabla\Gamma, \quad \text{and} \quad \mathcal{B}^{\#} = \mathcal{B}', \tag{4.1}$$

cf. the proof of similar statements when b is replaced by a Hermitian form in Section 10.4 of [7]. As $\mathcal{B}$ and $\mathcal{B}^{\#}$ have the same spectrum it then follows that

$$\eta(\mathcal{B}') = \eta(\mathcal{B}) \quad \text{and} \quad \operatorname{Det}_{\mathrm{gr},\theta}(\mathcal{B}') = \operatorname{Det}_{\mathrm{gr},\theta}(\mathcal{B}). \tag{4.2}$$

4.3. The duality theorem for the refined analytic torsion

The pairing (3.2) induces a non-degenerate bilinear form

$$H^j(M,E') \otimes H^{d-j}(M,E) \longrightarrow \mathbb{C}, \quad j = 0, \dots, d,$$

and, hence, identifies $H^j(M,E')$ with the dual space of $H^{d-j}(M,E)$. Using the construction of Subsection 3.4 of [7] (with $\tau : \mathbb{C} \to \mathbb{C}$ being the identity map) we thus obtain a linear isomorphism

$$\alpha : \operatorname{Det}\big(H^{\bullet}(M,E)\big) \longrightarrow \operatorname{Det}\big(H^{\bullet}(M,E')\big). \tag{4.3}$$

We have the following analogue of Theorem 10.3 from [7]

Theorem 4.1. — *Let $E \to M$ be a complex vector bundle over a closed oriented odd-dimensional manifold M endowed with a non-degenerate bilinear form b and let ∇ be a flat connection on E. Let ∇' denote the connection dual to ∇ with respect to b. Then*

$$\alpha\big(\rho_{\mathrm{an}}(\nabla)\big) = \rho_{\mathrm{an}}(\nabla'). \tag{4.4}$$

The proof is the same as the proof of Theorem 10.3 from [7] (actually, it is simple, since $\mathcal{B}$ and $\mathcal{B}'$ have the same spectrum and, hence, there is no complex conjugation involved) and will be omitted.

4.4. The Burghelea-Haller quadratic form and the dual connection

Let

$$\Delta' = (\nabla')^{\#}\nabla' + \nabla'(\nabla')^{\#}$$

denote the Laplacian of the connection ∇'. From (4.1) we conclude that

$$\Delta' = \Gamma \circ \Delta \circ \Gamma.$$

Hence, a verbatim repetition of the arguments in Subsection 11.6 of [7] implies that we have

$$\tau_{b,\nabla,(\lambda,\infty)} = \tau_{b,\nabla',(\lambda,\infty)}, \tag{4.5}$$

and, for each $h \in \operatorname{Det}\big(H^\bullet(M,E)\big)$,

$$\tau^{\mathrm{BH}}_{b,\nabla}(h) = \tau^{\mathrm{BH}}_{b,\nabla'}\big(\alpha(h)\big) \tag{4.6}$$

with α being the duality isomorphism (4.3).

From (4.4) and (4.6) we get

$$\tau^{\mathrm{BH}}_{b,\nabla}\big(\rho_{\mathrm{an}}(\nabla)\big) = \tau^{\mathrm{BH}}_{b,\nabla'}\big(\rho_{\mathrm{an}}(\nabla')\big). \tag{4.7}$$

4.5. Direct sum of a connection and its dual

Let

$$\widetilde{\nabla} = \begin{pmatrix} \nabla & 0 \\ 0 & \nabla' \end{pmatrix} \tag{4.8}$$

denote the flat connection on $E \oplus E$ obtained as a direct sum of the connections ∇ and ∇'. The bilinear form b induces a bilinear form $b \oplus b$ on $E \oplus E$. To simplify the notations we shall denote this form by b. For each $\lambda \geqslant 0$, one easily checks, *cf.* Subsection 11.7 of [7], that

$$\tau_{b,\widetilde{\nabla},(\lambda,\infty)} = \tau_{b,\nabla,(\lambda,\infty)} \cdot \tau_{b,\nabla',(\lambda,\infty)} \tag{4.9}$$

and

$$\tau^{\mathrm{BH}}_{b,\widetilde{\nabla}}\big(\rho_{\mathrm{an}}(\widetilde{\nabla})\big) = \tau^{\mathrm{BH}}_{b,\nabla}\big(\rho_{\mathrm{an}}(\nabla)\big) \cdot \tau^{\mathrm{BH}}_{b,\nabla'}\big(\rho_{\mathrm{an}}(\nabla')\big). \tag{4.10}$$

Combining the latter equality with (4.7), we get

$$\tau^{\mathrm{BH}}_{b,\widetilde{\nabla}}\big(\rho_{\mathrm{an}}(\widetilde{\nabla})\big) = \tau^{\mathrm{BH}}_{b,\nabla}\big(\rho_{\mathrm{an}}(\nabla)\big)^2. \tag{4.11}$$

Hence, (1.4) *is equivalent to the equality*

$$\tau^{\mathrm{BH}}_{b,\widetilde{\nabla}}\big(\rho_{\mathrm{an}}(\widetilde{\nabla})\big) = e^{-4\pi i\left(\eta(\nabla) - \operatorname{rank} E\cdot\eta_{\mathrm{trivial}}\right)}. \tag{4.12}$$

4.6. Deformation of the chirality operator

We will prove (4.12) by a deformation argument. For $t \in [-\pi/2, \pi/2]$ introduce the rotation U_t on

$$\Omega^\bullet := \Omega^\bullet(M,E) \oplus \Omega^\bullet(M,E),$$

given by

$$U_t = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

Note that $U_t^{-1} = U_{-t}$. Denote by $\widetilde{\Gamma}(t)$ the deformation of the chirality operator, defined by

$$\widetilde{\Gamma}(t) = U_t \circ \begin{pmatrix} \Gamma & 0 \\ 0 & -\Gamma \end{pmatrix} \circ U_t^{-1} = \Gamma \circ \begin{pmatrix} \cos 2t & \sin 2t \\ \sin 2t & -\cos 2t \end{pmatrix}. \tag{4.13}$$

Then

$$\widetilde{\Gamma}(0) = \begin{pmatrix} \Gamma & 0 \\ 0 & -\Gamma \end{pmatrix}, \quad \widetilde{\Gamma}(\pi/4) = \begin{pmatrix} 0 & \Gamma \\ \Gamma & 0 \end{pmatrix}. \tag{4.14}$$

4.7. Deformation of the odd signature operator

Consider a one-parameter family of operators $\widetilde{\mathcal{B}}(t) : \Omega^\bullet \to \Omega^\bullet$ with $t \in [-\pi/2, \pi/2]$ defined by the formula

$$\widetilde{\mathcal{B}}(t) := \widetilde{\Gamma}(t)\widetilde{\nabla} + \widetilde{\nabla}\widetilde{\Gamma}(t). \tag{4.15}$$

Then

$$\widetilde{\mathcal{B}}(0) = \begin{pmatrix} \mathcal{B} & 0 \\ 0 & -\mathcal{B}' \end{pmatrix} \tag{4.16}$$

and

$$\widetilde{\mathcal{B}}(\pi/4) = \begin{pmatrix} 0 & \Gamma\nabla' + \nabla\Gamma \\ \Gamma\nabla + \nabla'\Gamma & 0 \end{pmatrix}. \tag{4.17}$$

Hence, using (4.1), we obtain

$$\widetilde{\mathcal{B}}(\pi/4)^2 = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta' \end{pmatrix} = \widetilde{\Delta}. \tag{4.18}$$

Set

$$\Omega^\bullet_+(t) := \operatorname{Ker} \widetilde{\nabla}\widetilde{\Gamma}(t);$$
$$\Omega^\bullet_- := \operatorname{Ker} \widetilde{\nabla} = \operatorname{Ker} \nabla \oplus \operatorname{Ker} \nabla'.$$

Note that $\Omega^\bullet_-$ is independent of t. Since the operators $\widetilde{\nabla}$ and $\widetilde{\Gamma}(t)$ commute with $\widetilde{\mathcal{B}}(t)$, the spaces $\Omega^\bullet_+(t)$ and $\Omega^\bullet_-$ are invariant for $\widetilde{\mathcal{B}}(t)$.

Let $\mathcal{I}$ be an interval of the form $[0, \lambda]$ or (λ, ∞). Denote

$$\Omega^\bullet_{\mathcal{I}}(t) := \Pi_{\widetilde{\mathcal{B}}(t)^2, \mathcal{I}}\big(\Omega^\bullet(t)\big) \subset \Omega^\bullet(t),$$

where $\Pi_{\widetilde{\mathcal{B}}(t)^2,\mathcal{I}}$ is the spectral projection of $\widetilde{\mathcal{B}}(t)^2$ corresponding to $\mathcal{I}$. For $j = 0, \dots, d$, set $\Omega^j_{\mathcal{I}}(t) = \Omega^\bullet_{\mathcal{I}}(t) \cap \Omega^j$ and

$$\Omega^j_{\pm,\mathcal{I}}(t) := \Omega^j_{\pm}(t) \cap \Omega^j_{\mathcal{I}}(t). \tag{4.19}$$

As $\Pi_{\widetilde{\mathcal{B}}(t)^2,\mathcal{I}}$ and $\widetilde{\mathcal{B}}(t)$ commute, one easily sees, *cf.* Subsection 11.9 of [7], that

$$\Omega^\bullet_{(\lambda,\infty)}(t) = \Omega^\bullet_{+,(\lambda,\infty)}(t) \oplus \Omega^\bullet_{-,(\lambda,\infty)}(t), \quad t \in [-\pi/2, \pi/2]. \tag{4.20}$$

We define $\widetilde{\mathcal{B}}^{\mathcal{I}}_j(t), \widetilde{\mathcal{B}}^{\mathcal{I}}_{\mathrm{even}}(t), \widetilde{\mathcal{B}}^{\mathcal{I}}_{\mathrm{odd}}(t), \widetilde{\mathcal{B}}^{\pm,\mathcal{I}}_j(t), \widetilde{\mathcal{B}}^{\pm,\mathcal{I}}_{\mathrm{even}}(t), \widetilde{\mathcal{B}}^{\pm,\mathcal{I}}_{\mathrm{odd}}(t)$, etc. in the same way as the corresponding maps were defined in Subsection 2.4.

4.8. Deformation of the canonical element of the determinant line

Since the operators $\widetilde{\nabla}$ and $\widetilde{\mathcal{B}}(t)^2$ commute, the space $\Omega^\bullet_{\mathcal{I}}(t)$ is invariant under $\widetilde{\nabla}$, *i.e.*, it is a subcomplex of $\Omega^\bullet$. The complex $\Omega^\bullet_{(\lambda,\infty)}(t)$ is acyclic and, hence, the cohomology of the finite dimensional complex $\Omega^\bullet_{[0,\lambda]}(t)$ is naturally isomorphic to

$$H^\bullet(M, E \oplus E') \simeq H^\bullet(M,E) \oplus H^\bullet(M,E').$$

Let $\widetilde{\Gamma}_{[0,\lambda]}(t)$ denote the restriction of $\widetilde{\Gamma}(t)$ to $\Omega^\bullet_{[0,\lambda]}(t)$. As $\widetilde{\Gamma}(t)$ and $\widetilde{\mathcal{B}}(t)^2$ commute, it follows that $\widetilde{\Gamma}_{[0,\lambda]}(t)$ maps $\Omega^\bullet_{[0,\lambda]}(t)$ onto itself and, therefore, is a chirality operator for $\Omega^\bullet_{[0,\lambda]}(t)$. Let

$$\rho_{\widetilde{\Gamma}_{[0,\lambda]}(t)}(t) \in \mathrm{Det}\big(H^\bullet(M, E \oplus E')\big) \tag{4.21}$$

denote the refined torsion of the finite dimensional complex $\big(\Omega^\bullet_{[0,\lambda]}(t), \widetilde{\nabla}\big)$ corresponding to the chirality operator $\widetilde{\Gamma}_{[0,\lambda]}(t)$, *cf.* Definition 2.1.

For each $t \in (-\pi/2, \pi/2)$ fix an Agmon angle $\theta = \theta(t) \in (-\pi/2, 0)$ for $\widetilde{\mathcal{B}}_{\mathrm{even}}(t)$ and define the element $\rho(t) \in \mathrm{Det}\big(H^\bullet(M, E \oplus E')\big)$ by the formula

$$\rho(t) := \mathrm{Det}_{\mathrm{gr},\theta}\big(\widetilde{\mathcal{B}}^{(\lambda,\infty)}_{\mathrm{even}}(t)\big) \cdot \rho_{\widetilde{\Gamma}_{[0,\lambda]}(t)}(t), \tag{4.22}$$

where λ is any non-negative real number. It follows from Proposition 5.10 of [7] that $\rho(t)$ is independent of the choice of $\lambda \geqslant 0$.

For $t \in [-\pi/2, \pi/2]$, $\lambda \geqslant 0$, set

$$T_\lambda(t) := \prod_{j=0}^{d} \Big(\mathrm{Det}_{2\theta} \big[\widetilde{\mathcal{B}}^{(\lambda,\infty)}_{\mathrm{even}}(t)^2 \big|_{\Omega^j_{(\lambda,\infty)}(t)} \big] \Big)^{(-1)^{j+1} j}. \tag{4.23}$$

Then, from (4.22) and (2.20) we conclude that

$$\tau^{\mathrm{BH}}_{b,\widetilde{\nabla}}\big(\rho(t)\big) = \tau^{\mathrm{BH}}_{b,\widetilde{\nabla}}\Big(\rho_{\widetilde{\Gamma}_{[0,\lambda]}(t)}(t)\Big)\cdot T_\lambda(t)\cdot e^{-2i\pi\eta\left(\widetilde{\mathcal{B}}_{\mathrm{even}}(t)\right)}. \tag{4.24}$$

In particular,

$$\tau^{\mathrm{BH}}_{b,\widetilde{\nabla}}\Big(\rho_{\widetilde{\Gamma}_{[0,\lambda]}(t)}(t)\Big)\cdot T_\lambda(t)$$

is independent of $\lambda \geqslant 0$.

4.9. Computation for $t = 0$

From (2.4) and definition (2.6) of the element ρ, we conclude that

$$\rho_{-\Gamma_{[0,\lambda]}}(\nabla', g^M) = \pm\rho_{\Gamma_{[0,\lambda]}}(\nabla', g^M).$$

Thus,

$$\tau^{\mathrm{BH}}_{b,\widetilde{\nabla}}\big(\rho_{-\Gamma_{[0,\lambda]}}(\nabla', g^M)\big) = \tau^{\mathrm{BH}}_{b,\widetilde{\nabla}}\big(\rho_{\Gamma_{[0,\lambda]}}(\nabla', g^M)\big).$$

Hence, from (4.8) and (4.14) we obtain

$$\tau^{\mathrm{BH}}_{b,\widetilde{\nabla}}\big(\rho_{\widetilde{\Gamma}_{[0,\lambda]}(0)}(0)\big) = \tau^{\mathrm{BH}}_{b,\nabla}\big(\rho_{\Gamma_{[0,\lambda]}}(\nabla, g^M)\big)\cdot\tau^{\mathrm{BH}}_{b,\nabla'}\big(\rho_{\Gamma_{[0,\lambda]}}(\nabla', g^M)\big). \tag{4.25}$$

Using (4.16) and the definitions (2.19) and (4.23) of T_λ we get

$$T_\lambda(0) = T_\lambda(\nabla, g^M, \theta)\cdot T_\lambda(\nabla', g^M, \theta). \tag{4.26}$$

Combining the last two equalities with definitions (2.14), (4.22) of ρ and with (2.20), (4.2), and (4.7), we obtain

$$\tau^{\mathrm{BH}}_{b,\widetilde{\nabla}}\big(\rho_{\widetilde{\Gamma}_{[0,\lambda]}(0)}(0)\big)\cdot T_\lambda(0) = \tau^{\mathrm{BH}}_{b,\nabla}\big(\rho_{\mathrm{an}}(\nabla)\big)^2\cdot e^{4\pi i\left(\eta(\nabla)-\operatorname{rank}E\cdot\eta_{\mathrm{trivial}}\right)}. \tag{4.27}$$

Comparing this equality with (4.11) we see that *in order to prove* (4.12) *and, hence,* (1.4) *it is enough to show that*

$$\tau^{\mathrm{BH}}_{b,\widetilde{\nabla}}\big(\rho_{\widetilde{\Gamma}_{[0,\lambda]}(0)}(0)\big)\cdot T_\lambda(0) = 1. \tag{4.28}$$

4.10. Computation for $t = \pi/4$

From (4.18) and the definitions (3.4) and (4.23) of $\tau_{b,\widetilde{\nabla},(\lambda,\infty)}$ and $T_\lambda(t)$, we conclude

$$T_\lambda(\pi/4) = 1/\tau_{b,\widetilde{\nabla},(\lambda,\infty)}. \tag{4.29}$$

By (4.18) we have

$$\Omega^{\bullet}_{[0,\lambda]}(\pi/4) = \Omega^{\bullet}_{[0,\lambda]}(M,E) \oplus \Omega^{\bullet}_{[0,\lambda]}(M,E').$$

From (4.14) we see that the restriction of $\widetilde{\Gamma}(\pi/4)$ to $\Omega^{\bullet}_{[0,\lambda]}(\pi/4)$ preserves the bilinear form on $\Omega^{\bullet}_{[0,\lambda]}(\pi/4)$ induced by b. Hence we obtain from Lemma 3.1

$$\tau_{b,\widetilde{\nabla},[0,\lambda]}\big(\rho_{\widetilde{\Gamma}_{[0,\lambda]}(\pi/4)}(\pi/4)\big) = 1.$$

Therefore, from (4.29) and the definitions (3.5) of τ^{BH}, we get

$$\tau^{\mathrm{BH}}_{b,\widetilde{\nabla}}\big(\rho_{\widetilde{\Gamma}_{[0,\lambda]}(\pi/4)}(\pi/4)\big) \cdot T_\lambda(\pi/4) = 1. \tag{4.30}$$

4.11. Proof of Theorem 1.2

Fix an Agmon angle $\theta \in (-\pi/2, 0)$ and set

$$\xi_{\lambda,\theta}(t) := -\frac{1}{2}\sum_{j=0}^{d}(-1)^{j+1} j\zeta'_\theta\big(0, \widetilde{B}_{\mathrm{even}}(t)^2|_{\Omega^j_{(\lambda,\infty)}(t)}\big),$$

where $\zeta'_\theta(0,A)$ denotes the derivative at zero of the ζ-function of the operator operator A. Then $T_\lambda(t) = e^{2\xi_{\lambda,\theta}(t)}$. Hence, from (4.30) we conclude that *in order to prove* (4.28) *(and, hence,* (4.12) *and* (1.4)*) it suffices to show that*

$$\tau^{\mathrm{BH}}_{b,\widetilde{\nabla}}\big(\rho_{\widetilde{\Gamma}_{[0,\lambda]}(t)}(t)\big) \cdot e^{2\xi_{\lambda,\theta}(t)} \tag{4.31}$$

is independent of t.

Fix $t_0 \in [-\pi/2, \pi/2]$ and let $\lambda \geqslant 0$ be such that the operator $\widetilde{\mathcal{B}}_{\mathrm{even}}(t_0)^2$ has no eigenvalues with absolute value λ. Choose an angle $\theta \in (-\pi/2, 0)$ such that both θ and $\theta+\pi$ are Agmon angles for $\widetilde{\mathcal{B}}(t_0)$. Then there exists $\delta > 0$ such that for all $t \in (t_0-\delta, t_0+\delta)\cap[-\pi/2,\pi/2]$, the operator $\widetilde{\mathcal{B}}_{\mathrm{even}}(t)^2$ has no eigenvalues with absolute value λ and both θ and $\theta+\pi$ are Agmon angles for $\widetilde{\mathcal{B}}(t)$.

A verbatim repetition of the proof of Lemma 9.2 of [7] shows that

$$\frac{d}{dt}\rho_{\widetilde{\Gamma}_{[0,\lambda]}(t)}(t) \cdot e^{\xi_{\lambda,\theta}(t)} = 0. \tag{4.32}$$

Hence, (4.31) is independent of t. □

5. Properties of the Burghelea-Haller quadratic form

Combining Theorem 1.2 with results of our papers [4, 7, 6] we derive new properties and obtain new proofs of some known ones of the Burghelea-Haller quadratic form τ. In particular, we prove a weak version of Theorem 1.3 which relates the quadratic form (1.3) with the Farber-Turaev torsion — see Subsection 1.8 for a discussion of Theorem 1.3.

5.1. Independence of τ^{BH} of the Riemannian metric and the bilinear form

The following theorem was established by Burghelea and Haller [12, Th. 4.2] without the assumption that M is oriented and odd-dimensional.

Theorem 5.1. — **[Burghelea-Haller]** *Let M be an odd dimensional orientable closed manifold and let (E,∇) be a flat vector bundle over M. Assume that there exists a non-degenerate symmetric bilinear form b on E. Then the Burghelea-Haller quadratic form $\tau^{\mathrm{BH}}_{b,\nabla}$ is independent of the choice of the Riemannian metric g^M on M and is locally constant in b.*

Our Theorem 1.2 provides a new proof of this theorem and at the same time gives the following new result.

Theorem 5.2. — *Under the assumptions of Theorem 5.1 suppose that b' is another non-degenerate symmetric bilinear form on E not necessarily homotopic to b in the space of non-degenerate symmetric bilinear forms. Then $\tau^{\mathrm{BH}}_{b',\nabla}=\pm\tau^{\mathrm{BH}}_{b,\nabla}$.*

Proof of Theorems 5.1 and 5.2. — As the refined analytic torsion $\rho_{\mathrm{an}}(\nabla)$ does not depend on g^M and b, Theorem 1.2 implies that, modulo sign, $\tau^{\mathrm{BH}}_{b,\nabla}$ is independent of g^M and b. Since $\tau^{\mathrm{BH}}_{b,\nabla}$ is continuous in g^M and b it follows that it is locally constant in g^M and b. Since the space of Riemannian metrics is connected, $\tau^{\mathrm{BH}}_{b,\nabla}$ is independent of g^M. $\square$

5.2. Comparison with the Farber-Turaev torsion: proof of Theorem 1.6

Let $L(p)=L_M(p)$ denote the Hirzebruch L-polynomial in the Pontrjagin forms of a Riemannian metric on M. We write $\widehat{L}(p)\in H_\bullet(M,\mathbb{Z})$ for the

Poincaré dual of the cohomology class $[L(p)]$ and let $\widehat{L}_1 \in H_1(M,\mathbb{Z})$ denote the component of $\widehat{L}(p)$ in $H_1(M,\mathbb{Z})$.

Theorem 5.11 of [6] combined with formulae (5.4) and (5.6) of [6] implies that for each connected component $\mathcal{C} \subset \mathrm{Flat}(E)$, there exists a constant $F_{\mathcal{C}}$ such that for every flat connection $\nabla \in \mathcal{C}$ and every Euler structure ε we have

$$|F_{\mathcal{C}}| = \left| e^{-2\pi i \langle \mathrm{Arg}_\nabla, \widehat{L}_1\rangle + 2\pi i \eta(\nabla)} \right|, \tag{5.1}$$

and

$$\left(\frac{\rho_{\varepsilon,\mathfrak{o}}(\nabla)}{\rho_{\mathrm{an}}(\nabla)} \right)^2 = F_{\mathcal{C}} \cdot e^{2\pi i \langle \mathrm{Arg}_\nabla, \widehat{L}_1 + c(\varepsilon)\rangle}. \tag{5.2}$$

Hence, from the definition (1.1) of the quadratic form τ, we get

$$\tau_\nabla\big(\rho_{\varepsilon,\mathfrak{o}}(\nabla)\big) \cdot e^{-2\pi i \langle \mathrm{Arg}_\nabla, c(\varepsilon)\rangle} = F_{\mathcal{C}} \cdot e^{2\pi i \langle \mathrm{Arg}_\nabla, \widehat{L}_1\rangle - 2\pi i(\eta(\nabla) - \mathrm{rank}\, E \cdot \eta_{\mathrm{trivial}})}. \tag{5.3}$$

Assume now that ∇_t with $t \in [0,1]$ is a smooth path of flat connections. The derivative $\dot{\nabla}_t = \frac{d}{dt}\nabla_t$ is a smooth differential 1-form with values in the bundle of isomorphisms of E. We denote by $[\mathrm{Tr}\, \dot{\nabla}_t] \in H^1(M,\mathbb{C})$ the cohomology class of the closed 1-form $\mathrm{Tr}\, \dot{\nabla}_t$.

By Lemma 12.6 of [4], we have

$$2\pi i \frac{d}{dt} \mathrm{Arg}_{\nabla_t} = -\big[\, \mathrm{Tr}\, \dot{\nabla}_t \big] \in H^1(M,\mathbb{C}). \tag{5.4}$$

Let $\overline{\eta}(\nabla_t, g^M) \in \mathbb{C}/\mathbb{Z}$ denote the reduction of $\eta(\nabla_t, g^M)$ modulo $\mathbb{Z}$. Then $\overline{\eta}(\nabla_t, g^M)$ depends smoothly on t, *cf.* [17, §1]. From Theorem 12.3 of [4] we obtain[(2)]

$$-2\pi i \frac{d}{dt} \overline{\eta}(\nabla_t, g^M) = \int_M L(p) \wedge \mathrm{Tr}\, \dot{\nabla}_t = \big\langle \big[\, \mathrm{Tr}\, \dot{\nabla}_t \big], \widehat{L}_1 \big\rangle. \tag{5.5}$$

From (5.3)–(5.5) we then obtain

$$\frac{d}{dt}\Big[\tau_{\nabla_t}\big(\rho_{\varepsilon,\mathfrak{o}}(\nabla_t)\big) \cdot e^{-2\pi i \langle \mathrm{Arg}_\nabla, c(\varepsilon)\rangle} \Big] = 0, \tag{5.6}$$

proving that the right hand side of (5.3) is independent of $\nabla \in \mathcal{C}$. From (5.1) and the fact that $\eta_{\mathrm{trivial}} \in \mathbb{R}$ we conclude that the absolute value of the right hand side of (5.3) is equal to 1. Part (i) of Theorem 1.6 is proven.

Finally, consider the case when $\mathcal{C}$ contains an acyclic Hermitian connection ∇. In this case both, τ_∇ and $\rho_{\varepsilon,\mathfrak{o}}(\nabla)$, can be viewed as non-zero complex numbers. To prove part (ii) of Theorem 1.6 it is now enough to show that the numbers $\rho_{\varepsilon,\mathfrak{o}}(\nabla)^2$ and $\tau_\nabla \cdot e^{-2\pi i \langle \mathrm{Arg}_\nabla, c(\varepsilon)\rangle}$ have the same

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(2) This result was originally proven by Gilkey [17, Th. 3.7].

phase. Since ∇ is a Hermitian connection, the number $\eta(\nabla)$ is real. Hence, it follows from Theorem 10.3 of [7] that

$$\mathbf{Ph}\left(\rho_{\rm an}(\nabla)\right) \equiv -\pi i\big(\eta(\nabla) - \operatorname{rank} E\cdot\eta_{\rm trivial}\big) \mod \pi i.$$

Thus, by (1.1),

(5.7)
$$\mathbf{Ph}\left(\tau_\nabla\cdot e^{-2\pi i\langle \operatorname{Arg}_\nabla, c(\varepsilon)\rangle}\right) = \mathbf{Ph}\left(e^{-2\pi i\langle \operatorname{Arg}_\nabla, c(\varepsilon)\rangle}\right) = -2\pi\operatorname{Re}\langle \operatorname{Arg}_\nabla, c(\varepsilon)\rangle.$$

By formula (2.4) of [15],

$$\mathbf{Ph}\left(\rho_{\varepsilon,\mathfrak{o}}(\nabla)^2\right) = -2\pi\operatorname{Re}\langle \operatorname{Arg}_\nabla, c(\varepsilon)\rangle. \tag{5.8}$$

The proof of Theorem 1.6 is complete. □

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