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# SIMONS TYPE EQUATION IN $\mathbb{S}^2 \times \mathbb{R}$ AND $\mathbb{H}^2 \times \mathbb{R}$ AND APPLICATIONS

by Márcio Henrique BATISTA DA SILVA

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ABSTRACT. — Let  $\Sigma^2$  be an immersed surface in  $M^2(c) \times \mathbb{R}$  with constant mean curvature. We consider the traceless Weingarten operator  $\phi$  associated to the second fundamental form of the surface, and we introduce a tensor  $S$ , related to the Abresch-Rosenberg quadratic differential form. We establish equations of Simons type for both  $\phi$  and  $S$ . By using these equations, we characterize some immersions for which  $|\phi|$  or  $|S|$  is appropriately bounded.

RÉSUMÉ. — Soit  $\Sigma^2$  une surface immergée dans  $M^2(c) \times \mathbb{R}$  avec une courbure moyenne constante. Nous considérons l'opérateur de Weingarten à trace nulle  $\phi$  associé à la seconde forme fondamentale de la surface et nous introduisons un tenseur  $S$ , liés à la forme quadratique de Abresch-Rosenberg. Nous établissons les équations de type Simons pour  $\phi$  et  $S$ . En utilisant ces équations, nous caractérisons les immersions pour lesquelles  $|\phi|$  ou  $|S|$  sont bornés.

## 1. Introduction

In 1994, using the traceless Weingarten operator  $\phi = A - HI$  associated to an immersed hypersurface  $M^n \looparrowright \mathbb{S}^{n+1}$ , H. Alencar and M. do Carmo, see [2], proved that

THEOREM. — *Let  $M^n \looparrowright \mathbb{S}^{n+1}$  be an immersed hypersurface. If  $M^n$  is compact and orientable with constant mean curvature  $H$  and*

$$|\phi|^2 \leq B_H,$$

where  $B_H$  is the square of the positive root of

$$P_H(x) = x^2 + \frac{n(n-2)}{\sqrt{n(n-1)}}Hx - n(H^2 + 1).$$

Then:

*Keywords:* Surface with constant mean curvature, Simons type equation, Codazzi's equation.

*Math. classification:* 53A10, 53C42.

- (a) Either  $|\phi|^2 = 0$  (and  $M^n$  is totally umbilic) or  $|\phi|^2 = B_H$ .  
 (b) The  $H(r)$ -tori  $\mathbb{S}^{n-1}(r) \times \mathbb{S}^1(\sqrt{1-r^2})$  with  $r^2 \leq \frac{n-1}{n}$  are the only hypersurfaces with constant mean curvature  $H$  and  $|\phi|^2 = B_H$ .

Motivated by this result we study this problem for surfaces in  $M^2(c) \times \mathbb{R}$  with  $c = \pm 1$ , where  $M^2(-1) = \mathbb{H}^2$  and  $M^2(1) = \mathbb{S}^2$ .

We begin by using the traceless Weingarten operator  $\phi$  associated to an immersed surface  $\Sigma^2 \looparrowright M^2(c) \times \mathbb{R}$ .

In [1], the authors defined the quadratic differential form

$$Q(X, Y) = 2H\langle AX, Y \rangle - c\langle X, \partial_t \rangle \langle Y, \partial_t \rangle,$$

and its (2,0)-part

$$Q^{(2,0)}(X, Y) = \frac{1}{2}(Q(X, Y) - Q(JX, JY)) - \frac{1}{2}i(Q(JX, Y) + Q(X, JY)),$$

where  $J$  is the standard counter-clockwise rotation operator.

Using this notation, Abresch and Rosenberg proved

**THEOREM.** — (Thm. 1 in [1]) *Let  $\Sigma^2 \looparrowright M^2(c) \times \mathbb{R}$  be an immersed surface with constant mean curvature. Then its quadratic differential  $Q^{(2,0)}$  is holomorphic on the surface  $\Sigma^2$ .*

Inspired in the quadratic differential form  $Q$  introduced by Abresch and Rosenberg, we study, in section 3, a special tensor  $S$  defined by

$$(1.1) \quad SX = 2HAX - c\langle X, T \rangle T + \frac{c}{2}(1 - \nu^2)X - 2H^2X,$$

where  $X \in T_p\Sigma$ ,  $A$  is the Weingarten operator associated to the second fundamental form,  $H$  is the mean curvature,  $T$  is the tangential component of the parallel field  $\partial_t$ , tangent to  $\mathbb{R}$  in  $M^2(c) \times \mathbb{R}$ , and  $\nu = \langle N, \partial_t \rangle$ .

The tensor  $S$  is the traceless tensor associated with the quadratic differential  $Q$ . In fact,

$$\begin{aligned} \langle SX, Y \rangle &= 2H\langle AX, Y \rangle - c\langle X, T \rangle \langle Y, T \rangle + \frac{c}{2}(1 - \nu^2)\langle X, Y \rangle - 2H^2\langle X, Y \rangle \\ &= Q(X, Y) - \frac{trQ}{2}\langle X, Y \rangle. \end{aligned}$$

We will prove that this operator satisfies Codazzi's equation, provided  $H$  is constant, with vanishing trace. Moreover, we remark that any surface with  $|S| = 0$  and constant mean curvature is very interesting, because the  $Q^{(2,0)}$  of these surfaces vanishes.

In [1], Theorem 3, p. 143, the authors described four distinct classes of complete, possibly immersed, constant mean curvature surfaces  $\Sigma^2 \looparrowright M^2(c) \times \mathbb{R}$  with vanishing of their quadratic differential  $Q^{(2,0)}$ .

More precisely, the four classes are

- (i)  $\Sigma^2$  is an embedded rotationally invariant constant mean curvature sphere  $S_H^2$ ;
- (ii)  $\Sigma^2$  is a convex rotationally invariant constant mean curvature graph  $D_H^2$  over the horizontal leaf  $M^2(c) \times \{t_0\}$ ;
- (iii)  $\Sigma^2$  is an embedded annulus, rotationally invariant constant mean curvature surface  $C_H^2$  with two asymptotically conical ends;
- (iv)  $\Sigma^2$  is the embedded constant mean curvature surface  $P_H^2$ ; it is an orbit under some two dimensional solvable subgroup of ambient isometries.

The surface in (i) was known to W.T. Hsiang and W.Y. Hsiang, in [6], and to R. Pedrosa and M. Ritoré, in [7]. We shall refer to  $S_H^2$  as the embedded rotationally invariant constant mean curvature spheres. In this paper we will call the surfaces described in [1] by Abresch-Rosenberg surfaces.

*Remark. 1.* — In  $\mathbb{S}^2 \times \mathbb{R}$  only the spheres  $S_H^2$  occur.

We obtain an equation of Simons type for  $S$  and apply it in some particular cases:

**THEOREM 1.1.** — *Let  $\Sigma^2 \looparrowright M^2(c) \times \mathbb{R}$  be an immersed surface with non zero constant mean curvature  $H$  and  $S$  as defined in (1.1). Then,*

$$\begin{aligned} \langle (\nabla^2 S)x, y \rangle &= 2c\nu^2 \langle Sx, y \rangle + 2H \langle Ax, Sy \rangle - \langle A^2x, Sy \rangle + \\ &\quad + \langle Ay, SAx \rangle - \langle Ax, y \rangle \text{tr}(AS) \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2} \Delta |S|^2 &= |\nabla S|^2 - |S|^4 + |S|^2 \left( \frac{5c\nu^2}{2} - \frac{c}{2} + 2H^2 - \frac{c}{H} \langle ST, T \rangle \right) + \\ &\quad + c|ST|^2 - \frac{1}{4H^2} \langle ST, T \rangle^2. \end{aligned}$$

Let us consider the polynomial  $p_H(t) = -t^2 - \frac{1}{H}t + \left( \frac{4H^2 - 1}{2} \right)$ . When  $H$  is greater than one half there is a positive root for  $p_H$ . Let  $L_H$  be this positive root. One has:

**THEOREM 1.2.** — *Let  $\Sigma^2 \looparrowright \mathbb{S}^2 \times \mathbb{R}$  be an immersed surface with constant mean curvature  $H$  greater than one half. If*

$$\Sigma^2 \text{ is complete and } \sup_{\Sigma} |S| < L_H,$$

or

$$\Sigma^2 \text{ is closed and } |S| \leq L_H,$$

then  $\Sigma^2 = S_H^2$ , i.e,  $\Sigma^2$  is an embedded rotationally invariant constant mean curvature sphere.

*Remark. 2.* — The number  $L_H$  is  $\frac{\sqrt{2}H(4H^2 - 1)}{\sqrt{16H^4 - 4H^2 + 1} + 1}$ .

Let us consider the polynomial

$$q_H(t) = -t^2 - \frac{1}{\sqrt{2}H}t + \left( \frac{8H^4 - 12H^2 - 1}{4H^2} \right).$$

When  $H$  is greater than  $\sqrt{\frac{3 + \sqrt{11}}{4}}$ , there is a positive root for  $q_H$ . Let  $M_H$  be this positive root.

**THEOREM 1.3.** — *Let  $\Sigma^2 \looparrowright \mathbb{H}^2 \times \mathbb{R}$  be an immersed surface with constant mean curvature  $H$  greater than  $\sqrt{\frac{3 + \sqrt{11}}{4}} \approx 1.25664$ . If*

$$\Sigma^2 \text{ is complete and } \sup_{\Sigma} |S| < M_H,$$

or

$$\Sigma^2 \text{ is closed and } |S| \leq M_H,$$

then  $\Sigma^2 = S_H^2$ , i.e,  $\Sigma^2$  is an embedded rotationally invariant constant mean curvature sphere.

*Remark. 3.* — The number  $M_H$  is  $\frac{8H^4 - 12H^2 - 1}{\sqrt{2}H(\sqrt{16H^4 - 24H^2 - 1} + 1)}$ .

*Remark. 4.* — Besides Theorems 1.2 and 1.3, we obtain in section 4 further applications of Simons equation of Theorem 1.1.

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## 2. Preliminaries

Let  $\Sigma^2 \looparrowright M^3$  be an immersed surface. Let  $\bar{\nabla}$  denote the Levi-Civita connection on  $M^3$  and let  $\nabla$  denote the Levi-Civita connection on  $\Sigma$  for the induced metric.

Generally speaking, objects defined on  $M^3$  will be denoted by the same symbols as the corresponding objects defined on  $\Sigma$  plus a bar over the symbol.

The Riemannian metric extends to natural inner products on spaces of tensors and the above connections induce natural covariant derivatives of tensor fields. For example, for  $\{e_1, e_2\}$  a geodesic frame at  $p \in \Sigma^2$  and a tensor  $\psi$  on  $\Sigma^2$ , we have

$$\nabla^2\psi(p) = \sum_{i=1}^2 (\nabla_{e_i} \nabla_{e_i} \psi)(p).$$

For more details about covariant derivatives of tensor fields see [8], sections 1 and 2.

We adopt the following convention for the curvature tensor: if  $x, y, z \in T_p\Sigma$ , we define  $R_{x,y}z$  by

$$R_{x,y}z = R(X, Y)Z(p) = (\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z)(p),$$

for any local vector fields which extend the given vectors  $x, y, z$ .

The second fundamental form is defined by  $\alpha(X, Y) = (\bar{\nabla}_X Y)^\perp$  and the associated Weingarten operator is given by  $Av = -(\bar{\nabla}_v N)^T$ , where  $N$  is a unit normal field on  $\Sigma^2$ . We use the Weingarten operator to define the following operators

$$(2.1) \quad \langle \bar{R}(A)x, y \rangle := \sum_{i=1}^2 (-\langle Ax, \bar{R}_{e_i,y}e_i \rangle - \langle Ay, \bar{R}_{e_i,x}e_i \rangle + \langle Ay, x \rangle \langle N, \bar{R}_{e_i,N}e_i \rangle - 2\langle Ae_i, \bar{R}_{e_i,xy} \rangle)$$

and

$$\langle \bar{R}'x, y \rangle := \sum_{i=1}^2 \{ \langle (\bar{\nabla}_x \bar{R})_{e_i,y}e_i, N \rangle + \langle (\bar{\nabla}_{e_i} \bar{R})_{e_i,xy}, N \rangle \},$$

where  $\{e_1, e_2\}$  is a orthonormal basis of  $T_p\Sigma$ .

With this notation we have the following result:

**THEOREM 2.1.** — *Let  $\Sigma^2 \looparrowright M^3$  be an immersed surface with constant mean curvature  $H$ . For any  $x, y \in T_p\Sigma$  we have*

$$(2.2) \quad \langle (\nabla^2 A)x, y \rangle = -|A|^2 \langle Ax, y \rangle + \langle \bar{R}(A)x, y \rangle + \langle \bar{R}'x, y \rangle + 2H \langle \bar{R}_{N,xy}, N \rangle + 2H \langle Ax, Ay \rangle.$$

*Proof.* — See Theorem 2 in [3] and observe that the codimension is one. □

We will also use the result known as the Omori-Yau Maximum Principle whose proof can be found in [10], Theorem 1.

**THEOREM 2.2** (Omori-Yau Maximum Principle). — *Let  $M$  be a complete Riemannian manifold with Ricci curvature bounded from below. If  $u \in C^\infty(M)$  is bounded from above, then there exists a sequence of points  $\{p_j\} \in M$  such that*

$$\lim_{j \rightarrow \infty} u(p_j) = \sup_M u, \quad |\nabla u|(p_j) < \frac{1}{j}, \text{ and } \Delta u(p_j) < \frac{1}{j}.$$

Let us recall Gauss' equation for  $\Sigma^2$  in  $M^2(c) \times \mathbb{R}$ :

$$(2.3) \quad R(Y, X)Z = \langle AX, Z \rangle AY - \langle AY, Z \rangle AX + c(\langle X, Z \rangle Y - \langle Y, Z \rangle X + \\ - \langle Y, T \rangle \langle X, Z \rangle T - \langle X, T \rangle \langle Z, T \rangle Y + \\ + \langle X, T \rangle \langle Y, Z \rangle T + \langle Y, T \rangle \langle Z, T \rangle X),$$

where  $X, Y, Z$  in  $T_p\Sigma$ ,  $N$  is a unitary normal field on  $\Sigma^2$  and  $T$  is the tangential component of the parallel field  $\partial_t$ . For more details see [5].

### 3. Simons' equation in $M^2(c) \times \mathbb{R}$

In this section we will obtain an equation of Simons type for the traceless Weingarten operator  $\phi$  and for the tensor  $S$  defined in (1.1).

Let  $M^2(c) \times \mathbb{R}$ , where  $M^2(-1) = \mathbb{H}^2$  and  $M^2(1) = \mathbb{S}^2$ . In this case we have that  $\bar{R}'=0$ , because  $M^2(c) \times \mathbb{R}$  is locally symmetric.

In Lemmas 3.1 and 3.2 we will consider an immersed surface  $\Sigma^2 \looparrowright M^2(c) \times \mathbb{R}$  with constant mean curvature  $H$  where  $A$  is the Weingarten operator associated to the second fundamental form on  $\Sigma^2$ .

**LEMMA 3.1.** — *Denoting the identity by  $I$ , we have that*

$$\bar{R}(A) = c(5\nu^2 - 1)A - 4cH\nu^2I.$$

*Proof.* — Consider an orthonormal basis  $\{e_1, e_2\}$  in  $T_p\Sigma^2$  such that  $Ae_i = k_i e_i$ ,  $i = 1, 2$ . Consider  $x, y \in T_p\Sigma$ . We have

$$x = x_1 e_1 + x_2 e_2 \text{ e } y = y_1 e_1 + y_2 e_2.$$

Computing the first sum in (2.1)

$$\sum_{i=1}^2 \langle \bar{R}_{e_i, y} e_i, Ax \rangle = k_2 x_2 y_2 \langle \bar{R}_{e_1, e_2} e_1, e_2 \rangle + k_1 x_1 y_1 \langle \bar{R}_{e_2, e_1} e_2, e_1 \rangle \\ = -\bar{K}_\Sigma (k_2 x_2 y_2 + k_1 x_1 y_1) = -\bar{K}_\Sigma \langle Ax, y \rangle,$$

where  $\bar{K}_\Sigma = \langle \bar{R}_{e_1, e_2} e_2, e_1 \rangle$ .

Hence,

$$(3.1) \quad \sum_{i=1}^2 \langle \bar{R}_{e_i, y} e_i, Ax \rangle = -\bar{K}_\Sigma \langle Ax, y \rangle.$$

It's simple see that

$$(3.2) \quad \sum_{i=1}^2 \langle \bar{R}_{e_i, x} e_i, Ay \rangle = -\bar{K}_\Sigma \langle Ax, y \rangle.$$

In the third sum in (2.1) we have

$$\begin{aligned} \langle \bar{R}_{e_i, N} e_i, N \rangle &= -c \{ (1 - \langle e_i, \partial_t \rangle^2) (1 - \nu^2) - \nu^2 \langle e_i, \partial_t \rangle^2 \} \\ &= -c \{ 1 - \nu^2 - \langle e_i, \partial_t \rangle^2 \}. \end{aligned}$$

Therefore,

$$(3.3) \quad \sum_{i=1}^2 \langle \bar{R}_{e_i, N} e_i, N \rangle = -c(1 - \nu^2).$$

To finish, we computing the fourth sum.

$$\begin{aligned} \sum_{i=1}^2 \langle \bar{R}_{e_i, x} y, Ae_i \rangle &= \bar{K}_\Sigma (k_1 x_2 y_2 + k_2 x_1 y_1) \\ &= \bar{K}_\Sigma ([2H - k_2] x_2 y_2 + [2H - k_1] x_1 y_1) \\ &= \bar{K}_\Sigma (2H \langle x, y \rangle - \langle Ax, y \rangle), \end{aligned}$$

where we used that  $2H = k_1 + k_2$ .

Thus,

$$(3.4) \quad \sum_{i=1}^2 \langle \bar{R}_{e_i, x} y, Ae_i \rangle = \bar{K}_\Sigma (2H \langle x, y \rangle - \langle Ax, y \rangle).$$

Now, we need computing  $\bar{K}_\Sigma$ . Using the tensor of curvature in  $M^2(c) \times \mathbb{R}$  we have:

$$\bar{K}_\Sigma = \langle \bar{R}_{e_1, e_2} e_2, e_1 \rangle = c (1 - \langle e_1, T \rangle^2 - \langle e_2, T \rangle^2) = c(1 - |T|^2).$$

Therefore,

$$(3.5) \quad \bar{K}_\Sigma = c\nu^2.$$

Substituting (3.1), (3.2), (3.3) and (3.4) into (2.1), obtain

$$\langle \bar{R}(A)x, y \rangle = 2\bar{K}_\Sigma \langle Ax, y \rangle - c(1 - \nu^2) \langle Ax, y \rangle - 2\bar{K}_\Sigma (2H \langle x, y \rangle - \langle Ax, y \rangle).$$

Using (3.5) we obtain

$$\langle \bar{R}(A)x, y \rangle = 5c\nu^2 \langle Ax, y \rangle - c \langle Ax, y \rangle - 4c\nu^2 H \langle x, y \rangle.$$



Thus,

$$\bar{R}(A) = c(5\nu^2 - 1)A - 4cH\nu^2 I.$$

□

LEMMA 3.2. —  $\langle \bar{R}_{N,xy}, N \rangle = -c\{\langle x, T \rangle \langle y, T \rangle - \langle x, y \rangle \langle T, T \rangle\}.$

*Proof.* — We observe that

$$\langle x^*, y^* \rangle = \langle x, y \rangle - \langle x, T \rangle \langle y, T \rangle,$$

$$\langle x^*, N^* \rangle = \nu \langle x, T \rangle$$

and

$$\langle N^*, N^* \rangle = 1 - \nu^2,$$

where we have used  $v^* = v - \langle v, \partial_t \rangle \partial_t$  for any  $v \in T_p(M^2(c) \times \mathbb{R})$ .

It follows that

$$\begin{aligned} \langle \bar{R}_{N,xy}, N \rangle &= -c\{\langle N^*, x^* \rangle \langle N^*, y^* \rangle - \langle N^*, N^* \rangle \langle x^*, y^* \rangle\} \\ &= -c\{\langle x, T \rangle \langle y, T \rangle - \langle x, y \rangle \langle T, T \rangle\}. \end{aligned}$$

This concludes the proof. □

PROPOSITION 3.3. — *Let  $\Sigma^2 \looparrowright M^2(c) \times \mathbb{R}$  be an immersed surface with constant mean curvature  $H$  and let  $A$  be the Weingarten operator associated to the second fundamental form on  $\Sigma^2$ . Then,*

$$\begin{aligned} \langle (\nabla^2 A)x, y \rangle &= -|A|^2 \langle Ax, y \rangle + c(5\nu^2 - 1) \langle Ax, y \rangle - 4cH\nu^2 \langle x, y \rangle + \\ &\quad - 2cH\{\langle x, T \rangle \langle y, T \rangle - \langle x, y \rangle \langle T, T \rangle\} + 2H \langle Ax, Ay \rangle, \end{aligned}$$

where  $\nu = \langle N, \partial_t \rangle$ .

*Proof.* — Consider equation (2.2)

$$\begin{aligned} \langle (\nabla^2 A)x, y \rangle &= -|A|^2 \langle Ax, y \rangle + \langle \bar{R}(A)x, y \rangle \\ &\quad + \langle \bar{R}'x, y \rangle + 2H \langle \bar{R}_{N,xy}, N \rangle + 2H \langle Ax, Ay \rangle. \end{aligned}$$

Now, we use Lemmas 3.1 and 3.2 and the fact that  $\bar{R}' = 0$  to obtain

$$\begin{aligned} \langle (\nabla^2 A)x, y \rangle &= -|A|^2 \langle Ax, y \rangle + c(5\nu^2 - 1) \langle Ax, y \rangle - 4cH\nu^2 \langle x, y \rangle + \\ &\quad - 2Hc\{\langle x, T \rangle \langle y, T \rangle - \langle x, y \rangle \langle T, T \rangle\} + 2H \langle Ax, Ay \rangle. \end{aligned}$$

□

Consider two tensors  $V, W$  on  $\Sigma^2$ . We define the inner product  $\langle V, W \rangle$  at  $p \in \Sigma^2$  as

$$\langle V, W \rangle = \sum_{i=1}^2 \langle V e_i, W e_i \rangle,$$

where  $\{e_1, e_2\}$  is an orthonormal basis for  $T_p \Sigma$ .

**COROLLARY 3.4.** — *Let  $\Sigma^2 \looparrowright M^2(c) \times \mathbb{R}$  be an immersed surface with constant mean curvature and let  $A$  be the Weingarten operator associated to the second fundamental form on  $\Sigma^2$ . Then,*

- (a)  $\langle \nabla^2 A, I \rangle = 0$ .  
 (b)  $\langle \nabla^2 A, A \rangle = -|A|^4 + c(5\nu^2 - 1)|A|^2 - 8cH^2\nu^2 - 2cH\langle AT, T \rangle + 4cH^2|T|^2 + 2H\text{tr}(A^3)$ .

*Proof.* — Consider  $\{e_1, e_2\}$  an orthonormal basis of  $T_p \Sigma$ . We use the definition of the inner product between tensors and the expression in Proposition 3.3 to obtain

$$\begin{aligned} \langle \nabla^2 A, A \rangle &= \sum_{i=1}^2 \langle (\nabla^2 A) e_i, A e_i \rangle = -|A|^2 \sum_{i=1}^2 \langle A e_i, A e_i \rangle + \\ &+ c(5\nu^2 - 1) \sum_{i=1}^2 \langle A e_i, A e_i \rangle - 4cH\nu^2 \sum_{i=1}^2 \langle A e_i, e_i \rangle - 2cH \left\{ \sum_{i=1}^2 \langle AT, e_i \rangle \langle e_i, T \rangle + \right. \\ &\quad \left. - \langle T, T \rangle \sum_{i=1}^2 \langle A e_i, e_i \rangle \right\} + 2H \sum_{i=1}^2 \langle A^2 e_i, A e_i \rangle. \end{aligned}$$

Therefore,

$$\begin{aligned} \langle \nabla^2 A, A \rangle &= -|A|^4 + c(5\nu^2 - 1)|A|^2 - 8cH^2\nu^2 - 2cH\langle AT, T \rangle \\ &\quad + 4cH^2|T|^2 + 2H\text{tr}(A^3). \end{aligned}$$

Using the definition of the inner product and Proposition 3.3 we obtain

$$\begin{aligned} \langle \nabla^2 A, I \rangle &= \sum_{i=1}^2 \langle (\nabla^2 A) e_i, e_i \rangle = -|A|^2 \sum_{i=1}^2 \langle A e_i, e_i \rangle + \\ &+ c(5\nu^2 - 1) \sum_{i=1}^2 \langle A e_i, e_i \rangle - 8cH\nu^2 - 2cH \left\{ \sum_{i=1}^2 \langle T, e_i \rangle \langle e_i, T \rangle + \right. \\ &\quad \left. - 2\langle T, T \rangle \right\} + 2H \sum_{i=1}^2 \langle A^2 e_i, e_i \rangle. \end{aligned}$$

Therefore,

$$\langle \nabla^2 A, I \rangle = -2H|A|^2 + c(5\nu^2 - 1)2H - 8cH\nu^2 + 2cH\langle T, T \rangle + 2H|A|^2 = 0,$$

where we have used that  $\nu^2 + |T|^2 = 1$ . □

PROPOSITION 3.5. — *Let  $\Sigma \looparrowright M^2(c) \times \mathbb{R}$  be an immersed surface with constant mean curvature  $H$  and let  $\phi$  be the traceless Weingarten operator, then*

- (a)  $|\phi|^2 = |A|^2 - 2H^2$ .
- (b)  $\nabla\phi = \nabla A$ .
- (c)  $trA^3 = 3H|\phi|^2 + 2H^3$ .

*Proof.* — The proof of item (a) is:

$$\begin{aligned} |\phi|^2 &= \langle \phi, \phi \rangle = \langle A - HI, A - HI \rangle = \langle A, A \rangle - 2H\langle A, I \rangle + H^2\langle I, I \rangle \\ &= |A|^2 - 4H^2 + 2H^2 = |A|^2 - 2H^2, \end{aligned}$$

where  $\langle A, I \rangle = 2H$  and  $\langle I, I \rangle = 2$ .

To prove item (b), we consider tangent fields  $X, Y$ . Then,

$$\begin{aligned} (\nabla_X \phi)Y &= (\nabla_X A)Y - (\nabla_X(HI))Y = (\nabla_X A)Y - \nabla_X HI(Y) + H\nabla_X Y \\ &= (\nabla_X A)Y - H\nabla_X Y - X(H)Y + H\nabla_X Y = (\nabla_X A)Y, \end{aligned}$$

because  $H$  is constant.

Finally, the proof of item(c) is:

$$\begin{aligned} tr(A^3) &= \sum_{i=1}^2 \langle A^3 e_i, e_i \rangle = \sum_{i=1}^2 \langle (\phi + HI)^3 e_i, e_i \rangle \\ &= \sum_{i=1}^2 \langle (\phi^3 + 3H\phi^2 + 3H^2\phi + H^3I)e_i, e_i \rangle = 3H|\phi|^2 + 2H^3, \end{aligned}$$

because  $tr\phi = tr\phi^3 = 0$ . □

Next we shall derive an equation of Simons type for the traceless Weingarten operator  $\phi$ :

THEOREM 3.6. — *Let  $\Sigma \looparrowright M^2(c) \times \mathbb{R}$  be an immersed surface with constant mean curvature  $H$  and let  $\phi$  be the traceless Weingarten operator. Then*

$$\langle \nabla^2 \phi, \phi \rangle = -|\phi|^4 + (2H^2 + 5c\nu^2 - c)|\phi|^2 - 2cH\langle \phi T, T \rangle$$

and

$$\frac{1}{2}\Delta|\phi|^2 = |\nabla\phi|^2 - |\phi|^4 + (2H^2 + 5c\nu^2 - c)|\phi|^2 - 2cH\langle \phi T, T \rangle.$$

*Proof.* — We use Proposition 3.5 to show that

$$\langle \nabla^2 \phi, \phi \rangle = \langle \nabla^2 A, A - HI \rangle = \langle \nabla^2 A, A \rangle - H \langle \nabla^2 A, I \rangle.$$

Now, we use Corollary 3.4 to obtain

$$\begin{aligned} \langle \nabla^2 \phi, \phi \rangle = & -|A|^4 + c(5\nu^2 - 1)|A|^2 - 8cH^2\nu^2 + 2cH \langle AT, T \rangle + \\ & + 4cH^2|T|^2 + 2H \operatorname{tr}(A^3). \end{aligned}$$

Therefore,

$$\begin{aligned} \langle \nabla^2 \phi, \phi \rangle = & -(|\phi|^2 + 2H^2)^2 + c(5\nu^2 - 1)(|\phi|^2 + 2H^2) - 8cH^2\nu^2 + \\ & - 2cH \langle (\phi + HI)T, T \rangle + 4cH^2|T|^2 + 2H(3H|\phi|^2 + 2H^3), \end{aligned}$$

which brings us to

$$\langle \nabla^2 \phi, \phi \rangle = -|\phi|^4 + 2H^2|\phi|^2 + c(5\nu^2 - 1)|\phi|^2 - 2cH \langle \phi T, T \rangle.$$

To finish, we use that  $\frac{1}{2}\Delta|\phi|^2 = |\nabla\phi|^2 + \langle \nabla^2 \phi, \phi \rangle$ . □

Now we evaluate the Laplacian of  $|S|^2$  where  $S$  is defined by (1.1), i.e.,

$$S = 2HA - c \langle T, \cdot \rangle T + \frac{c}{2}(1 - \nu^2)I - 2H^2I.$$

We observe the fact that  $S$  is a traceless operator, i.e.,

$$\operatorname{tr}(S) = 2H \operatorname{tr}(A) - c|T|^2 + c(1 - \nu^2) - 4H^2 = 0,$$

where we used that  $|T|^2 + \nu^2 = 1$  and  $\operatorname{tr}(A) = 2H$ .

**PROPOSITION 3.7** (Codazzi's Equation). — *Let  $\Sigma^2 \looparrowright M^2(c) \times \mathbb{R}$  be an immersed surface with constant mean curvature and the  $S$  be the tensor defined in (1.1). Then*

$$(\nabla_X S)Y = (\nabla_Y S)X,$$

for all tangent fields  $X, Y$  on  $\Sigma^2$ .

*Proof.* — We consider  $(u, v)$  isothermal parameters of the surface  $\Sigma^2$ . Now, we consider the complex parameter,  $z = u + iv$ . Let us set

$$T_S(X, Y) := (\nabla_X S)Y - (\nabla_Y S)X = \nabla_X(SY) - \nabla_Y(SX) - S[X, Y].$$

We will prove that  $T_S$  is null. For this, consider the derivatives

$$\partial_z = \frac{1}{2}(\partial_u - i\partial_v) \text{ and } \partial_{\bar{z}} = \frac{1}{2}(\partial_u + i\partial_v).$$

We will compute  $T_S$  in the basis  $\{\partial_z, \partial_{\bar{z}}\}$ . First note that,

$$\begin{aligned} \langle T_S(\partial_z, \partial_{\bar{z}}), \partial_z \rangle &= \partial_z \langle S\partial_{\bar{z}}, \partial_z \rangle - \langle S\partial_{\bar{z}}, \nabla_{\partial_z} \partial_z \rangle + \\ &\quad - \partial_{\bar{z}} \langle S\partial_z, \partial_z \rangle + \langle S\partial_z, \nabla_{\partial_{\bar{z}}} \partial_z \rangle \\ &= -Q_{\bar{z}}^{(2,0)} = 0, \end{aligned}$$

because  $Q^{(2,0)}$  is holomorphic, Theorem 1 in [1], and using the fact that  $\nabla_{\partial_z} \partial_{\bar{z}} = 0$ ,  $\nabla_{\partial_z} \partial_z = \frac{\lambda_z}{\lambda} \partial_z$ ,  $\langle S\partial_z, \partial_z \rangle = Q^{(2,0)}$  and  $\langle S\partial_z, \partial_{\bar{z}} \rangle = 0$ , where  $\lambda = \langle \partial_z, \partial_{\bar{z}} \rangle$ .

Next,

$$\begin{aligned} \langle T_S(\partial_z, \partial_{\bar{z}}), \partial_{\bar{z}} \rangle &= -\partial_{\bar{z}} \langle \partial_{\bar{z}}, S\partial_z \rangle + \langle S\partial_z, \nabla_{\partial_{\bar{z}}} \partial_{\bar{z}} \rangle + \\ &\quad + \partial_z \langle S\partial_{\bar{z}}, \partial_{\bar{z}} \rangle - \langle S\partial_{\bar{z}}, \nabla_{\partial_z} \partial_{\bar{z}} \rangle \\ &= \overline{Q^{(2,0)}} = 0, \end{aligned}$$

where we have used that  $\nabla_{\partial_{\bar{z}}} \partial_{\bar{z}} = \frac{\lambda_{\bar{z}}}{\lambda} \partial_{\bar{z}}$  and  $\overline{Q^{(2,0)}}_z = \overline{Q_{\bar{z}}^{(2,0)}}$ . It follows that  $T_S = 0$ . □

LEMMA 3.8. — *Let  $Z$  be a symmetric operator satisfying Codazzi's equation and  $tr(Z) = 0$ , then*

$$(3.6) \quad \langle (\nabla^2 Z)x, y \rangle = \sum_{i=1}^2 \{ -\langle Zy, R_{e_i, x} e_i \rangle - \langle Ze_i, R_{e_i, x} y \rangle \},$$

where  $\{e_1, e_2\}$  is an orthonormal basis of  $T_p \Sigma$ .

*Proof.* — See Lemma a. in [8], p. 81, adapted for codimension 1. □

Let us evaluate each summand in expression (3.6).

LEMMA 3.9. — *Let  $Z$  be an operator as in Lemma 3.8. Then,*

$$i) \quad \sum_{i=1}^2 \langle Zy, R_{e_i, x} e_i \rangle = -c\nu^2 \langle Zx, y \rangle - 2H \langle Ax, Zy \rangle + \langle A^2 x, Zy \rangle.$$

and

$$ii) \quad \sum_{i=1}^2 \langle Ze_i, R_{e_i, x} y \rangle = -c\nu^2 \langle Zx, y \rangle - \langle Ay, ZAx \rangle + \langle Ax, y \rangle tr(AZ).$$

*Proof.* — Consider  $\{e_1, e_2\}$  an orthonormal basis of  $T_p\Sigma$ . Using Gauss' equation (2.3) we find

$$\begin{aligned} \langle Zy, R_{e_i, x}e_i \rangle &= -c\{\langle x, Zy \rangle - \langle x, e_i \rangle \langle Zy, e_i \rangle - \langle x, T \rangle \langle Zy, T \rangle + \\ &\quad - \langle e_i, T \rangle^2 \langle x, Zy \rangle + \langle e_i, T \rangle \langle x, e_i \rangle \langle Zy, T \rangle + \\ &\quad + \langle x, T \rangle \langle e_i, T \rangle \langle e_i, Zy \rangle\} - \langle Ae_i, e_i \rangle \langle Ax, Zy \rangle + \\ &\quad + \langle Ax, e_i \rangle \langle Ae_i, Zy \rangle. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{i=1}^2 \langle Zy, R_{e_i, x}e_i \rangle &= -c\{2\langle x, Zy \rangle - \sum_{i=1}^2 \langle x, e_i \rangle \langle Zy, e_i \rangle + \dots \\ &\quad \dots - 2\langle x, T \rangle \langle Zy, T \rangle - \langle x, Zy \rangle \sum_{i=1}^2 \langle e_i, T \rangle^2 + \\ &\quad + \langle Zy, T \rangle \sum_{i=1}^2 \langle e_i, T \rangle \langle x, e_i \rangle + \langle x, T \rangle \sum_{i=1}^2 \langle e_i, T \rangle \langle e_i, Zy \rangle\} + \\ &\quad - \langle Ax, Zy \rangle \sum_{i=1}^2 \langle Ae_i, e_i \rangle + \sum_{i=1}^2 \langle Ax, e_i \rangle \langle Ae_i, Zy \rangle, \end{aligned}$$

which implies that

$$\begin{aligned} \sum_{i=1}^2 \langle Zy, R_{e_i, x}e_i \rangle &= -c\{2\langle x, Zy \rangle - \langle Zx, y \rangle - 2\langle x, T \rangle \langle Zy, T \rangle + \\ &\quad - \langle x, Zy \rangle |T|^2 + \langle Zy, T \rangle \langle x, T \rangle + \langle x, T \rangle \langle T, Zy \rangle\} + \\ &\quad - \langle Ax, Zy \rangle 2H + \langle Ax, AZy \rangle. \end{aligned}$$

Hence,

$$\sum_{i=1}^2 \langle Zy, R_{e_i, x}e_i \rangle = -c(1 - |T|^2) \langle Zx, y \rangle - 2H \langle Ax, Zy \rangle + \langle A^2x, Zy \rangle,$$

which shows the validity of (i). Now, one may verify that

$$\begin{aligned} \langle Ze_i, R_{e_i, x}y \rangle &= -c\{\langle e_i, y \rangle \langle Ze_i, x \rangle - \langle x, y \rangle \langle Ze_i, e_i \rangle + \\ &\quad - \langle x, T \rangle \langle Ze_i, T \rangle \langle e_i, y \rangle - \langle e_i, T \rangle \langle y, T \rangle \langle x, Ze_i \rangle + \\ &\quad + \langle e_i, T \rangle \langle x, y \rangle \langle Ze_i, T \rangle + \langle x, T \rangle \langle y, T \rangle \langle e_i, Ze_i \rangle\} + \\ &\quad - \langle Ae_i, y \rangle \langle Ax, Ze_i \rangle + \langle Ax, y \rangle \langle Ae_i, Ze_i \rangle. \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{i=1}^2 \langle Ze_i, R_{e_i,xy} \rangle &= -c \left\{ \sum_{i=1}^2 \langle e_i, y \rangle \langle Ze_i, x \rangle - \langle x, y \rangle \sum_{i=1}^2 \langle Ze_i, e_i \rangle + \right. \\ &\quad - \langle x, T \rangle \sum_{i=1}^2 \langle Ze_i, T \rangle \langle e_i, y \rangle - \langle y, T \rangle \sum_{i=1}^2 \langle e_i, T \rangle \langle x, Ze_i \rangle + \dots \\ &\quad \dots + \langle x, y \rangle \sum_{i=1}^2 \langle e_i, T \rangle \langle Ze_i, T \rangle + \langle x, T \rangle \langle y, T \rangle \sum_{i=1}^2 \langle e_i, Ze_i \rangle \left. \right\} + \\ &\quad - \sum_{i=1}^2 \langle Ae_i, y \rangle \langle Ax, Ze_i \rangle + \langle Ax, y \rangle \sum_{i=1}^2 \langle Ae_i, Ze_i \rangle. \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{i=1}^2 \langle Ze_i, R_{e_i,xy} \rangle &= -c \{ \langle Zx, y \rangle - \langle x, T \rangle \langle Zy, T \rangle - \langle y, T \rangle \langle Zx, T \rangle + \\ &\quad + \langle ZT, T \rangle \langle x, y \rangle \} - \langle Ay, ZAx \rangle + \langle Ax, y \rangle \text{tr}(AZ), \end{aligned}$$

noting that  $trZ = 0$ .

Considering that

$$-\langle x, T \rangle \langle Zy, T \rangle - \langle y, T \rangle \langle Zx, T \rangle + \langle ZT, T \rangle \langle x, y \rangle = -(1 - \nu^2) \langle Zx, y \rangle,$$

we find

$$\sum_{i=1}^2 \langle Ze_i, R_{e_i,xy} \rangle = -c\nu^2 \langle Zx, y \rangle - \langle Ay, ZAx \rangle + \langle Ax, y \rangle \text{tr}(AZ),$$

which demonstrates (ii). □

**THEOREM 3.10.** — *Let  $\Sigma^2 \looparrowright M^2(c) \times \mathbb{R}$  be an immersed surface with non zero constant mean curvature  $H$  and let  $Z$  be an operator on  $\Sigma^2$  satisfying Codazzi's equation with  $tr(Z) = 0$ . Then,*

$$\begin{aligned} \langle (\nabla^2 Z)x, y \rangle &= 2c\nu^2 \langle Zx, y \rangle + 2H \langle Ax, Zy \rangle - \langle A^2x, Zy \rangle + \\ &\quad + \langle Ay, ZAx \rangle - \langle Ax, y \rangle \text{tr}(AZ). \end{aligned}$$

*Proof.* — We use the expressions of Lemma 3.9 in equation (3.6) obtained in Lemma 3.8. □

Next we derive an equation of Simons type for the operator  $S$  as defined in (1.1).

THEOREM 3.11 (Thm 1.1 in Introduction). — *Let  $\Sigma^2 \looparrowright M^2(c) \times \mathbb{R}$  be an immersed surface with non zero constant mean curvature  $H$  and  $S$  as defined in (1.1). Then,*

$$\begin{aligned} \langle (\nabla^2 S)x, y \rangle &= 2c\nu^2 \langle Sx, y \rangle + 2H \langle Ax, Sy \rangle - \langle A^2 x, Sy \rangle + \\ &\quad + \langle Ay, SAx \rangle - \langle Ax, y \rangle \operatorname{tr}(AS), \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2} \Delta |S|^2 &= |\nabla S|^2 - |S|^4 + |S|^2 \left( \frac{5c\nu^2}{2} - \frac{c}{2} + 2H^2 - \frac{c}{H} \langle ST, T \rangle \right) + \\ &\quad + c|ST|^2 - \frac{1}{4H^2} \langle ST, T \rangle^2. \end{aligned}$$

*Proof.* — First, since  $S$  satisfies Proposition 3.7, we can use the Theorem 3.10 with  $Z = S$ ,

Now, we know that  $\frac{1}{2} \Delta |S|^2 = |\nabla S|^2 + \langle \nabla^2 S, S \rangle$ . Furthermore, we find that

$$\langle \nabla^2 S, S \rangle = 2c\nu^2 |S|^2 + 2H \operatorname{tr}(AS^2) - [\operatorname{tr}(AS)]^2.$$

Now, we need to compute  $\operatorname{tr}(AS^2)$  and  $\operatorname{tr}(AS)$ , as follows:

$$\begin{aligned} \operatorname{tr}(AS^2) &= \operatorname{tr} \left\{ S^2 \left( S + \frac{c}{2H} \langle T, \cdot \rangle T - \frac{c}{4H} (1 - \nu^2) I + HI \right) \right\} \\ &= \operatorname{tr} S^3 + \frac{c}{2H} \operatorname{tr}(\langle T, S^2 \cdot \rangle T) - \left( \frac{c}{4H} (1 - \nu^2) - H \right) \operatorname{tr} S^2 \\ &= 0 + \frac{c}{2H} |ST|^2 - \left( \frac{c}{4H} (1 - \nu^2) - H \right) |S|^2 \end{aligned}$$

and

$$\begin{aligned} \operatorname{tr}(AS) &= \operatorname{tr} \left\{ S \left( S + \frac{c}{2H} \langle T, \cdot \rangle T - \frac{c}{4H} (1 - \nu^2) I - HI \right) \right\} \\ &= \operatorname{tr} S^2 + \frac{c}{2H} \operatorname{tr}(\langle T, S \cdot \rangle T) - \left( \frac{c}{4H} (1 - \nu^2) - H \right) \operatorname{tr} S \\ &= |S|^2 + \frac{c}{2H} \langle ST, T \rangle - 0, \end{aligned}$$

noting that  $\operatorname{tr} S = \operatorname{tr} S^3 = 0$ , also that

$$\operatorname{tr}(\langle T, S \cdot \rangle T) = \sum_{i=1}^2 \langle T, S e_i \rangle \langle T, e_i \rangle = \langle ST, T \rangle$$

and that

$$\operatorname{tr}(\langle T, S^2 \cdot \rangle T) = \sum_{i=1}^2 \langle T, S^2 e_i \rangle \langle T, e_i \rangle = \langle S^2 T, T \rangle.$$



Therefore,

$$\frac{1}{2}\Delta|S|^2 = |\nabla S|^2 + 2c\nu^2|S|^2 + 2H \left( \frac{c}{2H}|ST|^2 - \left( \frac{c}{4H}(1 - \nu^2) - H \right) |S|^2 \right) + \left( |S|^2 + \frac{c}{2H}\langle ST, T \rangle \right)^2,$$

in this way,

$$\frac{1}{2}\Delta|S|^2 = |\nabla S|^2 + 2c\nu^2|S|^2 + c|ST|^2 - \left( \frac{c}{2}(1 - \nu^2) - 2H^2 \right) |S|^2 + |S|^4 - \frac{c}{H}\langle ST, T \rangle|S|^2 - \frac{1}{4H^2}\langle ST, T \rangle^2.$$

Rearranging terms, we obtain finally

$$\frac{1}{2}\Delta|S|^2 = |\nabla S|^2 - |S|^4 + |S|^2 \left( \frac{5c\nu^2}{2} - \frac{c}{2} + 2H^2 - \frac{c}{H}\langle ST, T \rangle \right) + c|ST|^2 - \frac{1}{4H^2}\langle ST, T \rangle^2.$$

□

### 4. Applications

In this section, we will apply the results found in section 3 together with the Omori-Yau’s Theorem to classify some surfaces in  $M^2(c) \times \mathbb{R}$ .

**THEOREM 4.1.** — *Let  $\Sigma^2 \looparrowright \mathbb{H}^2 \times \mathbb{R}$  be an oriented complete immersed minimal surface. Assume that*

$$\sup_{\Sigma} (|A|^2 + 5\nu^2) < 1.$$

*Then  $\Sigma^2$  is a vertical plane  $\gamma \times \mathbb{R}$  for some geodesic  $\gamma$  in  $\mathbb{H}^2$ .*

*Proof.* — Using Theorem 3.6 with  $H = 0$  and  $c = -1$ , one finds

$$\frac{1}{2}\Delta|A|^2 = |\nabla A|^2 - |A|^4 + (1 - 5\nu^2)|A|^2 \geq |A|^2 (-|A|^2 + 1 - 5\nu^2).$$

Let  $\frac{d}{2} := -\sup_{\Sigma} (|A|^2 + 5\nu^2) + 1 > 0$ . Therefore,

$$(4.1) \quad \Delta|A|^2 \geq d \cdot |A|^2.$$

Using Gauss’ equation (2.3) in  $\mathbb{H}^2 \times \mathbb{R}$  we have

$$K_{\Sigma} = K_{ext} - \nu^2 = -\frac{|A|^2 + 5\nu^2}{2} + \frac{3\nu^2}{2} \geq -\frac{1}{2}.$$

Now we can use Theorem 2.2 with  $u = |A|^2$ , i.e, there exist  $\{p_j\}$  in  $\Sigma^2$  such that

$$\lim_{j \rightarrow \infty} |A|^2(p_j) = \sup_{\Sigma} |A|^2 \text{ and } \lim_{j \rightarrow \infty} \Delta |A|^2(p_j) \leq 0.$$

Next, we use inequality (4.1) to conclude that  $\sup_{\Sigma} |A|^2 = 0$ , i.e,  $\Sigma^2$  is totally geodesic with  $|\nu| < \sqrt{0.2}$ .

Since  $\Sigma^2$  is totally geodesic and  $|\nu| < \sqrt{0.2}$  it cannot be a slice, it must be a vertical plane  $\gamma \times \mathbb{R}$  for some geodesic  $\gamma$  in  $\mathbb{H}^2$ .  $\square$

**THEOREM 4.2.** — *Let  $\Sigma^2 \looparrowright \mathbb{H}^2 \times \mathbb{R}$  be a complete immersed surface with constant mean curvature  $H$ . Assume that*

$$\sup_{\Sigma} (|\phi|^2 + 5\nu^2) < 2H^2 + 1 \text{ and } \langle \phi T, T \rangle \geq 0.$$

*Then  $\Sigma^2$  is a vertical plane  $\gamma \times \mathbb{R}$  for some geodesic  $\gamma$  in  $\mathbb{H}^2$ .*

*Proof.* — We consider the expression in Theorem 3.6 for the particular case  $c = -1$ :

$$\frac{1}{2} \Delta |\phi|^2 = |\nabla \phi|^2 - |\phi|^4 + (2H^2 + 1 - 5\nu^2) |\phi|^2 + 2H \langle \phi T, T \rangle.$$

As  $\langle \phi T, T \rangle \geq 0$ , we find

$$\frac{1}{2} \Delta |\phi|^2 \geq -|\phi|^4 + (2H^2 + 1 - 5\nu^2) |\phi|^2.$$

Consider  $\frac{d}{2} := 2H^2 + 1 - \sup_{\Sigma} (|\phi|^2 + 5\nu^2) > 0$ . Then

$$\Delta |\phi|^2 \geq 2|\phi|^2 (2H^2 + 1 - 5\nu^2 - |\phi|^2) \geq d|\phi|^2,$$

which implies,

$$(4.2) \quad \Delta |\phi|^2 \geq d|\phi|^2.$$

Using Gauss' equation (2.3) in  $\mathbb{H}^2 \times \mathbb{R}$  we have

$$K_{\Sigma} = K_{ext} - \nu^2 = -\frac{|\phi|^2 + 5\nu^2 - 2H^2}{2} + \frac{3\nu^2}{2} \geq -\frac{1}{2}.$$

Now we can use Theorem 2.2 with  $u = |\phi|^2$ , i.e, there exist  $\{p_j\}$  in  $\Sigma^2$  such that

$$\lim_{j \rightarrow \infty} |\phi|^2(p_j) = \sup_{\Sigma} |\phi|^2 \text{ and } \lim_{j \rightarrow \infty} \Delta |\phi|^2(p_j) \leq 0.$$

Furthermore, we use inequality (4.2) to conclude that  $\sup_{\Sigma} |\phi|^2 = 0$ , i.e,  $\Sigma^2$  is totally umbilical.

Next, we use that if  $\Sigma^2$  is totally umbilical with constant mean curvature in  $\mathbb{H}^2 \times \mathbb{R}$  then  $\Sigma^2$  is totally geodesic, which follows from [9] section 4.

Since  $\Sigma^2$  is totally geodesic and  $|\nu| < \sqrt{0.2}$  it must be a vertical plane  $\gamma \times \mathbb{R}$  for some geodesic  $\gamma$  in  $\mathbb{H}^2$ . This concludes the proof.  $\square$

We need the following result:

LEMMA 4.3. — *Let  $\Sigma^2 \looparrowright M^2(c) \times \mathbb{R}$  be a complete immersed surface with non zero constant mean curvature  $H$ . Then  $|S| = 0$  if and only if  $\Sigma^2$  is an Abresch-Rosenberg surface.*

*Proof.* — We consider  $(u, v)$  isothermal parameters on the surface  $\Sigma^2$ . Now, we consider the complex parameter,  $z = u + iv$  and the  $(2,0)$ -part of the Abresch-Rosenberg differential

$$Q(x, y) = 2H \langle Ax, y \rangle - c \langle x, T \rangle \langle y, T \rangle.$$

We can rewrite  $Q$  as

$$Q(x, y) = \langle Sx, y \rangle - \frac{c}{2}(1 - \nu^2) \langle x, y \rangle + 2H^2 \langle x, y \rangle.$$

Next we evaluate  $Q(\partial_z, \partial_z)$  noting that  $\langle \partial_z, \partial_z \rangle = 0$ :

$$Q(\partial_z, \partial_z) = \langle S\partial_z, \partial_z \rangle = \left( \frac{\tilde{e} - \tilde{g}}{4} \right) - i \frac{\tilde{f}}{2},$$

where  $\tilde{e} = \langle S\partial_u, \partial_u \rangle = -\langle S\partial_v, \partial_v \rangle = -\tilde{g}$  and  $\tilde{f} = \langle S\partial_u, \partial_v \rangle$ . Therefore

$$|Q^{(2,0)}| = \sqrt{\left( \frac{\tilde{e} - \tilde{g}}{4} \right)^2 + \frac{\tilde{f}^2}{4}} = \sqrt{\frac{\tilde{e}^2}{4} + \frac{\tilde{f}^2}{4}} = \frac{E^2}{2\sqrt{2}}|S|,$$

where  $E = |\partial_u| > 0$ . This concludes the proof. □

Let us consider the polynomial  $p_H(t) = -t^2 - \frac{1}{\sqrt{2}H}t + \left( \frac{4H^2 - 1}{2} \right)$ . When  $H$  is greater than one half there is a positive root for  $p_H$ . Let  $L_H$  be the positive root. One has:

THEOREM 4.4 (Thm 1.2 in Introduction). — *Let  $\Sigma^2 \looparrowright \mathbb{S}^2 \times \mathbb{R}$  be an immersed surface with constant mean curvature  $H$  greater than one half. If*

$$\Sigma^2 \text{ is complete and } \sup_{\Sigma} |S| < L_H$$

or

$$\Sigma^2 \text{ is closed and } |S| \leq L_H,$$

then  $\Sigma^2 = S_H^2$ , i.e,  $\Sigma^2$  is an embedded rotationally invariant constant mean curvature sphere.

*Proof.* — Let consider two cases. First,  $\Sigma$  is complete and second,  $\Sigma$  is closed.

*First Case.* Consider the expression in Theorem 3.11 with  $c = 1$ :

$$\frac{1}{2} \Delta |S|^2 = |\nabla S|^2 - |S|^4 + |S|^2 \left( \frac{5\nu^2}{2} - \frac{1}{2} + 2H^2 - \frac{1}{H} \langle ST, T \rangle \right) +$$

$$+|ST|^2 - \frac{1}{4H^2} \langle ST, T \rangle^2.$$

As  $|\langle ST, T \rangle| \leq |ST| \leq \frac{1}{\sqrt{2}}|S|$ , we have

$$\frac{1}{2}\Delta|S|^2 \geq -|S|^4 + |S|^2 \left( \frac{5\nu^2}{2} + \frac{4H^2 - 1}{2} - \frac{1}{\sqrt{2}H}|S| \right) + \left( \frac{4H^2 - 1}{4H^2} \right) \langle ST, T \rangle^2,$$

hence,

$$(4.3) \quad \frac{1}{2}\Delta|S|^2 \geq |S|^2 \left( \frac{4H^2 - 1}{2} - \frac{1}{\sqrt{2}H}|S| - |S|^2 \right) + \frac{5}{2}\nu^2|S|^2,$$

because  $H > \frac{1}{2}$ .

Observe that

$$\frac{4H^2 - 1}{2} - \frac{1}{\sqrt{2}H}|S| - |S|^2 \geq p_H(\sup_{\Sigma}|S|) =: \frac{d}{2} > 0$$

and  $\nu^2|S|^2 \geq 0$ . Therefore

$$(4.4) \quad \Delta|S|^2 \geq d|S|^2.$$

Now we estimate  $|S|$ .

$$|S| \geq 2H|A| - |\langle T, \cdot \rangle T| - (1 - \nu^2) - 4H^2 \geq 2H|A| - 2(1 - \nu^2) - 4H^2,$$

that is,

$$L_H \geq |S| \geq 2H|A| - 2 - 4H^2.$$

Using Gauss' equation (2.3) in  $\mathbb{S}^2 \times \mathbb{R}$  we find

$$K_{\Sigma} = K_{ext} + \nu^2 = -\frac{|A|^2}{2} + 2H^2 + \nu^2 \geq -\frac{1}{2} \left( \frac{L_H + 2 + 4H^2}{2H} \right)^2.$$

Now we can use Theorem 2.2 with  $u = |S|^2$ , i.e., there exists a  $\{p_j\}$  in  $\Sigma^2$  such that

$$\lim_{j \rightarrow \infty} |S|^2(p_j) = \sup_{\Sigma} |S|^2 \text{ and } \lim_{j \rightarrow \infty} \Delta|S|^2(p_j) \leq 0.$$

By means of inequality (4.4) we conclude that  $\sup_{\Sigma} |S|^2 = 0$ , i.e.,  $|S| = 0$  in  $\Sigma^2$ . Using Lemma 4.3 and Remark 1 of the Introduction we conclude the proof.

*Second case.* Let us consider expression (4.3)

$$\frac{1}{2}\Delta|S|^2 \geq |S|^2 \left( \frac{4H^2 - 1}{2} - \frac{1}{\sqrt{2}H}|S| - |S|^2 \right) + \frac{5}{2}\nu^2|S|^2.$$

As  $|S| \leq L_H$ , we have  $\frac{4H^2 - 1}{2} - \frac{1}{\sqrt{2}H}|S| - |S|^2 \geq 0$ . Hence,

$$\frac{1}{2}\Delta|S|^2 \geq \frac{5}{2}\nu^2|S|^2.$$

Integrating and using Stokes' Theorem we find

$$0 \geq \frac{5}{2} \int_{\Sigma} \nu^2 |S|^2 d\Sigma \geq 0.$$

It follows that

$$(4.5) \quad |S| \cdot \nu = 0.$$

Let  $\Theta = \{p \in \Sigma^2 : \nu(p) = 0\} = \nu^{-1}(0)$  be the nodal lines of  $\nu$ . We know that

$$\Delta\nu + (|A|^2 + Ric(N, N))\nu = 0.$$

Hence, we can apply Theorem 2.5 in [4], p. 49, to conclude that  $\Theta$  has empty interior. Thus, using (4.5),  $|S|$  vanishes in an open and dense set. By continuity,  $|S| = 0$  in  $\Sigma$ .

Using Lemma 4.3 and Remark.1 of the Introduction we conclude the proof. □

**THEOREM 4.5.** — *There exists no  $\Sigma^2 \looparrowright \mathbb{S}^2 \times \mathbb{R}$  complete immersed surface with constant mean curvature greater than one half such that  $|S| = L_H$ .*

*Proof.* — Suppose that there exist  $\Sigma^2 \looparrowright \mathbb{S}^2 \times \mathbb{R}$  satisfying the condition of the theorem. Using expression (4.3)

$$\frac{1}{2}\Delta|S|^2 \geq |S|^2 \left( \frac{4H^2 - 1}{2} - \frac{1}{\sqrt{2}H}|S| - |S|^2 \right) + \frac{5}{2}\nu^2|S|^2,$$

with  $|S| = L_H$  one find that

$$0 \geq 0 + \frac{5}{2}\nu^2 L_H^2 \geq 0.$$

Hence  $\nu = 0$ , i.e,  $\Sigma^2 \looparrowright \mathbb{S}^2 \times \mathbb{R}$  is a cylinder  $\gamma \times \mathbb{R}$  for some  $\gamma \in \mathbb{S}^2$  with constant curvature  $2H$ .

On the other hand, for a cylinder  $\gamma \times \mathbb{R}$ , where  $\gamma \in \mathbb{S}^2$  with constant curvature  $2H$ , we may write

$$S = \begin{pmatrix} 2H^2 + \frac{1}{2} & 0 \\ 0 & -2H^2 - \frac{1}{2} \end{pmatrix}.$$

As  $|S| = \frac{\sqrt{2}}{2}(4H^2 + 1) > L_H$  we have a contradiction. □

In next theorem we need the following result:

LEMMA 4.6. — Any Abresch-Rosenberg surface  $\Sigma^2 \looparrowright \mathbb{H}^2 \times \mathbb{R}$  with  $H > \frac{1}{2}$  is an embedded rotationally invariant constant mean curvature sphere.

*Proof.* — See Proposition 4.3 in [1], p. 159. □

Let us consider the polynomial

$$q_H(t) = -t^2 - \frac{1}{\sqrt{2}H}t + \left( \frac{8H^4 - 12H^2 - 1}{4H^2} \right).$$

When  $H$  is greater than a positive root of the polynomial  $r(x) = 8x^4 - 12x^2 - 1$ , i.e,  $H$  is greater than  $\sqrt{\frac{3 + \sqrt{11}}{4}}$ , there is a positive root for  $q_H$ . Let  $M_H$  be the positive root.

THEOREM 4.7 (Thm 1.3 in Introduction). — Let  $\Sigma^2 \looparrowright \mathbb{H}^2 \times \mathbb{R}$  be an immersed surface with constant mean curvature  $H$  greater than  $\sqrt{\frac{3 + \sqrt{11}}{4}} \approx 1.25664$ . If

$$\Sigma^2 \text{ is complete and } \sup_{\Sigma} |S| < M_H$$

or

$$\Sigma^2 \text{ is closed and } |S| \leq M_H,$$

then  $\Sigma^2 = S^2_H$ , i.e,  $\Sigma^2$  is an embedded rotationally invariant constant mean curvature sphere.

*Proof.* — Let us consider two cases. First,  $\Sigma$  is complete and second,  $\Sigma$  is closed.

*First case.* Consider the expression in Theorem 3.11 with  $c = -1$

$$\begin{aligned} \frac{1}{2}\Delta|S|^2 &= |\nabla S|^2 - |S|^4 + |S|^2 \left( -\frac{5\nu^2}{2} + \frac{1}{2} + 2H^2 + \frac{1}{H}\langle ST, T \rangle \right) + \\ &\quad - |ST|^2 - \frac{1}{4H^2}\langle ST, T \rangle^2. \end{aligned}$$

As  $|\langle ST, T \rangle| \leq |ST| \leq \frac{1}{\sqrt{2}}|S|$ , we may write

$$\frac{1}{2}\Delta|S|^2 \geq -|S|^4 + |S|^2 \left( \frac{4H^2 + 1 - 5\nu^2}{2} - \frac{1}{\sqrt{2}H}|S| \right) - \left( \frac{4H^2 + 1}{4H^2} \right) |S|^2,$$

i.e,

$$\frac{1}{2}\Delta|S|^2 \geq |S|^2 \left( \frac{4H^2 - 4 + 5 - 5\nu^2}{2} - \frac{1}{\sqrt{2}H}|S| - \frac{4H^2 + 1}{4H^2} - |S|^2 \right).$$

This may be rewritten as,

$$(4.6) \quad \frac{1}{2}\Delta|S|^2 \geq |S|^2 \left( \frac{8H^4 - 12H^2 - 1}{4H^2} - \frac{1}{\sqrt{2}H}|S| - |S|^2 \right) + \frac{5}{2}(1 - \nu^2)|S|^2.$$

Observe that

$$\frac{8H^4 - 12H^2 - 1}{4H^2} - \frac{1}{\sqrt{2}H}|S| - |S|^2 \geq q_H(\sup_{\Sigma} |S|) =: \frac{d}{2} > 0$$

and  $(1 - \nu^2)|S|^2 \geq 0$ . Therefore,

$$(4.7) \quad \Delta|S|^2 \geq d|S|^2.$$

Next we estimate  $|S|$ .

$$|S| \geq 2H|A| - |\langle T, \cdot \rangle T| - (1 - \nu^2) - 4H^2 \geq 2H|A| - 2(1 - \nu^2) - 4H^2,$$

i.e.,

$$M_H \geq |S| \geq 2H|A| - 2 - 4H^2.$$

Using Gauss' equation (2.3) in  $\mathbb{H}^2 \times \mathbb{R}$  we find

$$K_{\Sigma} = K_{ext} - \nu^2 = -\frac{|A|^2}{2} + 2H^2 - \nu^2 \geq -\frac{1}{2} \left( \frac{M_H + 2 + 4H^2}{2H} \right)^2.$$

Now we can use Theorem 2.2 with  $u = |S|^2$ , i.e., there exists a  $\{p_j\}$  in  $\Sigma^2$  such that

$$\lim_{j \rightarrow \infty} |S|^2(p_j) = \sup_{\Sigma} |S|^2 \text{ and } \lim_{j \rightarrow \infty} \Delta|S|^2(p_j) \leq 0.$$

Inequality (4.7) allows us conclude that  $\sup_{\Sigma} |S|^2 = 0$ , i.e.,  $|S| = 0$  in  $\Sigma^2$ . Then, by using Lemmas 4.3 and 4.6, we conclude the proof.

*Second case.* Let us consider expression (4.6)

$$\frac{1}{2}\Delta|S|^2 \geq |S|^2 \left( \frac{8H^4 - 12H^2 - 1}{4H^2} - \frac{1}{\sqrt{2}H}|S| - |S|^2 \right) + \frac{5}{2}(1 - \nu^2)|S|^2.$$

As  $|S| \leq M_H$ , we have that  $\frac{8H^4 - 12H^2 - 1}{4H^2} - \frac{1}{\sqrt{2}H}|S| - |S|^2 \geq 0$ . Hence,

$$\frac{1}{2}\Delta|S|^2 \geq \frac{5}{2}(1 - \nu^2)|S|^2.$$

Integrating and using Stokes' Theorem we write

$$0 \geq \frac{5}{2} \int_{\Sigma} (1 - \nu^2)|S|^2 d\Sigma \geq 0.$$

Moreover

$$(4.8) \quad (1 - \nu^2) \cdot |S|^2 = 0.$$

Consider  $\Theta = \{p \in \Sigma^2; \nu^2(p) = 1\} \subset \mathbb{H}^2 \times \{t_0\}$ , for any  $t_0$ . Since  $H$  is positive we have that  $\Theta$  has empty interior. Thus, using (4.8), we conclude that  $|S|$  vanishes in an open and dense set. By continuity,  $|S| = 0$  in  $\Sigma$ . Using Lemma 4.3 and the fact that the only Abresch-Rosenberg closed surface is  $S_H^2$  we conclude the proof.  $\square$

**THEOREM 4.8.** — *There exists no  $\Sigma^2 \looparrowright \mathbb{H}^2 \times \mathbb{R}$  a complete immersed surface with constant mean curvature greater than  $\sqrt{\frac{3 + \sqrt{11}}{4}} \approx 1.25664$  such that  $|S| = M_H$ .*

*Proof.* — Suppose that there exists  $\Sigma^2 \looparrowright \mathbb{H}^2 \times \mathbb{R}$  satisfying the condition of the theorem. Using expression (4.6)

$$\frac{1}{2}\Delta|S|^2 \geq |S|^2 \left( \frac{8H^4 - 12H^2 - 1}{4H^2} - \frac{1}{\sqrt{2}H}|S| - |S|^2 \right) + \frac{5}{2}(1 - \nu^2)|S|^2$$

with  $|S| = M_H$  we obtain:

$$0 \geq 0 + \frac{5}{2}(1 - \nu^2)M_H^2 \geq 0.$$

Hence  $\nu^2 = 1$ , i.e,  $\Sigma^2 \looparrowright \mathbb{H}^2 \times \mathbb{R}$  is a slice  $\mathbb{H}^2 \times \{t_0\}$ . But  $\mathbb{H}^2 \times \{t_0\}$  has zero mean curvature, and this is impossible because  $H$  is positive.  $\square$

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