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ON FUNDAMENTAL GROUPS OF ALGEBRAIC VARIETIES AND VALUE DISTRIBUTION THEORY

by **Katsutoshi YAMANOI**

Dedicated to Professor Junjiro Noguchi on his 60th birthday

Abstract. — If a smooth projective variety X admits a non-degenerate holomorphic map $\mathbb{C} \to X$ from the complex plane $\mathbb{C}$, then for any finite dimensional linear representation of the fundamental group of X the image of this representation is almost abelian. This supports a conjecture proposed by F. Campana, published in this journal in 2004.

Résumé. — Si une variété X projective lisse admet une application holomorphe non-dégénérée $\mathbb{C} \to X$ du plan complexe $\mathbb{C}$, alors pour chaque représentation linéaire de dimension finie du groupe fondamental de X l'image de cette représentation est presque abélienne. Cela soutient une conjecture proposée par F. Campana, parue dans ce même journal en 2004.

1. Main results

Let X be a smooth projective variety. We say that a holomorphic map $f\colon \mathbb{C} \to X$ is non-degenerate if the image $f(\mathbb{C})$ is Zariski dense in X. A group G is called almost abelian if G has a finite index subgroup which is abelian. In this paper, we prove the following theorem.

Theorem 1.1. — *Let X be a smooth projective variety which admits a non-degenerate holomorphic map $f\colon \mathbb{C} \to X$. Then for any representation $\varrho\colon \pi_1(X) \to \mathrm{GL}_n(\mathbb{C})$, the image $\varrho(\pi_1(X))$ is almost abelian.*

This theorem shows that the following conjecture proposed by F. Campana [4, Conjecture 9.8] is true in the special case that $\pi_1(X)$ is linear.

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Conjecture 1.2. — *Let X be a smooth projective variety which admits a non-degenerate holomorphic map $f\colon \mathbb{C} \to X$. Then the fundamental group $\pi_1(X)$ is almost abelian.*

This conjecture comes from Campana's theory of "special" variety (*cf.* [4]). A complex manifold X which admits a holomorphic map $f\colon \mathbb{C} \to X$ with metrically dense image has vanishing Kobayashi pseudo-metric. It is Campana's view that a smooth projective variety X would have vanishing Kobayashi pseudo-metric if and only if X is "special" (*cf.* [4, Conjecture 9.2]), and that the fundamental group of a "special" variety would be almost abelian (*cf.* [4, Conjecture 7.1]). For more discussion about Conjecture 1.2, we refer the reader to [4].

A representation $\varrho\colon \pi_1(X) \to \mathrm{GL}_n(\mathbb{C})$ is called big if the following condition is satisfied (*cf.* [11]):

> If $Z \subset X$ is a positive dimensional subvariety containing a very general point of X, then the image $\varrho(\mathrm{Im}(\pi_1(\bar{Z}) \to \pi_1(X)))$ is infinite. Here $\bar{Z}$ is a desingularization of Z.

For example, if there exists an unramified Galois covering $\widetilde{X} \to X$ such that $\widetilde{X}$ is a Stein space and its Galois transformation group Γ is a linear group, then the corresponding surjection $\pi_1(X) \to \Gamma$ is a big representation.

Corollary 1.3. — *Let X be a smooth projective variety with a big representation $\varrho\colon \pi_1(X) \to \mathrm{GL}_n(\mathbb{C})$. If X admits a non-degenerate holomorphic map $f\colon \mathbb{C} \to X$, then there exists a finite unramified covering X' of X which is birationally equivalent to an Abelian variety.*

The strategy of the proof of Theorem 1.1 is roughly as follows. Based on results of [4] and [20], Campana proved the following ([4]): If there exists a representation $\varrho\colon \pi_1(X) \to \mathrm{GL}_n(\mathbb{C})$ such that the image $\varrho(\pi_1(X))$ is not almost abelian, then there exist a finite unramified covering $X' \to X$ and a dominant rational map $X' \dashrightarrow Z$ with Z of general type and positive dimensional. The proof of this result shows that Z is not only of general type, but has more precise structure. Thanks to this precise structure, we can show that every holomorphic map $g\colon \mathbb{C} \to Z$ is degenerate, *i.e.* the image $g(\mathbb{C})$ is not Zariski dense in Z. This implies our theorem.

2. A reduction of the proof of Theorem 1.1

The proof of Theorem 1.1 is based on the following proposition, which is a special case of the theorem.

Proposition 2.1. — *Let X be a smooth projective variety, and let G be an almost simple algebraic group defined over the complex number field. Assume that there exists a representation $\varrho\colon \pi_1(X) \to G(\mathbb{C})$ whose image $\varrho(\pi_1(X))$ is Zariski dense in G. Then every holomorphic map $f\colon \mathbb{C} \to X$ is degenerate.*

Proposition 2.1 implies Theorem 1.1. — Let X be a smooth projective variety which admits a non-degenerate holomorphic map $f\colon \mathbb{C} \to X$. Let $\varrho\colon \pi_1(X) \to \mathrm{GL}_n(\mathbb{C})$ be a representation. We shall prove that $\varrho(\pi_1(X))$ is almost abelian.

Let $H \subset \mathrm{GL}_n(\mathbb{C})$ be the Zariski closure of the image $\varrho(\pi_1(X))$. Let $H_0 \subset H$ be the connected component of H containing the identity element of H. Then $\Gamma = \varrho^{-1}(\varrho(\pi_1(X)) \cap H_0)$ is a finite index subgroup of $\pi_1(X)$. Replacing X by the finite unramified covering $X' \to X$ which corresponds to Γ, we may assume that H is connected.

Let $R(H) \subset H$ be the radical of H, *i.e.* $R(H)$ is the maximal connected solvable closed normal subgroup of H. Put $H_{s.s.} = H/R(H)$. We first prove that $H_{s.s.}$ is trivial.

Assume, for the sake of contradiction, that $H_{s.s.}$ is not trivial. Then $H_{s.s.}$ is a semi-simple algebraic group. Hence $H_{s.s.}$ is an almost direct product of almost simple algebraic groups $G_1, \dots, G_l$. Let $H_{s.s.} \to G_1$ be a projection, and let $\varrho'\colon \pi_1(X) \to G_1$ be the composition of ϱ and the two projections $H \to H_{s.s.} \to G_1$. Since the image $\varrho'(\pi_1(X))$ is Zariski dense in G_1, we may apply Proposition 2.1 to conclude that every holomorphic map $\mathbb{C} \to X$ is degenerate. This contradicts to our assumption that X admits a non-degenerate holomorphic map $f\colon \mathbb{C} \to X$. Hence we have proved that $H_{s.s.}$ is trivial, *i.e.* $H = R(H)$.

Now the image $\varrho(\pi_1(X))$ is a solvable group. We note that every finite unramified covering X' of X admits a non-degenerate holomorphic map $f'\colon \mathbb{C} \to X'$ coming from a lifting of $f\colon \mathbb{C} \to X$. Hence by [14, Theorem 6.4.1], the Albanese map of X' is surjective for every finite unramified covering X' of X. Hence by [3, Théorème 2.9], there exists a finite unramified covering X' of X such that ϱ factors the induced group homomorphism $\pi_1(X') \to \pi_1(\mathrm{Alb}(X'))$. From this, we conclude that $\varrho(\pi_1(X'))$ is abelian. Hence $\varrho(\pi_1(X))$ is almost abelian. □

3. Representations over non-archimedean local fields

Let K be a number field, and let $\mathcal{O}_K$ be the ring of integers in K. Given a prime ideal p from $\mathcal{O}_K$, we denote by K_p the completion of K with respect to the natural discrete valuation defined by p. Let G be an almost simple algebraic group defined over K_p, and let $\varrho\colon \pi_1(X) \to G(K_p)$ be a p-adic representation. We say that ϱ is p-bounded if the image $\varrho(\pi_1(X))$ is contained in a maximal compact subgroup of $G(K_p)$. If ϱ is not p-bounded, then we say that ϱ is p-unbounded.

In this section, we prove the following:

Proposition 3.1. — *Let X be a smooth projective variety. Let G be an almost simple algebraic group defined over the p-adic field K_p. Assume that there exists a p-unbounded representation $\varrho\colon \pi_1(X) \to G(K_p)$ whose image is Zariski dense in G. Then every holomorphic map $f\colon \mathbb{C} \to X$ is degenerate.*

The proof of this proposition is based on the consideration of the spectral covering $\pi\colon X^s \to X$. We follow the exposition of [20, Section 1]. The construction of X^s is based on the theory of equivariant harmonic maps to buildings due to Gromov and Schoen [8]; Since ϱ is reductive, there exists a non-constant ϱ-equivariant pluriharmonic map $u\colon \widetilde{X} \to \triangle(G)$ from the universal covering of X to the Bruhat-Tits building of G. Considering the complexified differential of u, we get a multi-valued holomorphic one form ω on X. We consider a finite ramified Galois covering $\pi\colon X^s \to X$ such that $\pi^*\omega$ splits into single-valued holomorphic one forms $\omega_1,\dots,\omega_l$. All the forms $\omega_1,\dots,\omega_l$ are contained in the space $H^0(X^s,\pi^*\Omega^1_X)$. The covering $\pi\colon X^s \to X$ is unramified outside $\cup_{\omega_i\neq\omega_j}(\omega_i-\omega_j)_0$ where $\omega_i-\omega_j$ are considered as forms from $H^0(X^s,\pi^*\Omega^1_X)$ (*cf.* [9, Lemma 2.1]). For more detail about the construction of the spectral covering, we refer the reader to [20], [21], [5] and [10].

We construct the Albanese map $\Phi\colon X^s \to A$ with respect to $\omega_1,\dots,\omega_l$ as follows (*cf.* [21, p. 64]): Let $a\colon \widehat{X^s} \to A(\widehat{X^s})$ be the Albanese map, where $\psi\colon \widehat{X^s} \to X^s$ is a desingularization of X^s. For $i=1,\dots,l$, let $\widetilde{\omega_i}$ be the holomorphic one form on $A(\widehat{X^s})$ such that $\psi^*(\omega_i) = a^*\widetilde{\omega}_i$. Let $B \subset A(\widehat{X^s})$ be the maximal Abelian subvariety such that all $\widetilde{\omega}_i$ vanish on B. We set $A = A(\widehat{X^s})/B$. Then since X^s is normal, the composition of $\widehat{X^s} \to A(\widehat{X^s}) \to A$ factors through $\psi\colon \widehat{X^s} \to X^s$. This induces the desired map $\Phi\colon X^s \to A$.

We summarize the needed properties of the spectral covering from [20, Section 1].

Proposition 3.2. — *Assume furthermore that ϱ is big. Then:*

(1) *$\Phi\colon X^s \to A$ is generically finite.*
(2) *X^s is of general type.*

The proof of (1) can be found in [20, p. 148]. Indeed the following stronger result is proved in [20, p. 148]: The Stein factorization of $\Phi\colon X^s \to A$ is a Shafarevich map for the pull-back representation $\pi^*\varrho\colon \pi_1(X^s) \to G(K_p)$. The implication of (1) is immediate; Since ϱ is big, $\pi^*\varrho$ is also big. Hence the Shafarevich map for the representation $\pi^*\varrho$ is birational, which implies (1). The proof of (2) can be found in [20, p. 151].

Notation. — Before going to prove Proposition 3.1, we introduce the notations of Nevanlinna theory (*cf.* [14], [13]). Let Y be a Riemann surface with a proper surjective holomorphic map $p_Y\colon Y \to \mathbb{C}$. For $r > 0$, we set $Y(r) = p_Y^{-1}(\{z;\ |z| < r\})$. We put

$$N_{\mathrm{ram}\, p_Y}(r) = \frac{1}{\deg p_Y}\int_1^r \left[\sum_{y\in Y(t)} \mathrm{ord}_y \,\mathrm{ram}\, p_Y\right]\frac{dt}{t},$$

where $\mathrm{ram}\, p_Y$ is the ramification divisor of p_Y.

Let X be a projective variety and let Z be a closed subscheme of X. Let $g\colon Y \to X$ be a holomorphic map with Zariski dense image. Since Y is one dimensional, the pull-back g^*Z is a divisor on Y. We set

$$N(r, g, Z) = \frac{1}{\deg p_Y}\int_1^r \left[\sum_{y\in Y(t)} \mathrm{ord}_y\, g^*Z\right]\frac{dt}{t},$$

$$\bar{N}(r, g, Z) = \frac{1}{\deg p_Y}\int_1^r \left[\sum_{y\in Y(t)} \min\{1, \mathrm{ord}_y\, g^*Z\}\right]\frac{dt}{t}.$$

Let $\psi\colon \widehat{X} \to X$ be a desingularization, let $\widehat{g}\colon Y \to \widehat{X}$ be the lifting of g. Let M be a line bundle on X. Let $||\cdot||$ be a smooth Hermitian metric on ψ^*M, let Ω be the curvature form of $(M, ||\cdot||)$. We define

$$T(r, g, M) = \frac{1}{\deg p_Y}\int_1^r \left[\int_{Y(t)} \widehat{g}^*\Omega\right]\frac{dt}{t} + O(1).$$

This definition is independent of the choice of desingularization and Hermitian metric up to bounded function $O(1)$. Given a divisor $D \in H^0(X, M)$, we have the following Nevanlinna inequality (*cf.* [14, p. 180], [12, p. 269]):

$$N(r, g, D) \leqslant T(r, g, M) + O(1). \tag{3.1}$$

Let M be an ample line bundle on X. Let $\omega \in H^0(X, \Omega_X^1)$ be a holomorphic one form. Set $\eta = g^*\omega/p_Y^*(dz)$. Then η is a meromorphic function on Y. We set

$$m(r, \eta) = \frac{1}{\deg p_Y}\int_{\partial Y(r)} \max\{\log|\eta(y)|, 0\}\frac{d\arg p_Y(y)}{2\pi}.$$

Then by the lemma on logarithmic derivative ([13, Lemma 1.6]), we have

$$m(r,\eta)=o(T(r,g,M))\ ||.$$

Here the symbol || means that the stated estimate holds for $r>0$ outside some exceptional interval with finite Lebesgue measure. By the first main theorem (*cf.* [12, p. 269]), we have

$$T(r,\eta,\mathcal{O}_{\mathbb{P}^1}(1))=N(r,\eta,\infty)+m(r,\eta)+O(1),$$

where we consider η as a holomorphic map from Y into $\mathbb{P}^1$. Thus we have

$$T(r,\eta,\mathcal{O}_{\mathbb{P}^1}(1))\leqslant N(r,\eta,\infty)+o(T(r,g,M))\ ||. \tag{3.2}$$

Proof of Proposition 3.1. — First we shall reduce to the case that ϱ is big. Put $H=\ker\rho$ and consider the H-Shafarevich map $\mathrm{sh}_X^H\colon X \dashrightarrow \mathrm{Sh}^H(X)$ ([11, p. 185]). We remark that $\mathrm{Sh}^H(X)$ is only defined up to birationally equivalent class. Replacing X and $\mathrm{Sh}^H(X)$ by suitable models, we may assume that $\mathrm{sh}_X^H\colon X\to \mathrm{Sh}^H(X)$ is a morphism. Let F be a general fiber of sh_X^H and let $\pi_1(F)_X$ be the image of the natural map $\pi_1(F)\to\pi_1(X)$. Then by the definition of the H-Shafarevich map, the image $\varrho(\pi_1(F)_X)\subset G(K_p)$ is finite. We apply [21, Lemma 2.2.3]. The conclusion is as follows: After passing to a blowing-up and a finite unramified covering $e\colon X'\to X$, and denoting $s\colon X'\to\Sigma$ the Stein factorization of $\mathrm{sh}_X^H\circ e$, there exists a representation $\varrho_\Sigma\colon\pi_1(\Sigma)\to G(K_p)$ such that the pullback representation $e^*\varrho\colon \pi_1(X')\to G(K_p)$ factors through ϱ_Σ. Replacing X' and Σ by suitable models, we may assume that Σ is smooth. By the construction of Σ, we remark that the representation ϱ_Σ is big and Zariski dense (*cf.* [21, Proposition 2.2.2]). Given a holomorphic map $f\colon\mathbb{C}\to X$, we may take a lifting $f'\colon\mathbb{C}\to X'$ of f. If the composite holomorphic map $s\circ f'\colon\mathbb{C}\to\Sigma$ is degenerate, then f is also degenerate. Thus replacing X by Σ, ϱ by ϱ_Σ and f by $s\circ f'$, we have reduced to the case when ϱ is big.

Now assume, for the sake of contradiction, that there exists non-degenerate holomorphic map $f\colon\mathbb{C}\to X$. Then we may construct a Riemann surface Y with proper, surjective holomorphic map $p_Y: Y\to\mathbb{C}$ such that:

- the lifting $g\colon Y\to X^s$ of f exists, and
- p_Y is unramified outside the discrete set $g^{-1}(R)\subset Y$, where R is the ramification divisor of $\pi\colon X^s\to X$. Hence we have

$$N_{\mathrm{ram}\,p_Y}(r)\leqslant(\deg p_Y)\bar{N}(r,g,R). \tag{3.3}$$

Since we are assuming that f is non-degenerate, we remark that

(3.4) the image $g(Y)$ is Zariski dense in X^s.

For $\omega_i \neq \omega_j$, we set $\Xi_{ij} = (\omega_i - \omega_j)_0$, where $\omega_i - \omega_j$ is considered as a form from $H^0(X^s, \pi^*\Omega^1_X)$. We have

$$R \subset \cup_{i,j}\Xi_{ij}. \tag{3.5}$$

□

Let M be an ample line bundle on X^s.

CLAIM. — *$\bar{N}(r, g, \Xi_{ij}) \leqslant \varepsilon T(r, g, M)$ || for all $\varepsilon > 0$.*

Proof of Claim. — We prove the claim in the two possible cases:

Case 1. — $g^*\omega_i \neq g^*\omega_j$. Since $\omega_i \in H^0(X^s, \pi^*\Omega^1_X)$, we may consider $g^*\omega_i$ as a holomorphic section of $p_Y^*\Omega^1_{\mathbb{C}}$. Thus $\eta_i = g^*\omega_i/p_Y^*(dz)$ is a holomorphic function on Y. Since $g^*\omega_i \neq g^*\omega_j$, we have $\eta_i \neq \eta_j$. Note that if $g(y) \in \Xi_{ij}$, we have $\eta_i(y) = \eta_j(y)$. Hence using the Nevanlinna inequality (3.1), we have

$$\begin{aligned}\bar{N}(r, g, \Xi_{ij}) &\leqslant N(r, \eta_i - \eta_j, 0)\\ &\leqslant T(r, \eta_i - \eta_j, \mathcal{O}_{\mathbb{P}^1}(1)) + O(1).\end{aligned}$$

Since $\eta_i - \eta_j$ has no poles, we have $N(r, \eta_i - \eta_j, \infty) = 0$. Thus, applying (3.2) to $\eta_i - \eta_j = g^*(\omega_i - \omega_j)/p_Y^*(dz)$, we have

$$T(r, \eta_i - \eta_j, \mathcal{O}_{\mathbb{P}^1}(1)) = o(T(r, g, M)) \;||.$$

We conclude that

$$\bar{N}(r, g, \Xi_{ij}) = o(T(r, g, M)) \;||.$$

Case 2. — $g^*\omega_i = g^*\omega_j$. Let $b\colon X^s \to B$ be the Albanese map with respect to $\omega_i - \omega_j$, which is constructed as follows: Let $\Phi\colon X^s \to A$ be the Albanese map with respect to $\omega_1, \ldots, \omega_l$. For $k = 1, \ldots, l$, let $\widetilde{\omega_k}$ be the holomorphic one form on A such that $\Phi^*\widetilde{\omega_k} = \omega_k$. Let $C \subset A$ be the maximal Abelian subvariety such that $\widetilde{\omega_i} - \widetilde{\omega_j}$ vanishes on C. Put $B = A/C$. We define the map $b\colon X^s \to B$ by the composition of $\Phi\colon X^s \to A$ and the quotient $A \to B$.

Let Ξ'_{ij} be an irreducible component of Ξ_{ij}. Since there are only finitely many irreducible components of Ξ_{ij}, it is enough to prove

$$\bar{N}(r, g, \Xi'_{ij}) \leqslant \varepsilon T(r, g, M) \;|| \quad \text{for all } \varepsilon > 0.$$

Since $\omega_i - \omega_j$ vanishes on Ξ'_{ij}, we see that $b(\Xi'_{ij})$ is a point on B. We take an open subset $U \subset B$ and a holomorphic function φ on U such that $\varphi(b(\Xi'_{ij})) = 0$ and $\omega_i - \omega_j = b^*(d\varphi)$ on $b^{-1}(U)$.

Let $S \to b(X^s)$ be the normalization. Since X^s is normal, b factors as

$$X^s \xrightarrow{c} S \xrightarrow{\psi} B.$$

Since ψ is finite, $c(\Xi'_{ij})$ is a point on S; We denote this point by P. Let $\mathcal{O}^{\mathrm{an}}_{S,P}$ be the stalk at P in the sense of analytic space, and let $\mathfrak{m} \subset \mathcal{O}^{\mathrm{an}}_{S,P}$ be the maximal ideal. Since S is normal, $\mathcal{O}^{\mathrm{an}}_{S,P}$ is integral. We remark that $\varphi\circ\psi \in \mathcal{O}^{\mathrm{an}}_{S,P}$ is neither zero nor a unit, which follows from $\omega_i - \omega_j = b^*(d\varphi)$ and $\varphi(b(\Xi'_{ij})) = 0$. Hence we have

$$\dim \mathcal{O}^{\mathrm{an}}_{S,P}/(\varphi\circ\psi) = \dim S - 1. \tag{3.6}$$

Set

$$V_n = \operatorname{Spec} \mathcal{O}^{\mathrm{an}}_{S,P}/\big((\varphi\circ\psi) + \mathfrak{m}^n\big).$$

Then V_n is a closed subscheme of S with $\operatorname{supp} V_n = P$.

Let L be an ample line bundle on S. Using (3.6), we have

$$h^0(V_n, \mathcal{O}_{V_n} \otimes L^{\otimes \ell}) = h^0(V_n, \mathcal{O}_{V_n}) = O(n^{\dim S - 1}).$$

On the other hand, there are positive constants $c > 0$ and $\ell_0 > 0$ such that

$$h^0(S, L^{\otimes \ell}) > c\ell^{\dim S}$$

for $\ell > \ell_0$. Thus we may take a positive integer $\ell(n)$ such that $\ell(n) = o(n)$ as $n \to \infty$, and that $h^0(V_n, \mathcal{O}_{V_n} \otimes L^{\otimes \ell(n)}) < h^0(S, L^{\otimes \ell(n)})$. For example, $\ell(n) \sim n^{1-\frac{1}{2\dim S}}$. Thus we may take a divisor D_n from $H^0(S, L^{\otimes \ell(n)})$ such that $V_n \subset D_n$.

Now we claim that if $c\circ g(y) = P$ for $y \in Y$, then $\operatorname{ord}_y(c\circ g)^* D_n \geqslant n$. To see this, we take $y \in Y$ such that $c\circ g(y) = P$. Let $O \subset Y$ be the connected component of $(b\circ g)^{-1}(U)$ containing y. By the assumption $g^*(\omega_i - \omega_j) = 0$, we have $\varphi\circ b\circ g = 0$ on O. Hence $(\varphi\circ\psi)\circ(c\circ g) = 0$ on O. Thus by the construction of V_n, we have $\operatorname{ord}_y(c\circ g)^* V_n \geqslant n$. Hence by $V_n \subset D_n$, we have $\operatorname{ord}_y(c\circ g)^* D_n \geqslant n$.

By (3.4), $c\circ g(Y)$ is Zariski dense in S. Hence we have

$$\begin{aligned} n\bar{N}(r, g, \Xi'_{ij}) &\leqslant n\bar{N}(r, c\circ g, P) \\ &\leqslant N(r, c\circ g, D_n) \\ &\leqslant l(n) T(r, c\circ g, L) + O(1), \end{aligned}$$

where the last estimate follows from the Nevanlinna inequality (3.1). Thus, by $l(n) = o(n)$ and $T(r, c\circ g, L) = O(T(r, g, M))$, we have

$$\bar{N}(r, g, \Xi'_{ij}) \leqslant \varepsilon T(r, g, M) \ ||$$

for all $\varepsilon > 0$. We have proved our claim.

Now we go back to the proof of Proposition 3.1. By (3.3) and (3.5), we have

$$N_{\mathrm{ram}\, p_Y}(r) \leqslant \varepsilon T(r, g, M) \ ||$$

for all $\varepsilon > 0$. Hence by Proposition 3.2 and Proposition 3.3 below, we conclude that the image $g(Y)$ is not Zariski dense in X^s, which contradicts to (3.4). This conclude the proof of Proposition 3.1. □

PROPOSITION 3.3. — *Let X be a smooth projective variety such that (1) the Albanese map is generically finite, and (2) X is of general type. Let M be an ample line bundle on X. Let $g\colon Y \to X$ be a holomorphic map from a Riemann surface Y with a proper surjective holomorphic map $p_Y : Y \to \mathbb{C}$. Assume that*

$$N_{\mathrm{ram}\, p_Y}(r) \leqslant \varepsilon T(r, g, M) \ ||$$

for all $\varepsilon > 0$. Then the image of g is not Zariski dense in X.

This is a generalization of [18, Corollary 3.1.14]. The proof is parallel to that of [18, Corollary 3.1.14]. See also [17] for a generalization of Proposition 3.3.

4. Proof of Proposition 2.1

In this section, we prove Proposition 2.1. A representation of the fundamental group into an algebraic group G is called rigid if every nearby representation is conjugate to it. A representation which is not rigid is called non-rigid. The proof of Proposition 2.1 divides into two cases according to whether the representation $\varrho\colon \pi_1(X) \to G$ is rigid or non-rigid.

4.1. Case 1: ϱ is rigid

In this case ϱ is defined over some number field K. Given a prime ideal p from $\mathcal{O}_p$, we denote by $\varrho_p\colon \pi_1(X) \to G(K_p)$ the composition of $\varrho\colon \pi_1(X) \to G(K)$ and the inclusion $G(K) \subset G(K_p)$. If there exists a prime ideal p such that ϱ_p is p-unbounded, then Proposition 2.1 is a direct consequence of Proposition 3.1. Hence in the following, we consider the case that ϱ_p is p-bounded for every prime ideal p.

In this case, we remark that $\varrho^{-1}(G(\mathcal{O}_K))$ is of finite index in $\pi_1(X)$ (*cf.* [19, p. 120]). This can be proved as follows: Since $\pi_1(X)$ is finitely generated, there are only finite prime ideals $p_1, \dots, p_k$ such that $\varrho(\pi_1(X))$ is not contained in $G(\mathcal{O}_{K_p})$. Since $\varrho(\pi_1(X))$ is p_i-bounded for all p_i, the image of $\varrho(\pi_1(X))$ in $G(K_p)/G(\mathcal{O}_{K_p})$ is finite for all p_i. This shows our assertion.

Thus, after passing to a finite unramified covering, we may assume that $\varrho(\pi_1(X)) \subset G(\mathcal{O}_K)$.

Now by a result of Simpson (*cf.* [16, p. 58]), ϱ is a complex direct factor of a $\mathbb{Z}$-variation of Hodge structure. In particular, there is the period mapping $c\colon X \to \Gamma \setminus \mathcal{D}$ of this variation of Hodge structure (*cf.* [7, p. 57]). Here $\mathcal{D}$ is the classifying space and Γ is the arithmetic group which preserves the polarization and the lattice of the variation of Hodge structure. Then c is a horizontal locally liftable holomorphic map (*cf.* [7, 3.13]). Hence, for a holomorphic map $f\colon \mathbb{C} \to X$, $c \circ f\colon \mathbb{C} \to \Gamma \setminus \mathcal{D}$ is also a horizontal locally liftable holomorphic map. Since $\mathcal{D}$ has negative curvature in the horizontal direction, $c \circ f$ is constant (*cf.* [6, Corollary 9.7]). Since $\Gamma \setminus \mathcal{D}$ has the structure of a normal analytic space (*cf.* [7, p. 56]), the fibers of c are Zariski closed subsets on X. This shows that f is degenerate. Hence we have proved Proposition 2.1 when ϱ is rigid.

4.2. Case 2: ϱ is non-rigid

It suffices to prove the following:

Lemma 4.1. — *Let G be an almost simple algebraic group defined over the complex number field. Assume that there exists a Zariski dense, non-rigid representation $\varrho\colon \pi_1(X) \to G(\mathbb{C})$. Then every holomorphic map $f\colon \mathbb{C} \to X$ is degenerate.*

Proof. — We remark that G is defined over some number field K after some conjugations. Since $\pi_1(X)$ is finitely presented, there exists an affine scheme R over K such that

$$R(L) = \mathrm{Hom}(\pi_1(X), G(L))$$

for every field extension L/K. This space is defined as follows: We choose generators $\gamma_1, \dots, \gamma_k$ for $\pi_1(X)$. Let $\mathcal{R}$ be the set of relations among the generators γ_i. Then

$$R \subset \underbrace{G \times \cdots \times G}_{k \text{ times}}$$

is the closed subscheme defined by the equations $r(m_1, \dots, m_k) = 1$ for $r \in \mathcal{R}$. A representation $\tau\colon \pi_1(X) \to G(L)$ corresponds to the point $(m_1, \dots, m_k) \in R(L)$ with $m_i = \tau(\gamma_i)$. Note that R is an affine scheme, since it is a closed subscheme of an affine variety. Let $R_{Z.D.} \subset R$ be the space of Zariski dense representations. Then by [1, Proposition 8.2], $R_{Z.D.}$

is a Zariski open subset of R. The group G acts on R by simultaneous conjugation. Put $M = R//G$, and let $p\colon R \to M$ be the quotient map. Then M is an affine scheme defined over K. Let $[\varrho] \in R_{Z.D.}(\mathbb{C})$ be the point which correspond to the Zariski dense representation $\varrho\colon \pi_1(X) \to G(\mathbb{C})$.

Since $R_{Z.D.}(\bar{\mathbb{Q}})$ is dense in $R_{Z.D.}(\mathbb{C})$, by deforming ϱ slightly and replacing K by its finite extension, we may assume that ϱ is defined over K. Let p be a prime ideal from $\mathcal{O}_K$ and let K_p be the completion. In the following, we shall work over this K_p.

Since ϱ is non-rigid, we have $\dim M > 0$. Hence there exists a morphism $\psi\colon M \to \mathbb{A}^1$ such that the image $\psi(M)$ is Zariski dense in $\mathbb{A}^1$. Since the image $\psi \circ p(R_{Z.D.})$ is also Zariski dense in $\mathbb{A}^1$, there exists an affine curve $C \subset R_{Z.D.}$ such that the restriction $\psi \circ p|_C\colon C \to \mathbb{A}^1$ is generically finite. We may take a Zariski open subset $U \subset \mathbb{A}^1$ such that $\psi \circ p|_C$ is finite over U. Let $x \in U(K_p)$ be a point, and let $y \in C(\bar{K}_p)$ be a point over x. Then y is defined over some extension of K_p whose extension degree is bounded by the degree of $\psi \circ p|_C\colon C \to \mathbb{A}^1$. Note that there are only finitely many such field extensions. Hence there exists a finite extension L/K_p such that the points over $U(K_p)$ are all contained in $C(L)$. Since $U(K_p) \subset \mathbb{A}^1(L)$ is unbounded, the image $\psi \circ p(R_{Z.D.}(L)) \subset \mathbb{A}^1(L)$ is unbounded.

Let $R_0 \subset R(L)$ be the subset whose points correspond to p-bounded representations. Let $M_0 \subset M(L)$ be the image of R_0 under the quotient $p\colon R \to M$. Then by Lemma 4.2 below, M_0 is compact. Hence $\psi(M_0)$ is compact. In particular it is bounded. On the other hand, $\psi \circ p(R_{Z.D.}(L)) \subset \mathbb{A}^1(L)$ is unbounded. Hence we have $R_{Z.D.}(L) \not\subset R_0$. Thus we may take a Zariski dense, p-unbounded representation $\tilde{\varrho}\colon \pi_1(X) \to G(L)$. By Proposition 3.1, every holomorphic map $f\colon \mathbb{C} \to X$ is degenerate. □

LEMMA 4.2. — *M_0 is compact.*

Proof. — Note that there are only finitely many conjugacy classes of maximal compact subgroups in $G(L)$. Hence all maximal compact subgroups are conjugate to one of maximal compact subgroups $H_1, \dots, H_k \subset G(L)$. Hence given a p-bounded representation $\tau\colon \pi_1(X) \to G(L)$, there is a $G(L)$-conjugation $\tilde{\tau}\colon \pi_1(X) \to G(L)$ of τ such that the image $\tilde{\tau}(\pi_1(X))$ is contained in one of $H_1, \dots, H_k$.

Now take a sequence $[\tau_1], [\tau_2], \dots \in M_0$. Then we may take representations $\tau_1, \tau_2, \dots$ from R_0 such that $\tau_j(\pi_1(X))$ is contained in one of $H_1, \dots, H_k$. By taking subsequence, we may assume that $\tau_j(\pi_1(X)) \subset H_i$ for all j. Now since H_i is compact, some subsequence τ_j should converge to $\tau_\infty\colon \pi_1(X) \to H_i$. Then the sequence $[\tau_j]$ converges to $[\tau_\infty] \in M_0$. This shows that M_0 is compact. □

5. Proof of Corollary 1.3

Let X be a smooth projective variety with a big representation $\varrho\colon \pi_1(X) \to \mathrm{GL}_n(\mathbb{C})$. Assume that X admits a non-degenerate holomorphic map $f\colon \mathbb{C} \to X$. Then by Theorem 1.1, the image $\varrho(\pi_1(X))$ is almost abelian. Hence after passing to a finite unramified covering X' of X we may assume that $\varrho(\pi_1(X))$ is a free abelian group, *i.e.* ϱ factors the Albanese map $a_X\colon X \to \mathrm{Alb}(X)$. We shall prove that the Albanese map a_X is birational.

Since X admits a non-degenerate holomorphic map, the Albanese map a_X is surjective ([14, Theorem 6.4.1]) and has connected fibers ([15]). Let F be a general fiber of a_X. Then $\varrho(\mathrm{Im}(\pi_1(F) \to \pi_1(X)))$ is trivial. Since ϱ is big, F should be a point. Hence the Albanese map a_X is birational. This conclude the proof of the corollary.

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