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VANISHING OF THE FIRST REDUCED COHOMOLOGY WITH VALUES IN AN L^p-REPRESENTATION

by Romain TESSERA

Abstract. — We prove that the first reduced cohomology with values in a mixing L^p-representation, $1 < p < \infty$, vanishes for a class of amenable groups including connected amenable Lie groups. In particular this solves for this class of amenable groups a conjecture of Gromov saying that every finitely generated amenable group has no first reduced ℓ^p-cohomology. As a byproduct, we prove a conjecture by Pansu. Namely, the first reduced L^p-cohomology on homogeneous, closed at infinity, Riemannian manifolds vanishes. We also prove that a Gromov hyperbolic geodesic metric measure space with bounded geometry admitting a bi-Lipschitz embedded 3-regular tree has non-trivial first reduced L^p-cohomology for large enough p. Combining our results with those of Pansu, we characterize Gromov hyperbolic homogeneous manifolds: these are the ones having non-zero first reduced L^p-cohomology for some $1 < p < \infty$.

Résumé. — Nous prouvons que pour une classe de groupes moyennables, incluant tous les groupes moyennables de Lie connexes, la cohomologie réduite en degré 1 à valeurs dans une représentation mélangeante sur un espace L^p, pour $p > 1$, est nulle. En particulier, cela démontre pour cette classe de groupes moyennables une conjecture de Gromov s'appliquant à tous les groupes de type fini moyennables. Nous obtenons également la version “de Lie” de cette conjecture, qui avait été formulée par Pansu. Nous montrons par ailleurs qu'un espace métrique hyperbolique possédant un arbre 3-regulier quasi-isométriquement plongé a un premier groupe de cohomologie L^p réduite non trivial pour p assez grand. Finalement, en combinant nos résultats avec ceux de Pansu, nous obtenons une caractérisation des variétés riemanniennes homogènes hyperboliques au sens de Gromov : ce sont celles qui possèdent de la cohomologie L^p réduite en degré 1 pour p assez grand.

Keywords: Reduced L^p-cohomology, amenable groups, Folner sequences, hyperbolic metric spaces, homogeneous Riemannian manifold.
Math. classification: 20F65, 22F30.

1. Introduction

1.1. A weak generalization of a result of Delorme.

In [8], Delorme proved the following deep result: every connected solvable Lie groups has the property that every weakly mixing[(1)] unitary representation π has trivial first reduced cohomology, *i.e.*, $\overline{H}^1(G,\pi) \neq 0$. This was recently extended to connected amenable Lie groups, see [17, Theorem 3.3], and to a large class of amenable groups including polycyclic groups by Shalom [25]. Shalom also proves that this property, that he calls Property H_{FD}, is invariant under quasi-isometry between amenable discrete groups. Property H_{FD} has nice implications in various contexts. For instance, Shalom shows that an amenable finitely generated group with Property H_{FD} has a finite index subgroup with infinite abelianization [25, Theorem 4.3.1]. In [6], we prove [6, Theorem 4.3] that an amenable finitely generated group with Property H_{FD} cannot quasi-isometrically embed into a Hilbert space unless it is virtually abelian.

It is interesting and natural to extend the definition of Property H_{FD} to isometric representations of groups on certain classes of Banach spaces.

In this paper, we prove that a weak version of Property H_{FD}, also invariant under quasi-isometry, holds for isometric L^p-representations of a large class of amenable groups including connected amenable Lie groups and polycyclic groups: for $1 < p < \infty$, every *strongly mixing* isometric L^p-representation π has trivial first reduced cohomology (see Section 2 for a precise statement).

1.2. L^p-cohomology.

The L^p-cohomology (for p not necessarily equal to 2) of a Riemannian manifold has been introduced by Gol'dshtein, Kuz'minov, and Shvedov in [10]. It has been intensively studied by Pansu [19, 22, 21] in the context of homogeneous Riemannian manifolds and by Gromov [12] for discrete metric spaces and groups. The L^p-cohomology is invariant under quasi-isometry in degree one [15]. But in higher degree, the quasi-isometry invariance requires some additional properties, like for instance the uniform contractibility of the space [12] (see also [3, 21]). Most authors focus on

(1) A unitary representation is called weakly mixing if it contains no finite dimensional sub-representation.

the first reduced L^p-cohomology since it is easier to compute and already gives a fine quasi-isometry invariant (used for instance in [1, 3]). The ℓ^2-Betti numbers of a finitely generated group, corresponding to its reduced ℓ^2-cohomology(2), have been extensively studied in all degrees by authors like Gromov, Cheeger, Gaboriau and many others. In particular, Cheeger and Gromov proved in [4] that the reduced ℓ^2-cohomology of a finitely generated amenable group vanishes in all degrees. In [12], Gromov conjectures that this should also be true for the reduced ℓ^p-cohomology. For a large class of finitely generated groups with infinite center, it is known [12, 16] that the reduced ℓ^p-cohomology vanishes in all degrees, for $1 < p < \infty$. The first reduced ℓ^p-cohomology for $1 < p < \infty$ is known to vanish [2, 18] for certain non-amenable finitely generated groups with "a lot of commutativity" (*e.g.*, groups having a non-amenable finitely generated normal subgroup with infinite centralizer).

A consequence of our main result is to prove that the first reduced ℓ^p-cohomology, $1 < p < \infty$, vanishes for large class of finitely generated amenable groups, including for instance polycyclic groups.

On the other hand, it is well known [12] that the first reduced ℓ^p-cohomology of a Gromov hyperbolic finitely generated group is non-zero for p large enough. Although the converse is false(3) for finitely generated groups, we will see that it is true in the context of connected Lie groups. Namely, a connected Lie group has non-zero reduced first L^p-cohomology for some $1 < p < \infty$ if and only if it is Gromov hyperbolic.

Acknowledgments. I would like to thank Pierre Pansu, Marc Bourdon and Hervé Pajot for valuable discussions about L^p-cohomology. Namely, Marc explained to me how one can extend a Lipschitz function defined on the boundary $\partial_\infty X$ of a Gromov hyperbolic space X to the space itself, providing a non-trivial element in $H^1_p(X)$ for p large enough (see the proof of Theorem 9.2 in Section 9). According to him, this idea is originally due to Gabor Elek. I would like to thank Yaroslav Kopylov for pointing out to me the reference [10] where the L^p-cohomology was first introduced. I am also grateful to Yves de Cornulier, Pierre Pansu, Gilles Pisier, and Michael Puls for their useful remarks and corrections.

(2) We write ℓ^p when the space is discrete.

(3) In [7] for instance, we prove that any non-amenable discrete subgroup of a semi-simple Lie group of rank one has non-trivial reduced L^p-cohomology for p large enough. On the other hand, non-cocompact lattices in $\mathrm{SO}(3,1)$ are not hyperbolic. See also [2] for other examples.

2. Main results

(The definitions of first L^p-cohomology, p-harmonic functions and of first cohomology with values in a representation are postponed to Section 4.)

Let G be a locally compact group acting by measure-preserving bijections on a measurable space equipped with an infinite measure (X, m). We say that the action is strongly mixing (or mixing) if for every measurable subset of finite measure $A \subset X$, $m(gA \cap A) \to 0$ when g leaves every compact subset of G. Let π be the corresponding continuous representation of G in $L^p(X, m)$, where $1 < p < \infty$. In this paper, we will call such a representation a mixing L^p-representation of G.

DEFINITION 2.1. — [26] *Let G be a locally compact, compactly generated group and let S be a compact generating subset of G. We say that G has Property (CF) (Controlled Følner) if there exists a sequence of compact subsets of positive measure (F_n) satisfying the following properties.*

- *$F_n \subset S^n$ for every n;*
- *there is a constant $C < \infty$ such that for every n and every $s \in S$,*

$$\frac{\mu(sF_n \vartriangle F_n)}{\mu(F_n)} \leqslant C/n.$$

Such a sequence F_n is called a controlled Følner[4] *sequence.*

In [26], we proved that following family[5] of groups are (CF).

(1) Polycyclic groups and connected amenable Lie groups;
(2) semidirect products $\mathbf{Z}[\frac{1}{mn}] \rtimes_{\frac{m}{n}} \mathbf{Z}$, with m, n co-prime integers with $|mn| \geqslant 2$ (if $n = 1$ this is the Baumslag-Solitar group $BS(1, m)$); semidirect products $\left(\bigoplus_{i\in I} \mathbf{Q}_{p_i}\right) \rtimes_{\frac{m}{n}} \mathbf{Z}$ with m, n co-prime integers, and $(p_i)_{i\in I}$ a finite family of primes (including ∞: $\mathbf{Q}_\infty = \mathbf{R}$)) dividing mn;
(3) wreath products $F \wr \mathbf{Z}$ for F a finite group.

Our main result is the following theorem.

THEOREM 2.2. — *Let G be a group with Property (CF) and let π be a mixing L^p-representation of G. Then the first reduced cohomology of G with values in π vanishes, i.e., $\overline{H^1}(G, \pi) = 0$.*

[4] A controlled Følner sequence is in particular a Følner sequence, so that Property (CF) implies amenability.
[5] This family of groups also appears in [6].

Invariance under quasi-isometry. The proof of [25, Theorem 4.3.3] that Property H_{FD} is invariant under quasi-isometry can be used identically in the context of L^p-representations and replacing the hypothesis "weak mixing" by "mixing" since the induced representation of a mixing L^p-representation is also a mixing L^p-representation. As a result, we obtain that the property that $\overline{H^1}(G,\pi)=0$ for every mixing L^p-representation is invariant under quasi-isometry between discrete amenable groups. It is also stable by passing to (and inherited by) co-compact lattices in amenable locally compact groups.

It is well known [24] that for finitely generated groups G, the first reduced cohomology with values in the left regular representation in $\ell^p(G)$ is isomorphic to the space $HD_p(G)$ of p-harmonic functions with gradient in ℓ^p modulo the constants. We therefore obtain the following corollary.

Corollary 2.3. — *Let G be a discrete group with Property (CF). Then every p-harmonic function on G with gradient in ℓ^p is constant.*

Using Von Neumann algebra technics, Cheeger and Gromov [4] proved that every finitely generated amenable group G has no nonconstant harmonic function with gradient in ℓ^2, the generalization to every $1<p<\infty$ being conjectured by Gromov.

To obtain a version of Corollary 2.3 for Lie groups, we prove the following result (see Theorem 5.1).

Theorem 2.4. — *Let G be a connected Lie group. Then for $1\leqslant p<\infty$, the first L^p-cohomology of G is topologically (canonically) isomorphic to the first cohomology with values in the right regular representation in $L^p(G)$, i.e.,*

$$H_p^1(G)\simeq H^1(G,\rho_{G,p}).$$

Now, since this isomorphism induces a natural bijection

$$HD_p(G)\simeq \overline{H^1}(G,\rho_{G,p}),$$

we can state the following result that was conjectured by Pansu in [22]. Recall that a Riemannian manifold is called closed at infinity if there exists a sequence of compact subsets A_n with regular boundary ∂A_n such that $\mu_{d-1}(\partial A_n)/\mu_d(A_n)\to 0$, where μ_k denotes the Riemannian measure on submanifolds of dimension k of M.

Corollary 2.5. — *Let M be a homogeneous Riemannian manifold. If it is closed at infinity, then for every $p>1$, every p-harmonic function on M with gradient in $L^p(TM)$ is constant. In other words, $HD_p(M)=0$.*

Together with Pansu's results [23, Théorème 1], we obtain the following dichotomy.

THEOREM 2.6. — *Let M be a homogeneous Riemannian manifold. Then the following dichotomy holds.*

- *Either M is quasi-isometric to a homogeneous Riemannian manifold with strictly negative curvature, and then there exists $p_0 \geqslant 1$ such that $HD_p(M) \neq 0$ if and only if $p > p_0$;*
- *or $HD_p(M) = 0$ for every $p > 1$.*

We also prove

THEOREM 2.7. — *(see Corollary 9.3) A homogeneous Riemannian manifold M has non-zero first reduced L^p-cohomology for some $1 < p < \infty$ if and only if it is non-elementary*[6] *Gromov hyperbolic.*

To prove this corollary, we need to prove that a Gromov hyperbolic Lie group has non-trivial first reduced L^p-cohomology for p large enough. This is done in Section 9. Namely, we prove a more general result.

THEOREM 2.8. — *(see Theorem 9.2) Let G be a Gromov hyperbolic metric measure space with bounded geometry having a bi-Lipschitz embedded 3-regular tree, then for p large enough, it has non-trivial first reduced L^p-cohomology.*

Corollary 9.3 and Pansu's contribution to Theorem 2.6 yield the following corollary.

COROLLARY 2.9. — *A non-elementary Gromov hyperbolic homogeneous Riemannian manifold is quasi-isometric to a homogeneous Riemannian manifold with strictly negative curvature.*

(See [14] for an algebraic description of homogeneous manifolds with strictly negative curvature).

3. Organization of the paper.

In the following section, we recall three definitions of first cohomology:

- a coarse definition of the first L^p-cohomology on a general metric measure space which is due to Pansu;

(6) By non-elementary, we mean not quasi-isometric to **R**.

- the usual definition of first L^p-cohomology on a Riemannian manifold;
- the first cohomology with values in a representation, which is defined for a locally compact group.

In Section 5, we construct a natural topological isomorphism between the L^p-cohomology of a connected Lie group G and the cohomology with values in the right regular representation of G in $L^p(G)$. We use this isomorphism to deduce Corollary 2.5 from Theorem 2.2.

The proof of Theorem 2.2 splits into two steps. First (see Theorem 6.1), we prove that for any locally compact compactly generated group G and any mixing L^p-representation π of G, every 1-cocycle $b \in Z^1(G, \pi)$ is *sublinear*, which means that for every compact symmetric generating subset S of G, we have

$$\|b(g)\| = o(|g|_S)$$

when $|g|_S \to \infty$, $|g|_S$ being the word length of g with respect to S. Then, we adapt to this context a remark that we made with Cornulier and Valette (see [6, Proposition 3.6]): for a group with Property (CF), a 1-cocycle belongs to $\overline{B}^1(G, \pi)$ if and only if it is sublinear. The part "only if" is an easy exercise and does not require Property (CF). To prove the other implication, we consider the affine action σ of G on E associated to the 1-cocycle b and use Property (CF) to construct a sequence of almost fixed points for σ.

In Section 8, we propose a more direct approach(7) to prove Corollary 2.5. The interest is to provide an explicit approximation of an element of $\mathbf{D}_p(G)$ by a sequence of functions in $W^{1,p}(G)$ using a convolution-type argument.

Finally, in Section 9, we prove that a Gromov hyperbolic homogeneous manifold has non-trivial L^p-cohomology for p large enough. This section can be read independently.

4. Preliminaries

4.1. A coarse notion of first L^p-cohomology on a metric measure space

The following coarse notion of (first) L^p-cohomology is essentially due to [21] (see also the chapter about L^p-cohomology in [12]).

(7) However, the ingredients are the same: sublinearity of cocycles, and existence of a controlled Følner sequence.

Let $X = (X, d, \mu)$ be a metric measure space, and let $p \geqslant 1$. For all $s > 0$, we write $\Delta_s = \{(x, y) \in X^2, d(x, y) \leqslant s\}$.

First, let us introduce the p-Dirichlet space $\mathbf{D}_p(X)$.

- The space $D_p(X)$ is the set of measurable functions f on X such that
$$\int_{\Delta_s} |f(x) - f(y)|^p d\mu(x)d\mu(y) < \infty$$
for every $s > 0$.
- Let $\mathbf{D}_p(X)$ be the Banach space $D_p(X)/\mathbf{C}$ equipped with the norm
$$\|f\|_{D_p} = \left(\int_{\Delta_1} |f(x) - f(y)|^p \mu(x)d\mu(y)\right)^{1/p}.$$
- By a slight abuse of notation, we identify L^p with its image in $\mathbf{D}_p$.

Definition 4.1. — *The first L^p-cohomology of X is the space*
$$H_p^1(X) = \mathbf{D}_p(X)/L^p(X),$$
and the first reduced L^p-cohomology of X is the space
$$\overline{H^1}_p(X) = \mathbf{D}_p(X)/\overline{L^p(X)}^{\mathbf{D}_p(X)}.$$

Definition 4.2 (1-geodesic spaces). — *We say that a metric space $X = (X, d)$ is 1-geodesic if for every two points $x, y \in X$, there exists a sequence of points $x = x_1, \dots x_m = y$, satisfying*

- *$d(x, y) = d(x_1, x_2) + \dots + d(x_{m-1}, x_m)$,*
- *for all $1 \leqslant i \leqslant m - 1$, $d(x_i, x_{i+1}) \leqslant 1$.*

Remark 4.3. — Let X and Y be two 1-geodesic metric measure spaces with bounded geometry in the sense of [21]. Then it follows from [21] that if X and Y are quasi-isometric, then $H_p^1(X) \simeq H_p^1(Y)$ and $\overline{H^1}_p(X) \simeq \overline{H^1}_p(Y)$.

Example 4.4. — Let G be a locally compact compactly generated group, and let S be a symmetric compact generating set. Then the word metric on G associated to S,
$$d_S(g, h) = \in \{n \in \mathbf{N}, g^{-1}h \in S^n\},$$
defines a 1-geodesic left-invariant metric on G. Moreover, one checks easily two such metrics (associated to different S) are bilipschitz equivalent. Hence, by Pansu's result, the first L^p-cohomology of (G, μ, d_S) does not depend on the choice of S.

DEFINITION 4.5 (coarse notion of p-harmonic functions). — *Let $f \in D_p(X)$ and assume that $p > 1$. The p-Laplacian*[8] *of f is*

$$\Delta_p f(x) = \frac{1}{V(x,1)} \int_{d(x,y)\leqslant 1} |f(x) - f(y)|^{p-2}(f(x) - f(y))d\mu(y),$$

where $V(x,1)$ is the volume of the closed ball $B(x,1)$. A function $f \in D_p(X)$ is called p-harmonic if $\Delta_p f = 0$. Equivalently, the p-harmonic functions are the minimizers of the variational integral

$$\int_{\Delta_1} |f(x) - f(y)|^p d\mu(x)d\mu(y).$$

DEFINITION 4.6. — *We say that X satisfies a Liouville D_p-Property if every p-harmonic function on X is constant.*

As $\mathbf{D}_p(X)$ is a strictly convex, reflexive Banach space, every $f \in \mathbf{D}_p(X)$ admits a unique projection $\tilde{f}$ on the closed subspace $\overline{L^p(X)}$ such that $d(f,\tilde{f}) = d(f,\overline{L^p(X)})$. One can easily check that $f - \tilde{f}$ is p-harmonic. In conclusion, the reduced cohomology class of $f \in \mathbf{D}_p(X)$ admits a unique p-harmonic representant modulo the constants. We therefore obtain

PROPOSITION 4.7. — *A metric measure space X has Liouville D_p-Property if and only if $\overline{H_p}^1(X) = 0$.*

4.2. First L^p-cohomology on a Riemannian manifold

Let M be Riemannian manifold, equipped with its Riemannian measure m. Let $1 \leqslant p < \infty$.

Let us first define, in this differentiable context, the p-Dirichlet space $\mathbf{D}_p$.

- Let D_p be the vector space of continuous functions whose gradient is (in the sense of distributions) in $L^p(TM)$.
- Equip $D_p(M)$ with a pseudo-norm $\|f\|_{D_p} = \|\nabla f\|_p$, which induces a norm on $D_p(M)$ modulo the constants. Denote by $\mathbf{D}_p(M)$ the completion of this normed vector space.
- Write $W^{1,p}(M) = L^p(M) \cap D_p(M)$. By a slight abuse of notation, we identify $W^{1,p}(M)$ with its image in $\mathbf{D}_p(M)$.

(8) Here we define a coarse p-Laplacian at scale 1: see [27, Section 2.2] for a more general definition.

Definition 4.8. — *The first L^p-cohomology of M is the quotient space*

$$H_p^1(M) = \mathbf{D}_p(M)/W^{1,p}(M),$$

and the first reduced L^p-cohomology of M is the quotient

$$\overline{H_p}^1(M) = \mathbf{D}_p(M)/\overline{W^{1,p}(M)},$$

where $\overline{W^{1,p}(M)}$ is the closure of $W^{1,p}(M)$ in the Banach space $\mathbf{D}_p(M)$.

Definition 4.9 (p-harmonic functions). — *A function $f \in D_p(M)$ is called p-harmonic if it is a weak solution of*

$$\mathrm{div}(|\nabla f|^{p-2}\nabla f) = 0,$$

that is,

$$\int_M \langle |\nabla f|^{p-2}\nabla f, \nabla\varphi\rangle dm = 0,$$

for every $\varphi \in C_0^\infty$. Equivalently, p-harmonic functions are the minimizers of the variational integral

$$\int_M |\nabla f|^p dm.$$

Definition 4.10. — *We say that M satisfies a Liouville D_p-Property if every p-harmonic function on M is constant.*

As $\mathbf{D}_p(M)$ is a strictly convex, reflexive Banach space, every $f \in \mathbf{D}_p(M)$ admits a unique projection $\tilde{f}$ on the closed subspace $\overline{W^{1,p}(M)}$ such that $d(f,\tilde{f}) = d(f,\overline{W^{1,p}(M)})$. One can easily check that $f - \tilde{f}$ is p-harmonic. In conclusion, the reduced cohomology class of $f \in \mathbf{D}_p(M)$ admits a unique p-harmonic representant modulo the constants. Hence, we get the following well-known fact.

Proposition 4.11. — *A Riemannian manifold M has Liouville D_p-Property if and only if $\overline{H_p}^1(M) = 0$.*

Remark 4.12. — In [21], Pansu proves (in particular) that if a Riemannian manifold has bounded geometry (which is satisfied by a homogeneous manifold), then the first L^p-cohomology defined as above is topologically isomorphic to its coarse version defined at the previous section. In particular, the Liouville D_p-Property is invariant under quasi-isometry between Riemannian manifolds with bounded geometry.

4.3. First cohomology with values in a representation

Let G be a locally compact group, and π a continuous linear representation on a Banach space $E = E_\pi$. The space $Z^1(G,\pi)$ is defined as the set of continuous functions $b : G \to E$ satisfying, for all g, h in G, the 1-cocycle condition $b(gh) = \pi(g)b(h) + b(g)$. Observe that, given a continuous function $b : G \to E$, the condition $b \in Z^1(G,\pi)$ is equivalent to saying that G acts by affine transformations on E by $\alpha(g)v = \pi(g)v + b(g)$. The space $Z^1(G,\pi)$ is endowed with the topology of uniform convergence on compact subsets.

The subspace of coboundaries $B^1(G,\pi)$ is the subspace (not necessarily closed) of $Z^1(G,\pi)$ consisting of functions of the form $g \mapsto v - \pi(g)v$ for some $v \in E$. In terms of affine actions, $B^1(G,\pi)$ is the subspace of affine actions fixing a point.

The first cohomology space of π is defined as the quotient space

$$H^1(G,\pi) = Z^1(G,\pi)/B^1(G,\pi).$$

The first *reduced* cohomology space of π is defined as the quotient space

$$\overline{H^1}(G,\pi) = Z^1(G,\pi)/\overline{B^1}(G,\pi),$$

where $\overline{B^1}(G,\pi)$ is the closure of $B^1(G,\pi)$ in $Z^1(G,\pi)$ for the topology of uniform convergence on compact subsets. In terms of affine actions, $\overline{B^1}(G,\pi)$ is the space of actions σ having almost fixed points, *i.e.*, for every $\varepsilon > 0$ and every compact subset K of G, there exists a vector $v \in E$ such that for every $g \in K$,

$$\|\sigma(g)v - v\| \leqslant \varepsilon.$$

If G is compactly generated and if S is a compact generating set, then this is equivalent to the existence of a sequence of almost fixed points, *i.e.*, a sequence v_n of vectors satisfying

$$\lim_{n\to\infty} \sup_{s\in S} \|\sigma(s)v_n - v_n\| = 0.$$

5. L^p-cohomology and affine actions on $L^p(G)$.

Let G be a locally compact group equipped with a left-invariant Haar measure. Let G act on $L^p(G)$ by right translations, which defines a representation $\rho_{G,p}$ defined by

$$\rho_{G,p}(g)f(x) = f(xg) \quad \forall f \in L^p(G).$$

Note that this representation is isometric if and only if G is unimodular, in which case $\rho_{G,p}$ is isomorphic to the left regular representation $\lambda_{G,p}$. In particular, in this case, the corresponding first reduced cohomologies are the same.

Now suppose that the group G is also compactly generated and equipped with a word metric d_S associated to a compact symmetric generating subset S. In this section, we prove that the first cohomology with values in the regular L^p-representation $\rho_{G,p}$ is topologically isomorphic to the first L^p-cohomology $H_p^1(G)$ (here, we mean the coarse version, see Section 4.1). By the result of Pansu mentioned in Remark 4.12, if G is a connected Lie group equipped with left-invariant Riemannian metric m, we can also identify $H^1(G,\rho_{G,p})$ with the first L^p-cohomology on (G,m) (see Section 4.2). We also obtain a direct proof of this fact.

We consider here the two following contexts: where G is a compactly generated locally compact group equipped with a length function d_S; or G is a connected Lie group, equipped with a left-invariant Riemannian metric.

Consider the linear map $J:\ \mathbf{D}_p(G)\to Z^1(G,\rho_{G,p})$ defined by

$$J(f)(g)=b(g)=f-\rho_{G,p}(g)f.$$

J is clearly well defined and induces a linear map

$$HJ:\ H_p^1(G)\to H^1(G,\rho_{G,p}).$$

Theorem 5.1. — *For $1\leqslant p<\infty$, the canonical map $HJ:\ H_p^1(G)\to H^1(G,\rho_{G,p})$ is an isomorphism of topological vector spaces.*

Let us start with a lemma.

Lemma 5.2. — *Let $1\leqslant p<\infty$ and $b\in Z^1(G,\rho_{G,p})$. Then there exists a 1-cocycle c in the cohomology class of b such that*

1. *the map $G\times G\to\mathbf{C}:\ (g,x)\mapsto c(g)(x)$ is continuous;*
2. *the continuous map $f(x)=c(x^{-1})(x)$ satisfies $c(g)=f-\rho_{G,p}(g)f$;*
3. *moreover if G is a Riemannian connected Lie group, then c can be chosen such that f lies in $D_p(G)$ (and in $C^\infty(G)$).*

Proof of the lemma. — Note that a cocycle b always satisfies $b(1)=0$. Let ψ be a continuous, compactly supported probability density on G. We define $c\in Z^1(G,\rho_{G,p})$ by

$$c(g)=\int_G b(gh)\psi(h)dh-\int_G b(h)\psi(h)dh=\int_G b(h)(\psi(g^{-1}h)-\psi(h))dh.$$

We have

$$\begin{aligned}c(gg') &= \int_G b(gg'h)\psi(h)dh - \int_G b(h)\psi(h)dh \\ &= \rho_{G,p}(g)\int_G b(g'h)\psi(h)dh + \int_G b(g)\psi(h)dh - \int_G b(h)\psi(h)dh\end{aligned}$$

But note that

$$\begin{aligned}\int_G b(g)\psi(h)dh &= \int_G b(ghh^{-1})\psi(h)dh \\ &= \rho_{G,p}(g)\int_G \rho_{G,p}(h)b(h^{-1})\psi(h)dh + \int_G b(gh)\psi(h)dh \\ &= -\rho_{G,p}(g)\int_G b(h)\psi(h)dh + \int_G b(gh)\psi(h)dh.\end{aligned}$$

So we obtain

$$\begin{aligned}c(gg') &= \rho_{G,p}(g)\left(\int_G b(g'h)\psi(h)dh - \int_G b(h)\psi(h)dh\right) \\ &\qquad + \int_G b(gh)\psi(h)dh - \int_G b(h)\psi(h)dh \\ &= \rho_{G,p}(g)c(g') + c(g).\end{aligned}$$

So c is a cocycle.

Let us check that c belongs to the cohomology class of b. Using the cocycle relation, we have

$$\begin{aligned}c(g) &= \int_G (\rho_{G,p}(g)b(h) + b(g))\psi(h)dh - \int_G b(h)\psi(h)dh \\ &= b(g) + \int_G (\rho_{G,p}(g)b(h) - b(h))\psi(h)dh \\ &= b(g) + \rho_{G,p}(g)\int_G b(h)\psi(h)dh - \int_G b(h)\psi(h)dh.\end{aligned}$$

But since $\int_G b(h)\psi(h)dh \in L^p(G)$, we deduce that c belongs to the cohomology class of b.

Now, let us prove that $(g,x) \mapsto c(g)(x)$ is continuous. It is easy to see from the definition of c that $g \mapsto c(g)(x)$ is defined and continuous for almost every x: fix such a point x_0. We conclude remarking that the cocycle relation implies

$$c(g)(x_0x) = c(xg)(x_0) - c(g)(x_0).$$

Now we can define $f(x) = c(x^{-1})(x) = -c(g)(1)$ and again the cocycle relation for c implies that $c(g) = f - \rho_{G,p}(g)f$.

Finally, assume that G is a Lie group and choose a smooth ψ. The function $\hat{\psi}$ defined by

$$\hat{\psi}(g) = \psi(g^{-1})$$

is also smooth and compactly supported. We have

$$c(g)(x) = f(x) - f(xg) = \int_G b(h)(x)(\hat{\psi}(h^{-1}g) - \hat{\psi}(h^{-1}))dh.$$

Hence, f is differentiable and

$$\nabla f(x) = -\int_G b(h)(x)(\nabla\hat{\psi})(h^{-1}))dh,$$

and so $\nabla f \in L^p(TG)$. □

Proof of Theorem 5.1. — The last statement of the lemma implies that HJ is surjective. The injectivity follows immediately from the fact that f is determined up to a constant by its associated cocycle $b = I(f)$.

We now have to prove that the isomorphism HJ is a topological isomorphism. This is immediate in the context of the coarse L^p-cohomology. Let us prove it for a Riemannian connected Lie group. Let S be a compact generating subset of G and define a norm on $Z^1(G, \rho_{G,p})$ by

$$\|b\| = \sup_{s\in S} \|b(s)\|_p.$$

Let ψ be a smooth, compactly generated probability density on G as in the proof of Lemma 5.2. Denote

$$f * \hat{\psi}(x) = \int_G f(k)\hat{\psi}(k^{-1}x)dk = \int_G f(xh)\psi(h)dh = \int_G f(k)\psi(x^{-1}k)dk.$$

We have

Lemma 5.3. — *There exists a constant $C < \infty$ such that for every $f \in \mathbf{D}_p(G)$,*

$$C^{-1}\|f * \hat{\psi}\|_{\mathbf{D}_p} \leqslant \|J(f)\| \leqslant C\|f\|_{\mathbf{D}_p}.$$

Proof of the lemma. — First, one checks easily that if b is the cocycle associated to f, then the regularized cocycle c constructed in the proof of Lemma 5.2 is associated to $f * \psi$.

We have

$$\begin{aligned}\nabla(f * \hat{\psi})(x) &= \int f(k)\nabla\hat{\psi}(k^{-1}x)dk \\ &= \int (f(k) - f(x))\nabla\hat{\psi}(k^{-1}x)dk \\ &= \int (f(xh) - f(x))\nabla\hat{\psi}(h^{-1})dh.\end{aligned}$$

So

$$\begin{aligned}\|\nabla(f*\hat{\psi})\|_p &\leqslant \sup_{h\in\mathrm{Supp}(\hat{\psi})}\int |f(xh)-f(x)|^p\|\nabla\hat{\psi}\|_\infty^p dx\\ &= \sup_{h\in\mathrm{Supp}(\hat{\psi})}\|b(h)\|^p\|\nabla\hat{\psi}\|_\infty^p,\end{aligned}$$

which proves the left-hand inequality of Lemma 5.3. Let $g\in G$ and $\gamma : [0,d(1,g)]\to G$ be a geodesic between 1 and g. For any $f\in\mathbf{D}_p(G)$ and $x\in G$, we have

$$(f-\rho_{G,p}(g)f)(x)=f(x)-f(xg)=\int_0^{d(1,g)}\nabla f(x)\cdot\gamma'(t)dt.$$

So we deduce that

$$\|f-\rho_{G,p}(g)f\|_p\leqslant d(1,g)\|\nabla f\|_p,$$

which proves the right-hand inequality of Lemma 5.3. □

Continuity of HJ follows from continuity of J which is an immediate consequence of Lemma 5.3.

Let us prove that the inverse of HJ is continuous. Let b_n be a sequence in $Z^1(G,\rho_{G,p})$, converging to 0 modulo $B^1(G,\rho_{G,p})$. This means that there exists a sequence a_n in $B^1(G,\rho_{G,p})$ such that $\|b_n+a_n\|\to 0$. By Lemma 5.2, we can assume that $b_n(g)=f_n-\rho_{G,p}(g)f_n$ with $f\in\mathbf{D}_p(G)$. On the other hand, $a_n=h-\rho_{G,p}(g)h$ with $h\in L^p(G)$. As compactly supported, regular[9] functions on G are dense in $L^p(G)$, we can assume that h is regular. So finally, replacing f_n by f_n+h_n, which is in $\mathbf{D}_p(G)$, we can assume that $J(f_n)\to 0$. Then, by Lemma 5.3, $\|f_n*\hat{\psi}\|_{\mathbf{D}_p}\to 0$. But by the proof of Lemma 5.2, $f_n*\hat{\psi}$ is in the class of L^p-cohomology of f_n. This finishes the proof of Theorem 5.1. □

6. Sublinearity of cocycles

THEOREM 6.1. — *Let G be a locally compact compactly generated group and let S be a compact symmetric generating subset. Let π be a mixing L^p-representation of G. Then, every 1-cocycle $b\in Z^1(G,\pi)$ is sublinear, i.e.,*

$$\|b(g)\|=o(|g|_S)$$

when $|g|_S\to\infty$, $|g|_S$ being the word length of g with respect to S.

(9) Regular here, means either continuous, or smooth if G is a Lie group.

Let $L^p(X,m)$ the L^p-space on which G acts. We will need the following lemma.

Lemma 6.2. — *Let us keep the assumptions of the theorem. For any fixed $j\in\mathbf{N}$,*

$$\|\pi(g_1)v_1+\ldots+\pi(g_j)v_j\|_p^p\to\|v_1\|_p^p+\ldots+\|v_j\|_p^p$$

when $d_S(g_k,g_l)\to\infty$ whenever $k\neq l$, uniformly with respect to $(v_1,\ldots,v_j)$ on every compact subset of $(L^p(X,m))^j$.

Proof of Lemma 6.2. — First, let us prove that if the lemma holds pointwise with respect to $\overline{v}=(v_1,\ldots,v_j)$, then it holds uniformly on every compact subset K of $(L^p(X,m))^j$. Let us fix some $\varepsilon>0$. Equip $(L^p(X,m))^j$ with the norm

$$\|\overline{v}\|=\max_i\|v_i\|_p,$$

and take a finite covering of K by balls of radius ε: $B(\overline{w},\varepsilon)$, $\overline{w}\in W$, where W is a finite subset of K. Take $\min_{1\leqslant k\neq i\leqslant j}d_S(g_k,g_l)$ large enough so that for any $\overline{w}\in W$, $\|\pi(g_1)v_1+\ldots+\pi(g_j)v_j\|_p^p$ is closed to $\|v_1\|_p^p+\ldots+\|v_j\|_p^p$ up to ε. As $\pi(g)$ preserves the L^p-norm for every $g\in G$, we immediately see that for any $\overline{v}$ in K, $\|\pi(g_1)v_1+\ldots+\pi(g_j)v_j\|_p^p$ is closed to $\|v_1\|_p^p+\ldots+\|v_j\|_p^p$ up to some ε' only depending on K, p and ε, and such that $\varepsilon'\to0$ when $\varepsilon\to0$.

So now, we just have to prove the lemma for $v_1,\ldots,v_j$ belonging to a dense subset of $L^p(X,m)$. Thus, assume that for every $1\leqslant k\leqslant j$, v_k is bounded and compactly supported. Let us denote by A_k the support of v_k. For every finite sequence $\overline{g}=g_1,\ldots,g_j$ of elements in G, we write, for every $1\leqslant i\leqslant j$,

- $U_{i,\overline{g}}=\left(\bigcup_{l\neq i}g_lA_l\right)\cap g_iA_i$;
- $A_{i,\overline{g}}=g_iA_i\smallsetminus U_{i,\overline{g}}$.

The key point of the proof is the following observation. □

Claim 6.3. — *For every $1\leqslant i\leqslant j$,*

$$m(U_{i,\overline{g}})\to0,$$

when the relative distance between the g_k goes to ∞.

Proof of the claim. — For $u, v \in L^2(G, m)$, write $\langle u, v\rangle = \int_X u(x)v(x)\, dm(x)$. For every $1 \leqslant i \leqslant j$,

$$\begin{aligned} m\left(\left(\bigcup_{l\neq i} g_l A_l\right) \cap g_i A_i\right) &= \left\langle \sum_{l\neq i} \pi(g_l)1_{A_l}, \pi(g_i)1_{A_i}\right\rangle \\ &= \sum_{l\neq i} \langle \pi(g_l)1_{A_l}, \pi(g_i)1_{A_i}\rangle \\ &= \sum_{l\neq i} \langle \pi(g_l^{-1}g_i)1_{A_i}, 1_{A_l}\rangle \\ &= \sum_{l\neq i} m(g_l^{-1}g_i A_i \cap A_l) \to 0 \end{aligned}$$

by mixing property of the action. □

Proof of the lemma. — First, observe that by the claim,

$$\|\pi(g_i)v_i 1_{U_{i,\bar g}}\|_p^p \leqslant \|v_i\|_\infty^p m(U_{i,\bar g}) \to 0,$$

when the relative distance between the g_k goes to ∞. In other words, as $\pi(g_i)v_i = \pi(g_i)v_i 1_{A_{i,\bar g}} + \pi(g_i)v_i 1_{U_{i,\bar g}}$,

$$\|\pi(g_i)v_i 1_{A_{i,\bar g}} - \pi(g_i)v_i\|_p^p \to 0.$$

In particular,

$$\|\pi(g_i)v_i 1_{A_{i,\bar g}}\|_p^p \to \|v_i\|_p^p.$$

On the other hand, the $A_{i,\bar g}$ are piecewise disjoint. So finally, we have

$$\begin{aligned} &\lim_{d_S(g_l,g_k)\to\infty} \|\pi(g_1)v_1 + \ldots + \pi(g_j)v_j\|_p^p \\ &\quad= \lim_{d_S(g_l,g_k)\to\infty} \|\pi(g_1)v_1 1_{A_{1,\bar g}} + \ldots + \pi(g_j)v_j 1_{A_{j,\bar g}}\|_p^p \\ &\quad= \lim_{d_S(g_l,g_k)\to\infty} \|\pi(g_1)v_1 1_{A_{1,\bar g}}\|^p + \ldots + \|\pi(g_j)v_j 1_{A_{j,\bar g}}\|_p^p \\ &\quad= \|v_1\|_p^p + \ldots + \|v_j\|_p^p, \end{aligned}$$

which proves the lemma. □

Proof of Theorem 6.1. — Fix some $\varepsilon > 0$. Let $g = s_1 \ldots s_n$ be a minimal decomposition of g into a product of elements of S. Let $m \leqslant n$, q and $r < m$ be positive integers such that $n = qm + r$. To simplify notation, we assume $r = 1$. For $1 \leqslant i < j \leqslant n$, denote by g_j the prefix $s_1 \ldots s_j$ of g and by $g_{i,j}$ the subword $s_{i+1} \ldots s_j$ of g. Developing $b(g)$ with respect to the cocycle relation, we obtain

$$b(g) = b(s_1) + \pi(g_1)b(s_2) + \ldots + \pi(g_{n-1})b(s_n).$$

Let us put together the terms in the following way

$$\begin{aligned}b(g) =&\big[b(s_1) + \pi(g_m)b(s_{m+1}) + \ldots + \pi(g_{(q-1)m})b(s_{(q-1)m+1})\big]\\&+ \big[\pi(g_1)b(s_2) + \pi(g_{m+1})b(s_{m+2}) + \ldots + \pi(g_{(q-1)m+1})b(s_{(q-1)m+2})\big]\\&+ \ldots + \big[\pi(g_{m-1})b(s_m) + \pi(g_{2m-1})b(s_{2m}) + \ldots + \pi(g_{qm})b(s_{qm+1})\big].\end{aligned}$$

In the above decomposition of $b(g)$, consider each term between $[\cdot]$, *e.g.*, of the form

$$\pi(g_k)b(s_{k+1}) + \ldots + \pi(g_{(q-1)m+k})b(s_{(q-1)m+k+1}) \tag{6.1}$$

for $0 \leqslant k \leqslant m-1$ (we decide that $s_0 = 1$). Note that since S is compact and π is continuous, there exists a compact subset K of E containing $b(s)$ for every $s \in S$. Clearly since $g = s_1 \ldots s_n$ is a minimal decomposition of g, the length of $g_{i,j}$ with respect to S is equal to $j-i-1$. For $0 \leqslant i < j \leqslant q-1$ we have

$$d_S(g_{im+k}, g_{jm+k}) = |g_{im+k,jm+k}|_S = (j-i)m \geqslant m.$$

So by Lemma 6.2, for $m = m(q)$ large enough, the p-power of the norm of (6.1) is less than

$$\|b(s_{k+1})\|_p^p + \|b(s_{m+k+1})\|_p^p + \ldots + \|b(s_{(q-1)m+k+1})\|_p^p + 1.$$

The above term is therefore less than $2q$. Hence, we have

$$\|b(g)\|_p \leqslant 2mq^{1/p}.$$

So for $q \geqslant q_0 = (2/\varepsilon)^{p/(p-1)}$, we have

$$\|b(g)\|_p/n \leqslant 2q^{1-1/p} \leqslant \varepsilon.$$

Now, let n be larger than $m(q_0)q_0$. We have $\|b(g)\|_p/|g| \leqslant \varepsilon$. □

7. Proof of Theorem 2.2

Theorem 2.2 results from Theorem 6.1 and the following result, which is an immediate generalization of [6, Proposition 3.6]. For the convenience of the reader, we give its short proof.

Proposition 7.1. — *Let G be a group with property (CF) and let π be a continuous isometric action of G on a Banach space E. Let b a 1-cocycle in $Z^1(G,\pi)$. Then b belongs to $\overline{B^1(G,\pi)}$ if and only if b is sublinear.*

Proof. — Assume that b is sublinear.

Let (F_n) be a controlled Følner sequence in G. Define a sequence $(v_n) \in E^{\mathbf{N}}$ by

$$v_n = \frac{1}{\mu(F_n)} \int_{F_n} b(g)dg.$$

We claim that (v_n) defines a sequence of almost fixed points for the affine action σ defined by $\sigma(g)v = \pi(g)v + b(g)$. Indeed, we have

$$\begin{aligned}\|\sigma(s)v_n - v_n\| &= \left\| \frac{1}{\mu(F_n)} \int_{F_n} \sigma(s)b(g)dg - \frac{1}{\mu(F_n)} \int_{F_n} b(g)dg \right\| \\ &= \left\| \frac{1}{\mu(F_n)} \int_{F_n} b(sg)dg - \frac{1}{\mu(F_n)} \int_{F_n} b(g)dg \right\| \\ &= \left\| \frac{1}{\mu(F_n)} \int_{s^{-1}F_n} b(g)dg - \frac{1}{\mu(F_n)} \int_{F_n} b(g)dg \right\| \\ &\leqslant \frac{1}{\mu(F_n)} \int_{s^{-1}F_n \triangle F_n} \|b(g)\| dg.\end{aligned}$$

Since $F_n \subset S^n$, we obtain that

$$\|\sigma(s)v_n - v_n\| \leqslant \frac{C}{n} \sup_{|g|_S \leqslant n+1} \|b(g)\|$$

which converges to 0. This proves the non-trivial implication of Proposition 7.1. □

8. Liouville D_p-Properties: a direct approach.

In this section, we propose a direct proof of Corollary 2.5. Instead of using Theorem 2.2 and Theorem 5.1, we reformulate the proof, only using Theorem 6.1 and [26, Theorem 11]. The interest is to provide an explicit approximation of an element of $\mathbf{D}_p(G)$ by a sequence of functions in $W^{1,p}(G)$ using a convolution-type argument. Since Liouville D_p-Property is equivalent to the vanishing of $\overline{H_p}^1(G)$, we have to show that for every p-Dirichlet function on G, there exists a sequence of functions (f_n) in $W^{1,p}(G)$ such that the sequence $(\|\nabla(f - f_n)\|_p)$ converges to zero. Let (F_n) be a *right* controlled Følner sequence. By a standard regularization argument, we can construct for every n, a smooth 1-Lipschitz function φ_n such that

- $0 \leqslant \varphi_n \leqslant 1$;
- for every $x \in F_n$, $\varphi_n(x) = 1$;
- for every y at distance larger than 2 from F_n, $\varphi_n(y) = 0$.

Denote by $F_n' = \{x \in G: \ d(x, F_n) \leqslant 2\}$. As F_n is a controlled Følner sequence, there exists a constant $C < \infty$ such that

$$\mu(F_n' \smallsetminus F_n) \leqslant C\mu(F_n')/n$$

and

$$F_n' \subset B(1, Cn).$$

Define

$$p_n = \frac{\varphi_n}{\int_G \varphi_n d\mu}.$$

Note that p_n is a probability density satisfying for every $x \in X$,

$$|\nabla p_n(x)| \leqslant \frac{1}{\mu(F_n)}.$$

For every $f \in D_p(G)$, write $P_n f(x) = \int_X f(y)p_n(y^{-1}x)d\mu(y)$. As G is unimodular,

$$P_n f(x) = \int_X f(yx^{-1})p_n(y^{-1})d\mu(y).$$

We claim that $P_n f - f$ is in $W^{1,p}$. For every $g \in G$ and every $f \in D_p$, we have

$$\|f - \rho(g)f\|_p \leqslant d(1, g)\|\nabla f\|_p.$$

Recall that the support of p_n is included in F_n' which itself is included in $B(1, Cn)$. Thus, integrating the above inequality, we get

$$\|f - P_n f\|_p \leqslant Cn\|\nabla f\|_p,$$

so $f - P_n f \in L^p(G)$.

It remains to show that the sequence $(\|\nabla P_n f\|_p)$ converges to zero. We have

$$\nabla P_n f(x) = \int_G f(y)\nabla p_n(y^{-1}x)d\mu(y).$$

Since $\int_G \nabla p d\mu = 0$, we get

$$\begin{aligned}\nabla P_n f(x) &= \int_G (f(y) - f(x^{-1}))\nabla p_n(y^{-1}x)d\mu(y)\\ &= \int_G (f(yx^{-1}) - f(x^{-1}))\nabla p_n(y^{-1})d\mu(y).\end{aligned}$$

Hence,

$$\begin{aligned}\|\nabla P_n f\|_p &\leqslant \int_G \|\lambda(y)f - f\|_p |\nabla p_n(y^{-1})| d\mu(y)\\ &\leqslant \frac{1}{\mu(F_n)} \int_{F'_n \smallsetminus F_n} \|\lambda(y)f - f\|_p d\mu(y)\\ &\leqslant \frac{\mu(F'_n \smallsetminus F_n)}{F_n} \sup_{|g|\leqslant Cn} \|b(g)\|_p\\ &\leqslant \frac{C}{n} \sup_{|g|\leqslant Cn} \|b(g)\|_p\end{aligned}$$

where $b(g) = \lambda(g)f - f$. Note that $b \in Z^1(G, \lambda_{G,p})$. Thus, by Theorem 6.1,

$$\|\nabla P_n f\|_p \to 0.$$

This completes the proof of Corollary 2.5.

9. Non-vanishing of the first reduced L^p-cohomology on a non-elementary Gromov hyperbolic space.

Let us start with a remark about first L^p-cohomology on a metric measure space.

Remark 9.1 (Coupling between 1*-cycles and* 1*-cocycles).* — A 1-chains on (X, d, μ) is a functions supported on $\Delta_r = \{(x, y) \in X^2, d(x, y) \leqslant r\}$ for some $r > 0$. The L^p-norm of a (measurable) 1-chain s is the norm

$$\left(\int_{X^2} |s(x, y)|^p d\mu(x) d\mu(y)\right)^{1/p}.$$

A 1-chain s is called a 1-cycle if $s(x, y) = s(y, x)$.

Given $f \in \mathbf{D}_p$, we define a 1-cocycle associated to f by $c(x, y) = f(x) - f(y)$, for every $(x, y) \in X^2$. Let s be a 1-cycle in L^q, with $1/p + 1/q = 1$. We can form a coupling between c and s

$$\langle c, s\rangle = \int_{X^2} c(x, y)s(x, y)d\mu(x)d\mu(y) = \int_{X^2} (f(x)-f(y))s(x, y)d\mu(x)d\mu(y).$$

Clearly, if $f \in L^p$, then as s is a cycle, we have $\langle c, s\rangle = 0$. This is again true for f in the closure of $L^p(X)$ for the norm of $\mathbf{D}_p(X)$. Hence, to prove that a 1-cocycle c is non-trivial in $\overline{H^1}_p(X)$, it is enough to find a 1-cycle in L^q whose coupling with c is non-zero.

The main result of this section is the following theorem.

THEOREM 9.2. — *Let X be a Gromov hyperbolic 1-geodesic metric measure space with bounded geometry having a bi-Lipschitz embedded 3-regular tree, then for p large enough, it has non-trivial first reduced L^p-cohomology.*

From this theorem, we will deduce

COROLLARY 9.3. — *A homogeneous Riemannian manifold M has non-zero first reduced L^p-cohomology for some $1 < p < \infty$ if and only if it is non-elementary Gromov hyperbolic.*

Proof of Corollary 9.3. — By Theorem 2.6, if M has non-zero $\overline{H}_p^1(M)$ for some $1 < p < \infty$, then being quasi-isometric to a negatively curved homogeneous manifold, it is non-elementary Gromov hyperbolic.

Conversely, let M be a Gromov hyperbolic homogeneous manifold. As M is quasi-isometric to its isometry group G, which is a Lie group with finitely many components, we can replace M by G, and assume that G is connected. If G has exponential growth, then [5, Corollary 1.3] it has a bi-Lipschitz embedded 3-regular tree T, and hence Theorem 9.2 applies. Otherwise G has polynomial growth, and we conclude thanks to the following classical fact. □

PROPOSITION 9.4. — *A non-elementary Gromov-hyperbolic connected Lie group has exponential growth.*

Proof. — Let G be a connected Lie group with polynomial growth. By [13], G is quasi-isometric to a simply connected nilpotent group G, whose asymptotic cone [20] is homeomorphic to another (graded) simply connected nilpotent Lie group with same dimension. Hence, unless G is quasi-isometric to $\mathbf{R}$, the asymptotic cone of G has dimension larger or equal than 2. But [12, page 37] the asymptotic cone of a Gromov hyperbolic space is an $\mathbf{R}$-tree, and therefore has topological dimension 1. □

Proof of Theorem 9.2. — The proof contains ideas that we found in [12, page 258]. Roughly speaking, we start by considering a non-trivial cycle defined on a bi-Lipschitz embedded 3-regular subtree T of X. To construct a 1-cocycle which has non-trivial reduced cohomology, we take a Lipschitz function F defined on the boundary of X, such that F is non-constant in restriction to the boundary of the subtree T. We then extend F to a function defined f on X which defines a 1-cocyle in $D_p(X)$. Coupling this cocycle with our cycle on T proves its non-triviality in $\overline{H}_p^1(X)$.

Boundary at infinity of a hyperbolic space. To denote the distance between to points in X or in its boundary, we will use indifferently the notation $d(x,y)$, or the notation of Gromov $|x-y|$. Let us fix a point $o \in X$. We will denote $|x| = |x-o| = d(x,o)$.

Consider the Gromov boundary (see [11, Chapter 1.8] or [9]) of X, *i.e.*, the set of geodesic rays issued from o up to Hausdorff equivalence.

For ε small enough, there exists [9] a distance $|\cdot|_\varepsilon$ on $\partial_\infty X$, and $C < \infty$ such that

$$|u-v|_\varepsilon \leqslant \limsup_{t\to\infty} e^{-\varepsilon(v(t)|w(t))} \leqslant C|u-v|_\varepsilon$$

for all $v, w \in \partial_\infty X$, where $(\cdot|\cdot)$ denotes the Gromov product, *i.e.*,

$$(x|y) = \frac{1}{2}(|x|+|y|-|x-y|).$$

Reduction to graphs. A 1-geodesic metric measure space with bounded geometry is trivially quasi-isometric to a connected graph with bounded degree (take a maximal 1-separated net, and join its points which are at distance 1 by an edge). Hence, we can assume that X is the set of vertices of a graph with bounded degree.

A Lipschitz function on the boundary. By [12, page 221], T has a cycle which has a non-zero pairing with every non-zero 1-cochain c on T supported on a single edge e. Hence, to prove that $\overline{H}_p^1(X) \neq 0$, it is enough to find an element c in $D_p(X)$ whose restriction to T is zero everywhere but on e.

The inclusion of T into X being bi-Lipschitz, it induces a homeomorphic inclusion of the boundary of T, which is a Cantor set, into the boundary of X. We therefore identify $\partial_\infty T$ with its image in $\partial_\infty X$. Consider T_1 and T_2 the two complementary subtrees of T which are separated by e. This induces a partition of the boundary $\partial_\infty T$ into two clopen non-empty subsets O_1 and O_2. As O_1 and O_2 are disjoint compact subsets of $\partial_\infty X$, they are at positive distance from one another. Hence, for $\delta > 0$ small enough, the δ-neighborhoods V_1 and V_2 of respectively O_1 and O_2 in $\partial_\infty X$ are disjoint.

Now, take a Lipschitz function F on $\partial_\infty X$ which equals 0 on V_1 and 1 on V_2.

Extension of F to all of X. Let us first assume that every point in X is at bounded distance from a geodesic ray issued from o.

Let us define a function f on X: for every x in $X \smallsetminus \{o\}$, we denote element by u_x a geodesic ray issued from o and passing at bounded distance, say C from x. Define

$$f(x) = F(u_x) \quad \forall x \in X \smallsetminus \{o\}.$$

Let us prove that for p large enough, $f \in D_p(X)$. Take two elements x and y in X such that $|x-y| \leqslant 1$, we have

$$\begin{aligned} |u_x(t)-u_y(t)| &\leqslant |u_x(t)-x|+|x-u_y(t)| \\ &\leqslant |u_x(t)-x|+|y-u_y(t)|+|x-y| \\ &\leqslant |u_x(t)-x|+|y-u_y(t)|+1. \end{aligned}$$

So for large t,

$$\begin{aligned} 2(u_x(t)|u_y(t)) &= |u_x(t)|+|u_y(t)|-|u_x(t)-u_y(t)| \\ &\geqslant |u_x(t)|+|u_y(t)|-|u_x(t)-x|-|y-u_y(t)|-1 \\ &\geqslant |x|+|y|-2|u_x(t)-x|-2|y-u_y(t)|-1 \\ &\geqslant |x|+|y|-4C-1 \\ &\geqslant 2|x|-4(C+1). \end{aligned}$$

Let $K>0$ be the Lipschitz norm of F, *i.e.*, $K=\sup_{u\neq v\in\partial_\infty T}\frac{|F(u)-F(v)|}{|u-v|_\varepsilon}$. We have

$$\begin{aligned} |f(x)-f(y)|^p &= |F(u_x)-F(u_y)|^p \\ &\leqslant K^p|u_x-u_y|^p \\ &\leqslant K^p \limsup_{t\to\infty} e^{-p\varepsilon(u_x(t)|u_y(t))} \\ &\leqslant K^p e^{-p\varepsilon|x|+2(C+1)p}. \end{aligned}$$

On the other hand, as $\mu(B(o,|x|)) \leqslant Ce^{\lambda|x|}$ for some λ, if $p\varepsilon>\lambda$, then f is in $D_p(X)$.

Now, let us consider the values of f along T. To fix the ideas, let us assume that o is a vertex of T. We will now show that up to modifying T, we can assume that f takes the value 0 on T_1, and 1 on T_2. Hence the coupling of the corresponding cocycle c with the cycle of [12, page 221] is non-zero, which implies that $\overline{H}_p^1(X)\neq 0$.

For $i=1,2$, take x_i a vertex of T_i. Let e_{x_i} be the edge whose one extremity is x_i and that separates o and x_i. Let T_{x_i} be the connected component of $T \smallsetminus \{e_{x_i}\}$ contained in T_i.

The point that we need to prove is that if both x_1 and x_2 are far enough from o, f equals 0 on T_{x_1} and 1 on T_{x_2}. Then, up to replacing T_1 and T_2 by T_{x_1} and T_{x_2}, and the geodesic segment between x_1 and x_2 by a single edge (which becomes e), we are done.

Let v be a geodesic ray of T_1 emanating from o and passing through some vertex y of T_{x_1}. Let z be the corresponding element of $\partial_\infty T\subset\partial_\infty X$. Let t be a geodesic ray in X from o to z. As T is bi-Lipschitz embedded in X, v

is a quasi-geodesic ray in X. Hence it stays at bounded distance, say less than C from t. In particular, t passes at distance less than C from y. So by choosing $d(o,y)$ large enough, $|u_y - z|$ can be made arbitrarily small, in particular $\leqslant \delta$. Hence, choosing x_1 far enough from o in T_1, we have that all u_y where $y \in T_{x_1}$ belong to V_1. Therefore $f(y) = 0$. The case $i = 2$ is similar.

Reduction to the case when every point in X is at bounded distance from a geodesic ray issued from o.

In this section, we embed X into a larger graph $\tilde{X}$ satisfying the property that every vertex is contained in a geodesic ray emanating from o.

Let Y be the graph whose set of vertices is $\mathbf{N}$ and such that n and m are joined by an edge if and only if $|n-m| = 1$. Consider the graph $\tilde{X}$ obtained by gluing a copy of Y to every vertex of X. This is done by identifying this vertex with the vertex 0 of the corresponding copy of Y. Clearly $\tilde{X}$ is a hyperbolic graph with bounded degree. It contains X as an isometrically embedded subgraph. In particular, T is bi-Lipschitz embedded into $\tilde{X}$. Finally, $\tilde{X}$ satisfies that every point in $\tilde{X}$ belongs to a geodesic ray issued from o.

Applying the above to $\tilde{X}$, we construct an element $\tilde{f}$ in $D_p(\tilde{X})$ that has a non-trivial coupling with the cycle that we considered on T. As the support of this cycle is contained in X, the restriction f of $\tilde{f}$ to X also has a non-trivial coupling with it. Moreover, f belongs to $D_p(X)$, so it defines a non-trivial cocycle in $\overline{H}_p^1(X)$. □

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