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ALMOST ÉTALE EXTENSIONS OF FONTAINE RINGS AND LOG-CRYSTALLINE COHOMOLOGY IN THE SEMI-STABLE REDUCTION CASE

by Rémi Shankar LODH

ABSTRACT. — Let K be a field of characteristic zero complete for a discrete valuation, with perfect residue field of characteristic $p > 0$, and let K^+ be the valuation ring of K . We relate the log-crystalline cohomology of the special fibre of certain affine K^+ -schemes $X = \text{Spec}(R)$ with good or semi-stable reduction to the Galois cohomology of the fundamental group $\pi_1(X_{\bar{K}})$ of the geometric generic fibre with coefficients in a Fontaine ring constructed from R . This is based on Faltings' theory of almost étale extensions.

RÉSUMÉ. — Soit K un corps de caractéristique nulle, complet pour une valuation discrète, à corps résiduel parfait de caractéristique $p > 0$, et soit K^+ son anneau d'entiers. Nous montrons que la cohomologie log-cristalline de la fibre spéciale de certains K^+ -schémas affines $X = \text{Spec}(R)$ à réduction bonne ou semi-stable se calcule comme la cohomologie galoisienne du groupe fondamental $\pi_1(X_{\bar{K}})$ de la fibre générique géométrique de X à valeurs dans un anneau de Fontaine construit à partir de R . Ce calcul est basé sur la théorie des revêtements presque étales de Faltings.

Introduction

The purpose of this article is to make precise the relationship between crystalline cohomology and Galois cohomology of certain Fontaine rings occurring in Faltings' approach to p -adic Hodge theory ([6], [7]). Let us very briefly recall this approach. Let K^+ be a complete discrete valuation ring of fraction field K of characteristic zero and perfect residue field k of characteristic $p > 0$. Let X be a proper smooth K^+ -scheme. One constructs a site (the "Faltings site"), usually denoted \mathcal{X} , whose cohomology formalizes the idea of glueing $\pi_1(X_{\bar{K}})$ -cohomology locally on X . One sheafifies a

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construction of Fontaine to obtain a sheaf of rings $\mathcal{A}_{\text{crys},n}$ on \mathcal{X} together with transformations

$$\begin{aligned}
 H^*(\mathcal{X}, \mathbb{Z}/p^n\mathbb{Z}) \otimes_{\mathbb{Z}_p} A_{\text{cris}} &\rightarrow H^*(\mathcal{X}, \mathcal{A}_{\text{crys},n}) \\
 &\leftarrow H_{\text{crys}}^*(X_k|W_n(k), \mathcal{O}) \otimes_{W(k)} A_{\text{cris}}
 \end{aligned}$$

where A_{cris} is the ring of p -adic periods constructed by Fontaine [8], and the group on the right denotes the crystalline cohomology of the special fibre X_k . Then one uses Faltings’ theory of almost étale extensions to show that the intermediate cohomology theory $H^*(\mathcal{X}, \mathcal{A}_{\text{crys},n})$ almost satisfies Poincaré duality and Künneth formula, hence by standard arguments is almost isomorphic to crystalline cohomology (here the term ‘almost’ is used in the sense of almost ring theory ([7], [9])). Since X is smooth, the group on the left is canonically isomorphic to étale cohomology of $X_{\bar{K}}$ tensored with A_{cris} and compatibility with Poincaré duality gives a one-sided inverse to the almost defined transformation to crystalline cohomology, up to a power of an element $t \in A_{\text{cris}}$. So after taking the projective limit and inverting t we obtain almost isomorphisms, which are in fact isomorphisms.

In this article we study closely the map

$$H_{\text{crys}}^*(X_k|W_n(k), \mathcal{O}) \otimes_{W(k)} A_{\text{cris}} \rightarrow H^*(\mathcal{X}, \mathcal{A}_{\text{crys},n})$$

locally on X . Our main result is that this map is *locally* an almost isomorphism up to t^d -torsion, where $d = \dim(X_K)$ and $t \in A_{\text{cris}}$ is an element which plays a role analogous to that of $2\pi i$ in the transcendental theory of periods. A similar result also holds in the case X has semi-stable reduction and in this paper we treat both the good and semi-stable reduction cases on the same footing. See the overview below for more details.

Finally, let us mention that F. Andreatta and O. Brinon [1] have independently found similar results in the good reduction case. Their proofs, although technically different, are based on the same idea.

Overview

§1. — We begin by reviewing the (log-) crystalline site. This is mainly to fix notation. Afterwards, we review the construction by Fontaine [8] of the final object of the crystalline site of a ring of characteristic p with surjective (absolute) Frobenius. Such final objects are called *Fontaine rings*. We give the proof for the more general log-crystalline site. Then we give some examples of Fontaine rings due to Fontaine and Kato.

§2. — We first recall the almost ring theory which we will use, the key input being Faltings’ Almost Purity Theorem [7]. Afterwards we apply this theorem to certain Fontaine rings, constructed as follows. Let $\text{Spec}(R)$ be a integral K^+ -scheme with semi-stable reduction. Up to localizing on $\text{Spec}(R)$ we may assume that it is étale over a ring of the form $K^+[T_1, \dots, T_{d+1}]/(T_1 \cdots T_r - c)$, where either $c = 1$ (good reduction) or $c \in K^+$ is a uniformizer (semi-stable reduction). One says that such an R is *small*. Let $\Sigma_n = W_n(k)[u]\langle u^e \rangle$, where e is the absolute ramification index of K and the angled brackets mean that we have added divided powers of u^e . Then there is a surjection $\Sigma_n \rightarrow K^+/pK^+$ whose kernel is a DP-ideal. Let \bar{R} be the normalization of R in the maximal profinite connected étale covering of $R[1/p]$. Then via the theory of almost étale extensions one can show that the ring $\bar{R}/p\bar{R}$ has surjective Frobenius, hence by a logarithmic version of Fontaine’s theorem (Theorem 1.1), we may construct the Fontaine ring

$$A_{\log}^+ := \varprojlim_n H_{\log\text{-crys}}^0(\bar{R}/p\bar{R}|\Sigma_n, \mathcal{O})$$

(log-crystalline cohomology). Also we can construct another Fontaine ring $A_{\log, \infty}^+$ as follows. Let \bar{K} denote the algebraic closure of K and \bar{K}^+ its valuation ring, i.e. the normalization of K^+ in \bar{K} . Let $\tilde{R} = R \otimes_{K^+} \bar{K}^+$ and let \tilde{R}_∞ denote the ring obtained from \tilde{R} by adding all p -power roots of the the T_i . Define

$$A_{\log, \infty}^+ := \varprojlim_n H_{\log\text{-crys}}^0(\tilde{R}_\infty/p\tilde{R}_\infty|\Sigma_n, \mathcal{O}).$$

Then the theory of almost étale extensions applied to these Fontaine rings implies that the canonical homomorphism

$$A_{\log, \infty}^+/p^n A_{\log, \infty}^+ \rightarrow A_{\log}^+/p^n A_{\log}^+$$

is the filtering inductive limit of almost Galois coverings and there are canonical almost isomorphisms for each i

$$H^i(\Delta_\infty, A_{\log, \infty}^+/p^n A_{\log, \infty}^+) \xrightarrow{\sim} H^i(\Delta, A_{\log}^+/p^n A_{\log}^+)$$

where $\Delta_\infty := \text{Gal}(\tilde{R}_\infty[1/p]/\tilde{R}[1/p]) \cong \mathbb{Z}_p(1)^d$ is a quotient of

$$\Delta := \text{Gal}(\bar{R}[1/p]/\tilde{R}[1/p])$$

(see Corollaries 2.17, 2.18). This also applies to p -adic divided power bases other than $\Sigma := \varprojlim_n \Sigma_n$.

§3. — We construct, via the formalism of log-crystalline cohomology, a canonical logarithmic de Rham resolution of the ring A_{\log}^+ . We then (almost) compute the Δ -cohomology of the components of this resolution, by

reducing to the case of $A_{\log, \infty}^+$ and making an explicit computation there. The result is the following (Corollary 3.12).

THEOREM 0.1. — *Let \mathcal{R}_n be a log-smooth Σ_n -lift of R/pR , and let $\omega_{\mathcal{R}_n/\Sigma_n}^\bullet$ be the logarithmic de Rham complex of \mathcal{R}_n/Σ_n . There is a canonical morphism of complexes in the derived category*

$$B_{\log}^+ \otimes_{\Sigma} \omega_{\mathcal{R}_n/\Sigma_n}^\bullet \rightarrow C^\bullet(\Delta, A_{\log}^+/p^n A_{\log}^+)$$

which is an almost quasi-isomorphism up to $t^d x$ -torsion, where $x \in B_{\log}^+$ is independent of n and R ($x = 1$ in the good reduction case).

Here

$$B_{\log}^+ = \varprojlim_n H_{\log\text{-cris}}^0(\bar{K}^+/p\bar{K}^+|\Sigma_n, \mathcal{O})$$

is a ring of p -adic periods constructed by Kato, $t \in A_{\text{cris}} \subset B_{\log}^+$ is the element alluded to above, and we write $C^\bullet(\Delta, -) = \text{Hom}_{\text{cont.}, \Delta}(\Delta^{\times \bullet}, -)$ for the usual functorial complex computing continuous group cohomology. The proofs of the main results of this section are straightforward, although they involve some lengthy computations.

§4. — In this appendix we include some results from on log-structures and commutative algebra which we need.

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Remarks on notation

- For any subset $R \subset \mathbb{R}$ we write R_+ (resp. $R_{>0}$) for the set of elements of R which are greater than or equal to zero (resp. greater than zero); $\mathbb{N} := \mathbb{Z}_+$ is the set of natural numbers.

- By *ring* we mean a commutative ring with unity. For any ring A , we denote by A^* its group of units. For any integral domain A , we denote by $Q(A)$ its fraction field.
- All monoids considered will be assumed to be with commutative. To a monoid M we write M^{gp} for the associated group.
- If A is a ring and M is an A -module, then we denote by $\Gamma_A(M)$ the divided power polynomial A -algebra defined by M (see [2] or [3] for a construction of this algebra). If $I \subset A$ is an ideal then we denote by $D_A(I)$ the divided power hull of A for the ideal I (*loc.cit.*). If A is a ring and $I \subset A$ is an ideal with a divided power structure $(\gamma_n : I \rightarrow A)_{n \in \mathbb{N}}$, then we will often write $x^{[n]} := \gamma_n(x)$ when it is clear which divided power structure is meant. Finally, if X_1, \dots, X_d denotes indeterminates, then we write $A \langle X_1, \dots, X_d \rangle$ for the divided power polynomial A -algebra in the variables X_1, \dots, X_d . It is the divided power hull of $A[X_1, \dots, X_d]$ for the ideal generated by X_1, \dots, X_d .
- If G is a profinite group and M is a discrete G -module, then we write $H^i(G, M)$ for the Galois cohomology groups (i.e. continuous group cohomology).

1. Crystalline cohomology of rings with surjective Frobenius

In this section we recall Fontaine's construction of the final object of the crystalline site of a ring of characteristic $p > 0$ with surjective Frobenius endomorphism. This will play the role of substitute for Poincaré's lemma in our approach.

1.1. Reminder on crystalline sites

1.1.1.

Let Z be a scheme and let $Z_0 \hookrightarrow Z$ be a closed immersion, such that the ideal defining the image of Z_0 in Z is nilpotent and has a divided power structure (we say that $Z_0 \hookrightarrow Z$ is a *divided power thickening*). We usually use the abbreviation DP for "divided power". Suppose that $T \rightarrow Z$ (resp. $T' \rightarrow Z$) is a morphism and $U \hookrightarrow T$ (resp. $U' \hookrightarrow T'$) is a DP-thickening such that the morphism $T \rightarrow Z$ (resp. $T' \rightarrow Z$) is a DP-morphism ([2, 1.9.4])

for the given DP-structures. A morphism of Z -schemes $T \rightarrow T'$ is called a Z -DP-morphism if it is a DP-morphism (for the given DP-structures) and such that the diagram

$$\begin{array}{ccc} T & \longrightarrow & T' \\ \downarrow & & \downarrow \\ Z & \xlongequal{\quad} & Z \end{array}$$

is a commutative diagram of DP-morphisms.

1.1.2.

All log-structures in this paper will be in the étale topology. Let $(X, M) \rightarrow (Z_0, N_0)$ be a morphism of log-schemes (see [12]). If $f : X \rightarrow Z_0$ is the underlying morphism, then we write f^*N_0 for the inverse image log-structure, in contrast with the inverse image sheaf $f^{-1}N_0$. If M is a pre-log-structure on X then we denote by M^a the associated log-structure. If X is a log-scheme and $U \rightarrow X$ is an étale morphism of schemes, then the restriction M_U of the log-structure M of X to U defines a natural log-structure on U and we will always consider U as a log-scheme for this log-structure.

1.1.3.

Let $(X, M) \rightarrow (Z_0, N_0)$ be a morphism of log-schemes and assume that the log-structure M is integral. Let $(Z_0, N_0) \hookrightarrow (Z, N)$ be an exact closed immersion such that $Z_0 \hookrightarrow Z$ is a DP-thickening. The *logarithmic crystalline site* of (X, M) over the DP-log-base (Z, N) (see [13, 2.4]) is the site whose underlying category has for objects exact closed immersions $(U, M_U) \hookrightarrow T$ of log-schemes, where:

- $U \rightarrow X$ is an étale morphism of schemes and the log-structure M_U is as in (1.1.2)
- T is a log- (Z, N) -scheme with integral log-structure such that the morphism of schemes underlying the structure morphism $T \rightarrow Z$ is a DP-morphism
- the morphism of schemes underlying $(U, M_U) \hookrightarrow T$ is a DP-thickening.

We usually write $U \hookrightarrow T$ instead of $(U, M_U) \hookrightarrow T$. A morphism of this category is a commutative diagram of log-schemes

$$\begin{array}{ccc} U' & \longrightarrow & T' \\ \downarrow & & \downarrow \\ U & \longrightarrow & T \end{array}$$

where morphism on the left is a morphism of $\log-(X, M)$ -schemes, and the morphism on the right is a morphism of $\log-(Z, N)$ -schemes such that the underlying morphism of schemes is a Z -DP-morphism. The pretopology on this category is given by defining covering families to be families of morphisms

$$(U_\alpha \hookrightarrow T_\alpha)_\alpha \rightarrow (U \hookrightarrow T)$$

such that the morphisms of schemes underlying $(U_\alpha \rightarrow U)_\alpha$ and $(T_\alpha \rightarrow T)_\alpha$ are coverings for the étale topology, and such that the squares

$$\begin{array}{ccc} U_\alpha & \longrightarrow & T_\alpha \\ \downarrow & & \downarrow \\ U & \longrightarrow & T \end{array}$$

are cartesian. Given such a covering, for any morphism $(U' \hookrightarrow T') \rightarrow (U \hookrightarrow T)$, note that the diagram

$$\begin{array}{ccc} U' \times_U U_\alpha & \longrightarrow & T' \times_T T_\alpha \\ \downarrow & & \downarrow \\ U' & \longrightarrow & T' \end{array}$$

is cartesian, hence $(U' \times_U U_\alpha \hookrightarrow T' \times_T T_\alpha)$ is an object of the category underlying the logarithmic crystalline site. This shows that the *log-crystalline site of X over the DP-base Z* is indeed a site denoted $((X, M)|(Z, N))_{\text{crys}}$ or simply by $(X|Z)_{\text{log-crys}}$ when it is clear which log-structures are meant for X and Z .

1.1.4.

To give a sheaf \mathcal{F} on $(X|Z)_{\text{log-crys}}$ is the same as giving for all $(U \hookrightarrow T) \in \text{ob}(X|Z)_{\text{log-crys}}$ a sheaf \mathcal{F}_T on the étale site of T , together with a morphism

$$g_{\mathcal{F}}^* : g^{-1}\mathcal{F}_T \rightarrow \mathcal{F}_{T'}$$

for any morphism $g : (U' \hookrightarrow T') \rightarrow (U \hookrightarrow T)$, such that the natural transitivity condition holds for morphisms $(U'' \hookrightarrow T'') \rightarrow (U' \hookrightarrow T') \rightarrow (U \hookrightarrow T)$ and moreover $g_{\mathcal{F}}^*$ is an isomorphism if the square defined by g is cartesian.

In this way we see that the presheaf defined

$$\mathcal{O}(U \hookrightarrow T) := \mathcal{O}_T(T)$$

is in fact a sheaf, called the *structure sheaf* of $(X|Z)_{\text{log-crys}}$. A *quasi-coherent \mathcal{O} -module* on $(X|Z)_{\text{log-crys}}$ is an \mathcal{O} -module \mathcal{E} such that for each

object $U \hookrightarrow T$ of $(X|Z)_{\log\text{-crys}}$, the restriction \mathcal{E}_T is a quasi-coherent \mathcal{O}_T -module in the usual sense.

1.1.5.

If $X = \text{Spec}(R)$ and $Z = \text{Spec}(S)$ are affine schemes, then we will usually write $(R|S)_{\log\text{-crys}}$ instead of $(X|Z)_{\log\text{-crys}}$.

1.1.6.

Assume in the sequel that Z is annihilated by a power of p and that (Z, N) is a fine log-scheme. By [13, 2.4.2], if $f : (X, M) \rightarrow (X', M')$ is a morphism of $\log\text{-}(Z_0, N_0)$ -schemes with M integral and M' fine, then f induces a morphism of log-crystalline topoi over the DP log-base (Z, N) .

1.1.7.

Given a projective system of fine $\log\text{-}(Z_0, N_0)$ -schemes $\{(X_i, M_i)\}_{i \in I}$ with affine transition morphisms and with projective limit (X, M) , then by [13, 2.4.3] we have

$$H_{\log\text{-crys}}^j(X|Z, \mathcal{O}) \cong \varinjlim_{i \in I} H_{\log\text{-crys}}^j(X_i|Z, \mathcal{O})$$

for all j .

1.2. Fontaine’s theorem

1.2.1.

Assume S is a ring such that $p^m S = 0$ for some $m \in \mathbb{N}$, and $J \subset S$ an ideal containing p with a divided power structure $\gamma : J \rightarrow S$. Let R be a S/J -algebra. Let N be an integral monoid defining a log-structure on $\text{Spec}(S)$ and M an integral monoid defining a log structure on $\text{Spec}(R)$ and a map $N \rightarrow M$ such that we have a morphism of log-schemes

$$(\text{Spec}(R), M^a) \rightarrow (\text{Spec}(S), N^a).$$

Then we may consider the log-crystalline site $(R|S)_{\log\text{-crys}}$.

For any monoid M , the endomorphism

$$M \rightarrow M : m \mapsto m^p$$

is called the *Frobenius* of M .

THEOREM 1.1. — *With the above notation and assumptions. Assume that the (absolute) Frobenius is surjective on R and on M . If M is a saturated log-structure, then the site $(R|S)_{\log\text{-crys}}$ has a final object.*

We give two proofs, the first generalizing that of Fontaine in the classical case and the second following Breuil [5].

First proof. — The proof is constructive and proceeds in several steps.

Step 1. — We construct a candidate for the final object. First note that for any étale map $U \rightarrow \text{Spec}(R)$, the (absolute) Frobenius is surjective on \mathcal{O}_U . Indeed, since the map is étale, its relative Frobenius is an isomorphism, so this follows from the factorization of the absolute Frobenius of U as the relative Frobenius followed by the pullback of the absolute Frobenius of $\text{Spec}(R)$. We first define a perfect ring $P(R)$ as being the projective limit of the diagram

$$\dots \xrightarrow{F} R \xrightarrow{F} R \xrightarrow{F} R$$

where F denotes the (absolute) Frobenius of R . An element of $P(R)$ is given by a sequence $r = (r^{(n)})$ of elements of R indexed by the natural numbers, such that $r^{(n+1)p} = r^{(n)}$ for all n . $P(R)$ is a perfect ring of characteristic p , so its ring of Witt vectors $W(P(R))$ is a flat \mathbb{Z}_p -algebra. We write $(r_0, r_1, r_2, \dots) \in W(P(R))$ and $r_i = (r_i^{(n)})$ for each $i = 0, 1, 2, \dots$. Let $\text{Spec}(R') \hookrightarrow \text{Spec}(A)$ be an object of the site $(R|S)_{\log\text{-crys}}$. If $r = (r^{(n)}) \in P(R)$ and $g : R \rightarrow R'$ is structure homomorphism, then we choose arbitrary lifts $\hat{r}^{(n)} \in A$ of $g(r^{(n)})$ for all n and set

$$\tilde{r}^{(m)} := \lim_{n \rightarrow \infty} \hat{r}^{(m+n)p^n} \quad \text{for all } m.$$

Since p is nilpotent on A and $I := \ker(A \rightarrow R')$ has a DP-structure, one sees easily that $\tilde{r}^{(m)}$ is a lift of $g(r^{(m)})$ which is independent of the choices made. Define a map

$$\theta_A : W(P(R)) \rightarrow A$$

by sending (r_0, r_1, \dots) to $\sum_{i=0}^{\infty} p^i \tilde{r}_i^{(i)}$. Since p is nilpotent on A , these are just the usual Witt polynomials, so the map is indeed a homomorphism of rings. In the case $A = R$, the map θ_R is none other than the projection $(r_0, \dots) \mapsto r_0^{(0)}$ and in this way we obtain a commutative diagram

$$\begin{array}{ccc} W(P(R)) & \xrightarrow{\theta_A} & A \\ \downarrow & & \downarrow \\ R & \xrightarrow{g} & R'. \end{array}$$

We claim that the map θ_A is unique for the maps $W(P(R)) \rightarrow A$ making the above diagram commute. Indeed, any map $\alpha : W(P(R)) \rightarrow A$ is determined

by its values on $[r]$, where $[\cdot]$ denotes the Teichmüller lift. If $r = (r^{(n)})$, then write $r(m) := (r^{(m+n)})$ as the sequence “shifted” by m . So for all m we have

$$\alpha([r]) = \alpha([r(m)])^{p^m}$$

and $\alpha([r(m)]) = \theta_A([r(m)]) + a$ for some $a \in \ker(A \rightarrow R)$. If $p^m A = 0$, then

$$\begin{aligned} \alpha([r]) &= \alpha([r(m)])^{p^m} \\ &= (\theta_A([r(m)]) + a)^{p^m} \\ &= \sum_{i=0}^{p^m} \binom{p^m}{i} i! a^{[i]} \theta_A([r(m)])^{p^m-i} \\ &= \theta_A([r(m)])^{p^m} = \theta_A([r]) \end{aligned}$$

thus proving the uniqueness claim.

We may extend this map to a unique homomorphism of S -algebras

$$\theta_{A,S} : W(P(R)) \otimes_{\mathbb{Z}} S \rightarrow A$$

thereby obtaining a commutative diagram

$$\begin{array}{ccc} W(P(R)) \otimes_{\mathbb{Z}} S & \xrightarrow{\theta_{A,S}} & A \\ \downarrow & & \downarrow \\ R & \xrightarrow{g} & R'. \end{array}$$

Define an integral monoid $P(M)$ by

$$P(M) := \left\{ (m^{(n)}) \in M^{\mathbb{N}} : m^{(n+1)p} = m^{(n)}, \forall n \in \mathbb{N} \right\}.$$

The Teichmüller lift defines a map $P(M) \rightarrow W(P(R))$. By assumption, the natural map $P(M) \oplus N \rightarrow M$ induced from the projection $P(M) \rightarrow M$ sending $(m^{(n)})$ to $m^{(0)}$, is surjective. Let $Q := P(M) \oplus N$ and define

$$L := \ker(Q^{\text{gp}} \rightarrow M^{\text{gp}}).$$

Define

$$(W(P(R)) \otimes_{\mathbb{Z}} S)_{\log} := W(P(R)) \otimes_{\mathbb{Z}} S \otimes_{\mathbb{Z}} \mathbb{Z}[L].$$

Let us show how to extend the map $\theta_{A,S}$ to a unique map

$$\theta_{A,S}^{\log} : (W(P(R)) \otimes_{\mathbb{Z}} S)_{\log} \rightarrow A.$$

Let M_A (resp. $M_{R'}$) denote the log-structure of $\text{Spec}(A)$ (resp. $\text{Spec}(R')$). By Lemma 4.1 (ii) we have

$$M_{R'}(R') = M_A(A)/(1 + I)$$

where $I = \ker(A \rightarrow R')$. Define a map $P(M) \rightarrow M_A(A)$ by sending $(m^{(n)})_{n \in \mathbb{N}}$ to $\hat{m}^{(n)p^n}$, where $\hat{m}^{(n)}$ is a global section of M_A lifting the image of $m^{(n)}$ in $M_{R'}$ and n is any integer such that $p^n A = 0$. Because of the divided power structure of I it is easy to check that this section is independent of all choices. The map $P(M) \rightarrow M_A(A)$ extends uniquely to a map $\lambda_A : Q = P(M) \oplus N \rightarrow M_A(A)$. Now if $l \in L$, consider $\lambda_A(l) \in M_A(A)^{\text{gp}}$. Since the image of $\lambda_A(l)$ in $M_{R'}(R')^{\text{gp}}$ is the identity element, by exactness we deduce that $\lambda_A(l) \in A^*$. So λ_A extends uniquely to a map $L \rightarrow M_A(A)$. This defines the map

$$\theta_{A,S}^{\text{log}} : (W(P(R)) \otimes_{\mathbb{Z}} S)_{\text{log}} \rightarrow A.$$

In the special case $R' = R = A$ we see that the map $\theta_{R,S}^{\text{log}}$ is none other than induced by the "projection" map $\theta_{R,S}$ and the natural map $L \rightarrow M : l \mapsto 1$. It follows from the above that the map $\alpha = \theta_{A,S}^{\text{log}}$ is the unique map for which the diagram

$$\begin{CD} (W(P(R)) \otimes_{\mathbb{Z}} S)_{\text{log}} @>\alpha>> A \\ @V\theta_{R,S}^{\text{log}}VV @VVV \\ R @>>> R'. \end{CD}$$

commutes in a way compatible with the commutative diagram of monoids

$$\begin{CD} Q = P(M) \oplus N @>\lambda_A>> M_A(A) \\ @VVV @VVV \\ M @>>> M_{R'}(R'). \end{CD}$$

By the universal property of divided power hulls, $\theta_{A,S}^{\text{log}}$ factors over a homomorphism

$$(W(P(R)) \otimes_{\mathbb{Z}} S)_{\text{log}}^{\text{DP}} \rightarrow A$$

where $(W(P(R)) \otimes_{\mathbb{Z}} S)_{\text{log}}^{\text{DP}}$ is the divided power hull of $(W(P(R)) \otimes_{\mathbb{Z}} S)_{\text{log}}$ for the ideal $\ker(\theta_{R,S}^{\text{log}})$ defined above, compatible with $\gamma : J \rightarrow S$ (for the construction of divided power hulls see [2, Ch. I, §2.3] or [3, 3.19 Theorem]).

We can view $W(P(R))$ canonically as a $\mathbb{Z}[P(M)]$ -algebra via the Teichmüller lift of the map $P(M) \rightarrow P(R)$. Hence $W(P(R)) \otimes_{\mathbb{Z}} S$ is canonically a $\mathbb{Z}[Q] = \mathbb{Z}[P(M)] \otimes_{\mathbb{Z}} \mathbb{Z}[N]$ -algebra. Thus $(W(P(R)) \otimes_{\mathbb{Z}} S)_{\text{log}}$ is canonically a $\mathbb{Z}[Q \oplus L] = \mathbb{Z}[Q] \otimes_{\mathbb{Z}} \mathbb{Z}[L]$ -algebra, and so is $(W(P(R)) \otimes_{\mathbb{Z}} S)_{\text{log}}^{\text{DP}}$. Let $\overline{\ker(\theta_{R,S}^{\text{log}})}$ be the DP-ideal of $(W(P(R)) \otimes_{\mathbb{Z}} S)_{\text{log}}^{\text{DP}}$ generated by the divided powers of all elements of $\ker(\theta_{R,S}^{\text{log}})$, and $K \subset \overline{\ker(\theta_{R,S}^{\text{log}})}$ be the sub-DP-ideal

generated by the divided powers of all elements of the form $q_l \otimes l - p_l \otimes 1$, where $p_l, q_l \in Q$ and $l = p_l/q_l \in L$. Define

$$B := (W(P(R)) \otimes_{\mathbb{Z}} S)_{\log}^{\text{DP}} / K.$$

Note that by [2, Ch. I, Prop. 1.6.2] the ideal $\ker(B \rightarrow R)$ has a canonical DP-structure, compatible with $\gamma : J \rightarrow S$. Moreover, the map $\theta_{A,S}^{\log}$ factors uniquely over B , since the ideal K maps to zero in A . The closed immersion

$$\text{Spec}(R) \hookrightarrow \text{Spec}(B)$$

is our candidate for final object.

Step 2. — We define the log-structure on $\text{Spec}(B)$ as follows. Let QL be the submonoid of Q^{gp} consisting of elements of the form ql with $q \in Q$, $l \in L$. The map $Q \rightarrow W(P(R)) \otimes_{\mathbb{Z}} S \rightarrow B$ factors over a map $QL \rightarrow B$. We define the log-structure on B as that associated to $QL \rightarrow B$.

We claim that this log-structure makes the closed immersion

$$\text{Spec}(R) \hookrightarrow \text{Spec}(B)$$

exact. To verify this claim we use the criterion of Lemma 4.1 (i). This criterion is local for the étale topology, so let $\bar{x} \rightarrow \text{Spec}(R)$ be a geometric point, and $R_{\bar{x}}, B_{\bar{x}}$ the strict localizations of R and B respectively at \bar{x} . In this case the claim will follow from Lemma 4.3 if we can show:

- (a) the image of L in $((QL)^a)^{\text{gp}}$ lies in B^*
- (b) the map $(QL)^{\text{gp}} \cap B_{\bar{x}}^* \rightarrow M^{\text{gp}} \cap R_{\bar{x}}^*$ is surjective.

Since (a) is clear by construction, let us show (b). Suppose that $m = \frac{m_1}{m_2} \in M^{\text{gp}} \cap R_{\bar{x}}^*$ with $m_i \in M$ for $i = 1, 2$. Since the Frobenius is surjective on M , for $i = 1, 2$ we can choose a sequence

$$\underline{m}_i := (m_i = m_i^{(0)}, m_i^{(1)}, m_i^{(2)}, \dots, m_i^{(n)}, \dots)$$

of elements of M such that $m_i^{(n+1)p} = m_i^{(n)}$ for all $n \in \mathbb{N}$. If we define $m^{(n)} := \frac{m_1^{(n)}}{m_2^{(n)}}$, then we have $m^{(n)p^n} = m$ for all $n \in \mathbb{N}$. Since the log structure M^a is saturated, this implies that $m^{(n)} \in M_{\bar{x}}^a$ for all $n \in \mathbb{N}$. But then if $\alpha : M^a \rightarrow R$ is the structure morphism of the log structure, we have $\alpha(m^{(n)p^n} \in R_{\bar{x}}^*$, whence $\alpha(m^{(n)}) \in R_{\bar{x}}^*$, so $m^{(n)} \in R_{\bar{x}}^* \subset M_{\bar{x}}^a$ for all $n \in \mathbb{N}$. So we can naturally identify $\underline{m} := \underline{m}_1/\underline{m}_2$ with an element of $P(R_{\bar{x}})^*$ and, via the Teichmüller map, with an element of $W(P(R_{\bar{x}}))^*$. Now let $\theta_{B_{\bar{x}}} : W(P(R_{\bar{x}})) \rightarrow B_{\bar{x}}$ be the homomorphism constructed in Step 1. We claim that the map $P(M) \rightarrow B_{\bar{x}}$ factors $P(M) \rightarrow W(P(R_{\bar{x}})) \rightarrow B_{\bar{x}}$.

This follows from the commutative diagram

$$\begin{CD} W(P(R)) @>\theta_B>> B \\ @VVV @VVV \\ W(P(R_{\bar{x}})) @>\theta_{B_{\bar{x}}}>> B_{\bar{x}}. \end{CD}$$

So in $(QL)^a$ we have $\underline{m}_1 = \theta_{B_{\bar{x}}}([\underline{m}]) \cdot \underline{m}_2$, i.e. $\underline{m}_1/\underline{m}_2 \in P(M)^{\text{gp}} \cap B_{\bar{x}}^*$, which implies (b). Thus, the closed immersion $\text{Spec}(R) \hookrightarrow \text{Spec}(B)$ is exact, hence $\text{Spec}(R) \hookrightarrow \text{Spec}(B)$ is an object of the site $(R|S)_{\text{log-crys}}$.

Step 3. — We have shown that if $\text{Spec}(R') \hookrightarrow \text{Spec}(A)$ is an S -DP-thickening we have unique morphisms of $(R|S)_{\text{log-crys}}$

$$\begin{CD} (\text{Spec}(R') \hookrightarrow \text{Spec}(A)) \\ @VVV \\ (\text{Spec}(R) \hookrightarrow \text{Spec}(B)) \end{CD}$$

and hence for any object $U \hookrightarrow T$ and any affine covering $(U_\alpha \hookrightarrow T_\alpha)_\alpha \rightarrow (U \hookrightarrow T)$ we may define unique morphisms

$$(U_\alpha \hookrightarrow T_\alpha) \rightarrow (\text{Spec}(R) \hookrightarrow \text{Spec}(B))$$

and since the uniqueness allows us to glue, we are done. □

Now we sketch the second proof, cf. [5, §4-5].

Second proof (sketch). — Suppose $p^m S = 0$. For any S -DP-thickening $\text{Spec}(R) \hookrightarrow \text{Spec}(A)$, define a map

$$\theta_A : W_m(R) \rightarrow A$$

by sending (r_0, \dots, r_{m-1}) to $\sum_{i=0}^{m-1} p^i \hat{r}_i^{p^{m-i}}$, where \hat{r}_i denotes an arbitrary lift of $r_i \in R$ to A . Because of the divided power structure of $\ker(A \rightarrow R)$, this is a well-defined homomorphism of rings. In this way we obtain a commutative square, and to check the uniqueness of θ_A for such commutative squares, we first reduce to checking it for Teichmüller lifts and then use the fact that the Frobenius is surjective on R . Let $Q := M \oplus N$, $L = \ker(\lambda^{\text{gp}} : Q^{\text{gp}} \rightarrow M^{\text{gp}})$, where the map $\lambda : Q \rightarrow M$ is induced by the maps $M \rightarrow M : m \mapsto m^{p^n}$ and the structure map $N \rightarrow M$. We view $W_m(R) \otimes_{\mathbb{Z}} S$ as a $\mathbb{Z}[Q]$ -algebra via the map

$$M \oplus N \rightarrow W_m(R) \otimes_{\mathbb{Z}} S : (r, n) \mapsto \hat{r}^{p^m} \otimes n.$$

Let QL be the submonoid of Q^{gp} consisting of elements of the form ql with $q \in Q, l \in L$. Then the map λ extends uniquely to $\lambda' : QL \rightarrow M : L \ni$

$l \mapsto 1$. Set

$$(W_m(R) \otimes_{\mathbb{Z}} S)_{\log} := (W_m(R) \otimes_{\mathbb{Z}} S) \otimes_{\mathbb{Z}} \mathbb{Z}[L].$$

Take the divided power hull $(W_m(R) \otimes_{\mathbb{Z}} S)_{\log}^{\text{DP}}$ for the kernel of the unique surjection onto R extending the S -linearization $\theta_{R,S}$ of θ_R by λ' . Define

$$B = (W_m(R) \otimes_{\mathbb{Z}} S)_{\log}^{\text{DP}} / \mathcal{K}$$

where \mathcal{K} is the DP-ideal generated by the divided powers of $q_l \otimes l - p_l$, $l \in L$, where $q_l, p_l \in Q$ and $l = p_l/q_l \in L$. Then there is a map $QL \rightarrow B$ and we endow B with the log-structure associated to this map, thereby obtaining an object

$$\text{Spec}(R) \hookrightarrow \text{Spec}(B)$$

of the site $(R|S)_{\log\text{-crys}}$. Then we claim that this is the final object: one can verify the universal property as in the first proof. \square

1.2.2. Remarks.

- (a) Note that in the construction of the final object, we may replace the group L by any subgroup $L' \subset L$ such that for all $l \in L$ we have $l'u = l$ for some $l' \in L'$ and $u \in P(R)^*$ in the first proof (resp. $u \in R^*$ in the second proof), and obtain a ring B' . Then exactly the same argument as in Step 2 of the proof of Theorem 1.1 shows that the closed immersion

$$\text{Spec}(R) \hookrightarrow \text{Spec}(B')$$

is exact, hence defines an object of $(R|S)_{\log\text{-crys}}$, and it is easy to see that it is the final object.

- (b) The main advantage of the first proof is the uniformity of the construction. If $(S, J) \rightarrow (S/p^n S, J \cdot S/p^n S)$ is a DP-homomorphism, then it follows from [3, 3.20, Remark 8] that the final object of the crystalline site $(R|S/p^n S)_{\log\text{-crys}}$ is given by the $S/p^n S$ -DP-thickening

$$\text{Spec}(R) \hookrightarrow \text{Spec}(B) \otimes_{\mathbb{Z}} \mathbb{Z}/p^n \mathbb{Z}.$$

In particular, if S is a p -adically complete ring (we say that S is p -adic base), then as in [3] one may define a crystalline site $(R|S)_{\log\text{-crys}}$ whose cohomology automatically computes the derived projective limit of the cohomology of each $(R|S/p^n S)_{\log\text{-crys}}$. Then the proof given in the theorem also works for such S and shows that

$$\text{Spec}(R) \hookrightarrow \text{Spf}(\hat{B})$$

is the final object of the site $(R|S)_{\log\text{-crys}}$, where the hat denotes the p -adic completion.

- (c) The existence of the final object of the site $(R|S)_{\log\text{-crys}}$ implies that the cohomology of any sheaf is canonically isomorphic to the cohomology of its restriction to the étale site of the final object. In particular, the crystalline cohomology of any quasi-coherent \mathcal{O} -module vanishes in non-zero degree and we have

$$H^0_{\log\text{-crys}}(R|S, \mathcal{O}) = B.$$

In the case S is a p -adic base, using the previous remark we see that

$$\varprojlim_n H^0_{\log\text{-crys}}(R|S/p^{n+1}S, \mathcal{O}) = \hat{B}.$$

- (d) If one compares the 1st and 2nd proofs of Theorem 1.1, then it is not difficult to see that the canonical isomorphism

$$(W_m(P(R)) \otimes_{\mathbb{Z}} S)_{\log}^{\text{DP}}/K \cong (W_m(R) \otimes_{\mathbb{Z}} S)_{\log}^{\text{DP}}/\mathcal{K}$$

sends the image of an element $[x] \in W_n(P(R))$, where

$$x = (x^{(0)}, x^{(1)}, \dots) \in P(R),$$

to the image of the element $[x^{(m)}] \in W_m(R)$.

1.2.3.

Let R, M , and S be as in Theorem 1.1. Let $\text{Spec}(A)$ be a fine saturated $\log\text{-}S/J$ -scheme and let $h : \text{Spec}(R) \rightarrow \text{Spec}(A)$ be a homomorphism of logarithmic S/J -schemes. Denote also by h the associated morphism of \log -crystalline sites

$$h : (R|S)_{\log\text{-crys}} \rightarrow (A|S)_{\log\text{-crys}}.$$

Then for any sheaf of abelian groups \mathcal{F} on $(R|S)_{\log\text{-crys}}$, the i th direct image sheaf $R^i h_* \mathcal{F}$ is the sheaf on $(A|S)_{\log\text{-crys}}$ associated to the presheaf

$$(U \hookrightarrow T) \rightsquigarrow H^i_{\log\text{-crys}}(\text{Spec}(R) \times_{\text{Spec}(A)} U|T, \mathcal{F}).$$

If \mathcal{E} is a quasi-coherent sheaf of \mathcal{O} -modules on $(R|S)_{\log\text{-crys}}$ then it follows immediately from Theorem 1.1 that for $i \neq 0$

$$R^i h_* \mathcal{E} = 0$$

and hence for all i

$$H^i_{\log\text{-crys}}(R|S, \mathcal{E}) \cong H^i_{\log\text{-crys}}(A|S, h_* \mathcal{E})$$

which is again zero for $i \neq 0$.

PROPOSITION 1.2. — *With the above notation and assumptions, $h_*\mathcal{O}$ is a quasi-coherent crystal of \mathcal{O} -modules on $(A|S)_{\log\text{-crys}}$.*

Proof. — Assume $p^n S = 0$. Let \mathcal{P} be the presheaf on $(A|S)_{\log\text{-crys}}$ defined

$$\mathcal{P}(U \hookrightarrow T) := H_{\log\text{-crys}}^0(\mathrm{Spec}(R) \times_{\mathrm{Spec}(A)} U|T, \mathcal{O}).$$

Consider a morphism $g : (U' \hookrightarrow T') \rightarrow (U \hookrightarrow T)$ of $(A|S)_{\log\text{-crys}}$. Then $g : T' \rightarrow T$ is an open map, and hence

$$(g^{-1}\mathcal{P}|_T \otimes_{g^{-1}\mathcal{O}_T} \mathcal{O}_{T'}) (T') = \mathcal{P}(g(T')) \otimes_{\mathcal{O}_T(g(T))} \mathcal{O}_{T'}(T').$$

We claim that, up to localizing on U' , we have

$$g^{-1}\mathcal{P}|_T \otimes_{g^{-1}\mathcal{O}_T} \mathcal{O}_{T'} \cong \mathcal{P}|_{T'}.$$

We have a commutative square

$$\begin{array}{ccc} U' & \longrightarrow & g(U') \\ \downarrow & & \downarrow \\ T' & \longrightarrow & g(T') \end{array}$$

Let C be the unique flat $g(T')$ -scheme making the following square cartesian

$$\begin{array}{ccc} U' & \longrightarrow & g(U') \\ \downarrow & & \downarrow \\ C & \longrightarrow & g(T'). \end{array}$$

Then $C \rightarrow g(T')$ is étale and therefore the morphism $U' \hookrightarrow C$ has a unique structure of a DP thickening which is compatible with that of $g(U') \hookrightarrow g(T')$. Moreover, we have a commutative square

$$\begin{array}{ccc} U' & \longrightarrow & C \\ \downarrow & & \downarrow \\ T' & \longrightarrow & g(T'). \end{array}$$

Since the left vertical arrow is a nilpotent thickening, there exists a unique morphism $T' \rightarrow C$, necessarily a DP-morphism, making the resulting diagram commute. Hence we have a commutative diagram

$$\begin{array}{ccccc} U' & \xlongequal{\quad} & U' & \longrightarrow & g(U') \\ \downarrow & & \downarrow & & \downarrow \\ T' & \longrightarrow & C & \longrightarrow & g(T') \end{array}$$

and so we reduce to proving the claim in the following two cases

- I. $g : T' \rightarrow g(T')$ is étale

II. $g : U' \rightarrow g(U')$ is an isomorphism.

In case I, we must show the following: if $(U' \hookrightarrow T') \rightarrow (U \hookrightarrow T)$ is an étale map with U and U' affine and $U \times_T T' \cong U'$, then the canonical map

$$(1.1) \quad \mathcal{P}(U \hookrightarrow T) \otimes_{\mathcal{O}(T)} \mathcal{O}(T') \rightarrow \mathcal{P}(U' \hookrightarrow T')$$

is an isomorphism. I thank the referee for the following argument (my original argument was a direct verification). To simplify, let $\mathcal{P} = \mathcal{P}(U \hookrightarrow T)$. Note that by Theorem 1.1, there is a closed immersion $(\mathrm{Spec}(R) \hookrightarrow \mathrm{Spec}(\mathcal{P}))$ which is in fact the final object of the site $(\mathrm{Spec}(R) \times_{\mathrm{Spec}(A)} U|T)_{\mathrm{log-crys}}$. It suffices to show that the closed immersion $X := (\mathrm{Spec}(R) \times_{\mathrm{Spec}(S)} U' \hookrightarrow \mathrm{Spec}(\mathcal{P}) \times_T T')$ is the final object of the site $(\mathrm{Spec}(R) \times_{\mathrm{Spec}(S)} U'|T')_{\mathrm{log-crys}}$, where $\mathrm{Spec}(R) \times_{\mathrm{Spec}(S)} U'$ is endowed with the inverse image log-structure under the structure morphism to $\mathrm{Spec}(R)$ and T' the inverse image log-structure under the structure morphism to T . We endow $\mathrm{Spec}(\mathcal{P}) \times_T T'$ with the inverse image log-structure under the structure morphism to $\mathrm{Spec}(\mathcal{P})$. Note that by flatness of $T' \rightarrow T$, the divided power structure of the closed immersion $\mathrm{Spec}(R) \times_{\mathrm{Spec}(A)} T \hookrightarrow \mathrm{Spec}(\mathcal{P})$ extends uniquely to X , so that X is indeed an object of the site in question. Since any object Y of this site can be naturally considered as an object of the site $(\mathrm{Spec}(R) \times_{\mathrm{Spec}(A)} U|T)_{\mathrm{log-crys}}$, there is a unique morphism $Y \rightarrow (\mathrm{Spec}(R) \hookrightarrow \mathrm{Spec}(\mathcal{P}))$. This morphism factors uniquely $Y \rightarrow X \rightarrow (\mathrm{Spec}(R) \hookrightarrow \mathrm{Spec}(\mathcal{P}))$, thus proving that X is the final object.

In case II, we can assume U' is affine, and hence so are T' and $g(T')$. Let $U' = \mathrm{Spec}(E_0)$, $g(T') = \mathrm{Spec}(E)$ and $T' = \mathrm{Spec}(F)$. Let $M \rightarrow \mathcal{O}_{U'}$ be the log-structure of U' . Up to localizing further we may assume that, in a neighbourhood of a geometric point $\bar{x} \rightarrow U'$, the log-structure of U' is given by the fine saturated monoid $\Lambda := M_{\bar{x}} / \mathcal{O}_{U', \bar{x}}^*$. Since $U' \hookrightarrow g(T')$ is an exact closed immersion, it follows from Lemma 4.1 (iii) that there is a map $\Lambda \rightarrow E$ defining the log-structure of E , and moreover the map $\Lambda \rightarrow E \rightarrow F$ also defines the log-structure of F . We will construct $g^{-1}\mathcal{P}|_T \otimes_{g^{-1}\mathcal{O}_T} \mathcal{O}_{T'}$ and $\mathcal{P}|_{T'}$ as in the proof of Theorem 1.1 and prove that they are canonically isomorphic. Let $Q := P(M) \oplus \Lambda$. As in the proof of Theorem 1.1, there is a canonical map $Q \rightarrow M$. Define

$$L := \ker(Q^{\mathrm{gp}} \rightarrow M^{\mathrm{gp}}).$$

Let J_F (resp. J_E) denote the ideal sheaf of U' in T' (resp. $g(U')$ in $g(T')$). Define the pairs

$$\begin{aligned} (B_E, I_E) &= \\ &((W(P(R \otimes_A E_0)) \otimes_{\mathbb{Z}} E) \otimes_{\mathbb{Z}} \mathbb{Z}[L], J_E \cdot (W(P(R \otimes_A E_0)) \otimes_{\mathbb{Z}} E) \otimes_{\mathbb{Z}} \mathbb{Z}[L]) \\ (B_F, I_F) &= \\ &((W(P(R \otimes_A E_0)) \otimes_{\mathbb{Z}} F) \otimes_{\mathbb{Z}} \mathbb{Z}[L], J_F \cdot (W(P(R \otimes_A E_0)) \otimes_{\mathbb{Z}} F) \otimes_{\mathbb{Z}} \mathbb{Z}[L]). \end{aligned}$$

Note that since the ring of Witt vectors of a perfect ring of characteristic p is \mathbb{Z} -torsion free, hence flat over \mathbb{Z} , it follows that $E \rightarrow B_E$ (resp. $F \rightarrow B_F$) is flat. Hence the ideal I_E (resp. I_F) is a DP-ideal. Since $(E, J_E) \rightarrow (F, J_F)$ is a DP-morphism, so is the canonical map $(B_E, I_E) \rightarrow (B_F, I_F)$. Moreover we have

$$B_E/I_E \cong B_F/I_F.$$

Define

$$\begin{aligned} \mathcal{J}_E &= \ker(B_E \rightarrow R \otimes_A E_0) \\ \mathcal{J}_F &= \ker(B_F \rightarrow R \otimes_A E_0) \end{aligned}$$

for the canonical maps. Then $I_E \subset \mathcal{J}_E$ and $I_F \subset \mathcal{J}_F$, and via the map $B_E \rightarrow B_F$, \mathcal{J}_E maps to \mathcal{J}_F , and by definition we have $B_E/\mathcal{J}_E \cong B_F/\mathcal{J}_F$. So by [2, Ch. I, Prop. 2.8.2], we have a canonical isomorphism

$$D_{B_F}(\mathcal{J}_F) \cong D_{B_E}(\mathcal{J}_E) \otimes_{B_E} B_F = D_{B_E}(\mathcal{J}_E) \otimes_E F$$

where $D_{B_F}(\mathcal{J}_F)$ (resp. $D_{B_E}(\mathcal{J}_E)$) denotes the divided power hull of B_F (resp. B_E) for the ideal \mathcal{J}_F (resp. \mathcal{J}_E) compatible with the divided power structure on J_F (resp. J_E). Let $K \subset D_{B_E}(\mathcal{J}_E)$ be the DP-ideal generated by the divided powers of all elements of the form $q_l \otimes l - p_l \otimes 1$ as in Theorem 1.1. Then $\text{Im}(K \otimes_E F) \subset D_{B_F}(\mathcal{J}_F)$ is the ideal generated by the divided powers of all elements of the form $q_l \otimes l - p_l \otimes 1$ (where $l = p_l/q_l \in L \subset Q^{\text{gp}}$), hence in particular is a DP-ideal. Taking the quotient by K we obtain an isomorphism

$$(D_{B_E}(\mathcal{J}_E)/K) \otimes_E F \cong D_{B_F}(\mathcal{J}_F)/\text{Im}(K \otimes_E F)$$

which is precisely

$$g^{-1}\mathcal{P}|_T \otimes_{g^{-1}\mathcal{O}_T} \mathcal{O}_{T'} \cong \mathcal{P}|_{T'}$$

and hence we have shown the claim. Sheafifying we see that

$$h_*\mathcal{O}|_{T'} \cong g^*(h_*\mathcal{O}|_T)$$

so $h_*\mathcal{O}$ is a crystal of \mathcal{O} -modules. Finally, the quasi-coherence of \mathcal{P} follows from the fact that the map (1.1) is an isomorphism. □

1.3. Some Fontaine rings

1.3.1.

Let K^+ be a complete discrete valuation ring of characteristic zero and of perfect residue field k of characteristic $p > 0$. Denote by K its fraction field. Let \bar{K} be an algebraic closure of K and let \bar{K}^+ be its valuation ring. Following Fontaine we set

$$A_{\text{cris}} := A_{\text{cris}}(K^+) := \varprojlim_n H_{\text{crys}}^0(\bar{K}^+/p\bar{K}^+ | W_{n+1}(k), \mathcal{O}).$$

This ring may be constructed as in the proof of Thm. 1.1. Let $P(\bar{K}^+/p\bar{K}^+)$ be as in the proof of *loc. cit.* Consider the element

$$\underline{1} := (1, \zeta_p, \zeta_{p^2}, \dots) \in P(\bar{K}^+/p\bar{K}^+)$$

where ζ_{p^n} denotes a primitive p^n th root of unity and $\zeta_{p^{n+1}}^p = \zeta_{p^n}$ for all n . Define

$$A_{\text{inf}}(K^+) := W(P(\bar{K}^+/p\bar{K}^+))$$

so that $[\underline{1}] \in A_{\text{inf}}(K^+)$. Define an element of A_{cris} by

$$t := \log([\underline{1}]) = - \sum_{n>0} (n-1)! (1 - [\underline{1}])^{[n]}.$$

This element plays a very important role in p -adic Hodge theory. By functoriality of crystalline cohomology, A_{cris} has a Frobenius endomorphism Φ .

LEMMA 1.3. — $A_{\text{cris}}(K^+)$ is a flat $W(k)$ -algebra.

Proof. — Let $P := P(\bar{K}^+/p\bar{K}^+)$. By [8, 5.1.2] P is a valuation ring, in particular an integral domain. Since P is perfect, it follows that $W(P)$ is a flat and p -adically separated $W(k)$ -algebra, hence is also an integral domain. As is well-known, $A_{\text{cris}}(K^+)$ is the p -adic completion of $W(P)^{\text{DP}} =$ the divided power hull of $W(P)$ for the ideal (ξ) , where $\xi = [\underline{p}] - p$ and

$$\underline{p} = (p, p^{1/p}, p^{1/p^2}, \dots) \in P(\bar{K}^+/p\bar{K}^+)$$

is a compatible system of p -power roots of p in $\bar{K}^+/p\bar{K}^+$. So it suffices to show that $W(P)^{\text{DP}}$ is p -torsion free. Now by construction of divided power hulls, we have an exact sequence

$$0 \rightarrow \mathcal{I} \rightarrow \Gamma_{W(P)}((\xi)) \rightarrow W(P)^{\text{DP}} \rightarrow 0$$

where $\Gamma_{W(P)}((\xi)) = W(P)\langle X \rangle$ is the divided power polynomial algebra on one variable X and \mathcal{I} is the principal ideal generated by $\xi - X$. Since $\Gamma_{W(P)}((\xi))$ is flat over $W(k)$, it suffices to show that the map $\mathcal{I} \otimes_{W(k)}$

$W_n(k) \rightarrow \Gamma_{W(P)}((\xi)) \otimes_{W(k)} W_n(k)$ is injective for all n . Suppose $(\xi - X)f \equiv 0 \pmod{p^n}$ for some $f \in \Gamma_{W(P)}((\xi))$. Then $(\xi - X)f = p^n g$ for some $g \in \Gamma_{W(P)}((\xi))$. Write $f = \sum_{i \geq 0} f_i X^{[i]}$ and $g = \sum_{i \geq 0} g_i X^{[i]}$ for some uniquely determined $f_i, g_i \in W(P)$. Then we have $\xi f_i - (i + 1)f_{i-1} = p^n g_i$ for all i . In particular, for $i = 0$ we get $\xi f_0 = p^n g_0$. Since ξ generates a prime ideal which does not contain p , the identity $\xi f_0 = p^n g_0$ implies that $g_0 \in (\xi)$, whence $f_0 \equiv 0 \pmod{p^n}$ since $W(P)$ is an integral domain. For $i = 1$ we have $\xi f_1 - 2p^n f'_0 = p^n g_1$, hence again by the same argument $f_1 = p^n f'_1$. Continuing in the manner we see that $f = p^n f'$, which implies the claim. \square

1.3.2.

Because of the lack of a lifting of the Frobenius endomorphism of K^+/pK^+ to $K^+/p^n K^+$, we will need to consider a formal divided power lifting Σ of K^+/pK^+ , defined as follows. Firstly, making a choice of uniformizer π of K^+ determines a presentation $K^+ = W(k)[u]/(E(u))$, where $E(u)$ is the minimal equation of π over $W(k)$, i.e. $E(u)$ is an Eisenstein equation of degree e , where e is the ramification index of K^+ over $W(k)$. So $W_n(k)[u]$ is a smooth $W_n(k)$ -lift of $K^+/pK^+ = k[u]/(u^e)$. Taking the divided power hull for the kernel of the surjection $W_n(k)[u] \rightarrow K^+/pK^+$, we obtain $\Sigma_n = W_n(k)[u] \langle u^e \rangle$. It has a lifting of the absolute Frobenius of K^+/pK^+ defined as the unique homomorphism sending u to u^p and restricting to the canonical Frobenius on $W_n(k)$. Finally we define

$$\Sigma := \lim_n \Sigma_{n+1}.$$

1.3.3.

Let us consider the log-crystalline cohomology of $\bar{K}^+/p\bar{K}^+$ over the log-arithmic base Σ_n , where the log-structures are as follows. We endow Σ_n with the *canonical log-structure* on Σ_n , i.e. the fine log-structure associated to the pre-log-structure

$$\mathcal{L}(u) : \mathbb{N} \rightarrow \Sigma_n : 1 \mapsto u.$$

Composing this with the canonical map $\Sigma_n \rightarrow K^+/p^n K^+$ defines a pre-log-structure on the latter, making it into a log- Σ_n -scheme. The associated log-structure is also the inverse image of the canonical log-structure on K^+ defined

$$K^+ - \{0\} \rightarrow K^+.$$

Similarly, the canonical log-structure on \bar{K}^+ is given by

$$\bar{K}^+ - \{0\} \rightarrow \bar{K}^+$$

and we endow $\bar{K}^+/p\bar{K}^+$ with the inverse image log-structure. If we fix roots of π , then it is the log-structure associated to the pre-log-structure

$$M : \mathbb{Q}_+ \rightarrow \bar{K}^+/p\bar{K}^+ : \alpha \mapsto \pi^\alpha.$$

(Note that although this pre-log-structure depends on choices of roots of π , its associated log-structure does not, as two choices of roots of π will differ by a unit of $\bar{K}^+/p\bar{K}^+$.) Let

$$L := \ker(P(M)^{\text{gp}} \oplus \mathcal{L}(u)^{\text{gp}} \rightarrow M^{\text{gp}}).$$

Since $P(M)$ consists of sequences (x_n) of non-negative rational numbers such that $p \cdot x_{n+1} = x_n$ for all n , we see that $P(M)^{\text{gp}}$ consists of sequences (x_n) of rational numbers such that $p \cdot x_{n+1} = x_n$ for all n , i.e. $P(M)^{\text{gp}} \cong \mathbb{Q}$. So we see that L is the kernel of the map

$$\mathbb{Q} \oplus \mathbb{Z} \rightarrow \mathbb{Q} : (\alpha, m) \mapsto \alpha + m$$

i.e. L consists of pairs $(m, -m) \in \mathbb{Z}^2$. Note that under the map to

$$A_{\text{inf}}(K^+) \otimes_{\mathbb{Z}} \Sigma, (m \cdot p^{-n})_n \in P(M)$$

maps to $[\underline{\pi}]^m \otimes 1$, where $\underline{\pi} = (\pi, \pi^{1/p}, \pi^{1/p^2}, \dots) \in P(\bar{K}^+/p\bar{K}^+)$ is the sequence of p -power roots of π determined by M , and $m \in \mathcal{L}(u)$ maps to $1 \otimes u^m$. So one should think of $(m, -m)$ as $[\underline{\pi}]^m \otimes u^{-m}$.

Define

$$B_{\text{log}}^+ := \lim_n H_{\text{log-crys}}^0(\bar{K}^+/p\bar{K}^+ | \Sigma_n, \mathcal{O}).$$

By construction (cf. proof of Thm. 1.1), B_{log}^+ is a quotient of the p -adic completion of the divided power hull of $A_{\text{inf}}(K^+) \otimes_{\mathbb{Z}} \Sigma \otimes_{\mathbb{Z}} \mathbb{Z}[L]$ for the kernel of the canonical surjection onto \hat{K}^+ . In fact, by [13, Prop. 3.3] we know that every choice of sequence of p -power roots of π determines an isomorphism

$$B_{\text{log}}^+/p^n B_{\text{log}}^+ \simeq A_{\text{cris}}/p^n A_{\text{cris}} \langle X \rangle$$

where $A_{\text{cris}}/p^n A_{\text{cris}} \langle X \rangle$ is a divided power polynomial ring in one indeterminate X over the ring $A_{\text{cris}}/p^n A_{\text{cris}}$. One should think of X as $[\underline{\pi}] \otimes u^{-1} - 1$. B_{log}^+ is also endowed with a Frobenius endomorphism.

Now, let us fix a sequence $(c_0 = \pi, c_1, c_2, \dots)$ of p -power roots of π in \bar{K}^+ such that $c_{n+1}^p = c_n$ for all $n \in \mathbb{N}$. Consider the log-structure associated to

$$\mathbb{N}[1/p] \rightarrow \bar{K}^+/p\bar{K}^+ : \frac{n}{p^m} \mapsto c_m^n.$$

If we define

$$L' := \ker (P(\mathbb{N}[1/p])^{\text{gp}} \oplus \mathcal{L}(u)^{\text{gp}} \rightarrow \mathbb{N}[1/p]^{\text{gp}})$$

then $L' = L$ if the p -power roots of π which we fixed in order to define M are the same as (c_0, c_1, \dots) . In fact, the log-crystalline cohomology of $\bar{K}^+ / p\bar{K}^+$ is the same with this log-structure as with M (this can be checked as in the proof of Prop. 3.3). So in the sequel we will use this latter log-structure, rather than the canonical one (cf. §2.2.5).

1.3.4.

We now make a computation, due to Fontaine, which will be crucial to us.

PROPOSITION 1.4 (Fontaine). — *Let $\alpha \in \mathbb{Q}$. We have*

- (i) $t^{p-1} \in p \cdot A_{\text{cris}}(K^+)$
- (ii) if $\alpha \in \mathbb{Z}$ then $[\mathbb{1}]^\alpha - 1 = \alpha \cdot t \cdot u_\alpha$, where u_α is a unit of $A_{\text{cris}}(K^+)$
- (iii) $\frac{t \cdot p^{\max\{v_p(\alpha), 0\}}}{[\mathbb{1}]^{\alpha-1}} \in A_{\text{cris}}(K^+)$.

Proof. —

(i). — Clearly $\zeta = \frac{1 - [\mathbb{1}]}{1 - [\mathbb{1}]^{1/p}}$ lies in the kernel of $A_{\text{inf}}(K^+) \rightarrow \hat{K}^+$. We find $\zeta^p \equiv (1 - [\mathbb{1}])^{p-1} \pmod{p}$, hence $(1 - [\mathbb{1}])^{p-1} \in p \cdot A_{\text{cris}}(K^+)$. Now, we have

$$t = - \sum_{n=1}^p (n-1)! (1 - [\mathbb{1}])^{[n]} - \sum_{n=p+1}^\infty (n-1)! (1 - [\mathbb{1}])^{[n]}$$

and the second sum is divisible by p , so it suffices to consider

$$\left(\sum_{n=1}^p (n-1)! (1 - [\mathbb{1}])^{[n]} \right)^{p-1} = \sum_{j_1 + \dots + j_p = p-1} \frac{(p-1)! (1 - [\mathbb{1}])^{\sum i \cdot j_i}}{j_1! \cdots j_p! \prod_{i=1}^p i^{j_i}}.$$

Here each summand has p -adic valuation j_p in the denominator and at least $\left\lceil \frac{\sum i \cdot j_i}{p-1} \right\rceil \geq j_p + 1$ in the numerator. This proves (i).

(ii). — $[\mathbb{1}]^\alpha - 1 \in \ker(A_{\text{cris}}(K^+) \rightarrow \hat{K}^+)$, so the divided power series $\log([\mathbb{1}]^\alpha)$ exists and converges to $\alpha \cdot t$. We have

$$[\mathbb{1}]^\alpha - 1 = \exp(\alpha \cdot t) - 1 = \alpha \cdot t \cdot \sum_{n>0} \frac{(\alpha \cdot t)^{n-1}}{n!}$$

and

$$\frac{t^{n-1}}{n!} = \frac{p^{q_n}}{n!} q_n! (t^{p-1}/p)^{[q_n]} t^{r_n}$$

where $n - 1 = q_n(p - 1) + r_n$, $0 \leq r_n < p - 1$, and $q_n = \left\lfloor \frac{n - 1}{p - 1} \right\rfloor \geq v_p(n!)$.

So $u_\alpha := \sum_{n>0} \frac{(\alpha \cdot t)^{n-1}}{n!}$ converges to a unit of $A_{\text{cris}}(K^+)$ for all $\alpha \in \mathbb{Z}$.

Hence assertion (ii).

(iii). — We separate in two cases:

I. $v_p(\alpha) \geq 0$

II. $v_p(\alpha) < 0$.

In case I, if $\alpha = \frac{x}{y}$ with x, y integers and $v_p(y) = 0$, then $\frac{[1]^{x-1}}{[1]^{\alpha-1}} \in A_{\text{cris}}(K^+)$, so assertion (iii) holds in this case by (ii).

In case II, write $\alpha = \frac{x}{yp^n}$ with x, y coprime integers and $v_p(x/y) = 0$. By case I we have $([1]^x - 1)^{-1} \in t^{-1} \cdot A_{\text{cris}}(K^+)$, and $\frac{[1]^{x-1}}{[1]^{\alpha-1}} \in A_{\text{cris}}(K^+)$, hence assertion (iii) in this case. \square

2. Almost ring theory

In this section we recall some results in almost ring theory and then apply them to Fontaine rings.

2.1. Reminder on almost ring theory

2.1.1.

We refer to [9] for the fundamentals of almost ring theory, in particular the categories of almost modules and almost rings. We will always refer to morphisms in these almost categories with the adjective “almost”, e.g. “almost homomorphism”, etc. So if we do not use the adjective “almost” we will be referring to usual morphisms. Finally, we reserve the notation $M \approx N$ for almost isomorphisms.

The almost ring theory we will consider will be relative to a pair (Λ, \mathfrak{m}) where:

- Λ is a rank 1 valuation ring of characteristic zero
- $\mathfrak{m} \subset \Lambda$ is its maximal ideal, such that $p \in \mathfrak{m}$
- $\mathfrak{m}^2 = \mathfrak{m}$.

Note that since Λ is a valuation ring, \mathfrak{m} is the filtered union of principal ideals, the index set for the limit being the set of valuations of elements of \mathfrak{m} .

2.1.2.

- LEMMA 2.1. — (i) *Almost étale coverings are stable by base change.*
 (ii) *If $A \rightarrow B \rightarrow C$ are homomorphisms such that $A \rightarrow B$ and the composition $B \rightarrow C$ are almost étale coverings, then so is $B \rightarrow C$.*
 (iii) *An almost projective module of finite rank which is everywhere of non-zero rank is almost faithfully flat. In particular, an almost étale covering which is everywhere of non-zero rank is almost faithfully flat.*

Proof. — This follows in a straightforward way from the results of [9]. □

The behaviour of almost étale morphisms of \mathbb{F}_p -algebras under the Frobenius endomorphism is studied in some detail in [9].

THEOREM 2.2 (Gabber-Ramero). — *Let $A \rightarrow B$ be an almost étale homomorphism of $\Lambda/p\Lambda$ -algebras. Then the commutative diagram*

$$\begin{array}{ccc} A & \longrightarrow & B \\ F \downarrow & & F \downarrow \\ A & \longrightarrow & B \end{array}$$

is almost cocartesian, where F denotes the (absolute) Frobenius endomorphism. In other words, the relative Frobenius of B over A is an almost isomorphism.

This is [9, Thm. 3.5.13]. We will need the following corollaries.

In the next corollary, we consider the almost ring theory relative to the pair $(W(\Lambda/p\Lambda), [\mathfrak{m}])$, where the ideal $[\mathfrak{m}]$ is the ideal generated by $[\bar{x}]$ for all $x \in \mathfrak{m}$, where $\bar{x} \in \Lambda/p\Lambda$ denotes the image of $x \in \Lambda$, and $[-]$ is the Teichmüller map. We claim that $[\mathfrak{m}]$ is the filtered union of principal ideals and $[\mathfrak{m}]^2 = [\mathfrak{m}]$. Since Λ is a valuation ring and $[\underline{x}]$ is nilpotent for all $x \in \mathfrak{m}$, one sees easily that every ideal generated by a finite number of generators of $[\mathfrak{m}]$ is principal. From this the first claim follows easily. For the second, suppose that $x = \sum_{i=0}^s [\bar{x}_i]a_i \in [\mathfrak{m}]$. Fix some elements $x_i \in \mathfrak{m}$ such that $\bar{x}_i \equiv x_i \pmod p$. Without loss of generality, we may assume that x_0 has smallest valuation of all x_i . Then $x_i = x_0 y_i$ for some $y_i \in \Lambda$, so $x = [\bar{x}_0](a_0 + \sum_{i>0} [\bar{y}_i]a_i)$. Now, since $\mathfrak{m}^2 = \mathfrak{m}$, it follows that $x_0 = x_{-1}x_{-2}$ for some $x_{-1}, x_{-2} \in \mathfrak{m}$. So $x = [\bar{x}_{-1}] \cdot [\bar{x}_{-2}](a_0 + \sum_{i>0} [\bar{y}_i]a_i) \in [\mathfrak{m}]^2$, which proves the claim.

COROLLARY 2.3. — *If $A \rightarrow B$ is an almost étale homomorphism of $\Lambda/p\Lambda$ -algebras, then the induced homomorphism of Witt vectors*

$$W_n(A) \rightarrow W_n(B)$$

is an almost étale homomorphism of $W(\Lambda/p\Lambda)$ -algebras.

Proof. — The proof is the same as the classical case [11, 0. Prop. 1.5.8]. We reproduce it here for the reader. We prove the assertion by induction on n , the case $n = 1$ being Theorem 2.2. Suppose that the result is true for $n - 1$. Let V and F be the Verschiebung and Frobenius morphisms of the functor $W_n(-)$. For any \mathbb{F}_p -algebra R , $V^i W_{n-i}(R)$ defines a filtration of $W_n(R)$, called the V -adic filtration, whose graded has a natural graded ring structure. We may consider R as a module over itself via the i th power F^i of the Frobenius endomorphism $F : R \rightarrow R$. We denote it $F_*^i R$. Then according to *loc. cit.* there is a canonical isomorphism $F_*^i R \cong \mathrm{gr}_V^i W_n(R) := V^i W_{n-i}(R)/V^{i+1} W_{n-(i+1)}(R)$. Now the natural map $B \otimes_A \mathrm{gr}_V^i W_n(A) \rightarrow \mathrm{gr}_V^i W_n(B)$ can be identified with the map $B \otimes_A F_*^i A \rightarrow F_*^i B$. It follows from Theorem 2.2 that it is an almost isomorphism. Hence so is the map $B \otimes_A \mathrm{gr}_V^i W_n(A) \rightarrow \mathrm{gr}_V^i W_n(B)$. Since $V^n = 0$, it follows that the natural homomorphism $V^{n-1}(A) \otimes_{W_n(A)} W_n(B) \rightarrow V^{n-1}(B)$ is an almost isomorphism. Writing $I := V^{n-1}(A) \subset W_n(A)$, we get that the map $I \otimes_{W_n(A)} W_n(B) \rightarrow I \cdot W_n(B)$ is an almost isomorphism and $W_n(B) \otimes_{W_n(A)} W_n(A)/I \approx W_n(B)/V^{n-1}(B) = W_{n-1}(B)$ is almost flat over $W_{n-1}(A)$ by inductive hypothesis. By the next lemma this implies that $W_n(A) \rightarrow W_n(B)$ is almost flat. It follows from [9, proof of Thm. 3.2.18 (ii)] that $W_n(A) \rightarrow W_n(B)$ must be the unique almost étale lift of $W_{n-1}(A) \rightarrow W_{n-1}(B)$. \square

In the proof we have used the following almost analogue of Bourbaki's local flatness criterion.

LEMMA 2.4. — *Suppose that A is a $W_n(\Lambda/p\Lambda)$ -algebra, $I \subset A$ a nilpotent ideal, and M an A -module. If the natural map $I \otimes_A M \rightarrow IM$ is an almost isomorphism and M/IM is almost flat over A/I , then M is an almost flat A -module.*

Proof. — We refer to [9, 2.4.10] for details on Tor-functors and their relation to flatness in the almost setting. It suffices to show that $0 \approx \mathrm{Tor}_i^A(M, N)$ for $i > 0$ and any A -module N , and since any module can be written as a quotient of a free module this reduces to showing that $\mathrm{Tor}_1^A(M, N) \approx 0$. Firstly, it follows from the hypothesis that $\mathrm{Tor}_1^A(M, A/I) \approx 0$. Since we may write any A/I -module as a quotient of a free module, it

follows easily that $\text{Tor}_1^A(M, N) \approx 0$ for any A -module N such that $IN = 0$. For general N an easy induction on m such that $I^m N = 0$ using the exact sequences

$$0 \rightarrow IN \rightarrow N \rightarrow N/IN \rightarrow 0$$

shows that $\text{Tor}_1^A(M, N) \approx 0$. □

COROLLARY 2.5. — *Assume that there is a sequence $x_n \in \mathfrak{m}$ such that $x_{n+1}^p = x_n$ for all $n \in \mathbb{N}$. Suppose that $A \rightarrow B$ is an almost étale homomorphism of Λ -algebras. Suppose B is a flat Λ -algebra which is integrally closed in $B \otimes_{\Lambda} Q(\Lambda)$. If the Frobenius is surjective on A/pA , then the Frobenius is also surjective on B/pB .*

Proof (Faltings). — Let v be the valuation of Λ and $\Gamma := v(\Lambda)$. By Theorem 2.2 we know that the Frobenius is almost surjective on B/pB . So for all $b \in B$ and all $\alpha \in \Gamma$, $0 < \alpha < v(p)$, there is n such that $v(x_0)/p^n \leq \alpha$ and such that we may write $x_n b = c^p + pd$. Rewriting this as $c^p = x_n(b - \frac{p}{x_n}d)$, we have $c^p = x_{n+1}^p(b - \frac{p}{x_n}d)$. Since B is integrally closed in $B \otimes_{\Lambda} Q(\Lambda)$, we have $c = x_{n+1}e$ for some $e \in B$, so $b = e^p + \frac{p}{x_n}d$. Similarly, $\frac{p}{x_n}d = f^p + pg$, hence $b = e^p + f^p + pg \equiv (e + f)^p \pmod{p}$. □

2.1.3.

If R is a ring and X is a finite set, then we write $R \times X$ for the product $\prod_{x \in X} R_x$, where $R_x = R$ for all $x \in X$. We say that an almost étale homomorphism $A \rightarrow B$ of Λ -algebras is an *almost Galois homomorphism* of group G if G is a finite group acting by A -algebra automorphisms on B such that the canonical map

$$B \otimes_A B \rightarrow B \times G$$

induced by the maps $B \rightarrow B \times G : b \mapsto (b, b, \dots, b)$ and $B \rightarrow B \times G : b \mapsto (g(b))_{g \in G}$, is an almost isomorphism.

We say that an almost Galois homomorphism $A \rightarrow B$ of Λ -algebras is an *almost Galois covering* if it is almost faithfully flat.

Note that an almost Galois homomorphism (resp. covering) is preserved under arbitrary base change.

PROPOSITION 2.6. — *Let $A \rightarrow B$ be an almost Galois covering of group G . Let M a B -module with semi-linear G -action. Then*

- (i) $B^G \approx A$
- (ii) $M^G \otimes_A B \approx M$

(iii) $H^i(G, M) \approx 0$ for all $i \neq 0$.

Proof. — See [7]. □

2.1.4.

Let A be a Λ -algebra and $I \subset A$ an ideal such that $I^2 = 0$.

THEOREM 2.7 (Faltings). — *The category of almost étale coverings of A/I is equivalent to the category of almost étale coverings of A .*

For the proof, see [7, 3. Theorem]. More generally, we have ([9, Thms. 3.2.18, 3.2.28])

THEOREM 2.8 (Gabber-Ramero). — *The category of almost étale homomorphisms of A/I is equivalent to the category of almost étale homomorphisms of A .*

PROPOSITION 2.9. — *The equivalence in Theorem 2.8 preserves almost faithful flatness.*

Proof. — Let $A \rightarrow B$ be an almost étale covering such that $A/I \rightarrow B/I \cdot B$ is almost faithfully flat. Let M be an A -module such that $M \otimes_A B \approx 0$. Then $(M/IM) \otimes_{A/IA} (B/IB) \approx 0$, hence $M/I \cdot M \approx 0$, whence $M \approx 0$. □

COROLLARY 2.10. — *If $A \rightarrow B$ is an almost étale homomorphism and G is a finite group of A -automorphisms of B , then it is an almost Galois homomorphism (resp. covering) of group G if and only if $A/I \rightarrow B/I \cdot B$ is an almost Galois homomorphism (resp. covering) of group G .*

Proof. — Since almost faithful flatness is preserved under the equivalence, it suffices to check that almost Galois homomorphisms are preserved. It suffices to prove that the map $B \otimes_A B \rightarrow B \times G$ is almost flat for then it must be almost étale (see the proof of [9, Thm. 3.2.18 (ii)]), hence an almost isomorphism. Note that

$$(B \otimes_A B \longrightarrow B \times G) = \bigoplus_{g \in G} (B \otimes_A B \xrightarrow{1 \otimes g} B \otimes_A B \xrightarrow{x \otimes y \mapsto xy} B).$$

Since $A \rightarrow B$ is almost étale, B is an almost projective $B \otimes_A B$ -module, hence so is $B \times G$. □

2.2. Almost purity

2.2.1.

Fix $c \in \{1, \pi\} \subset K^+$ (where $\pi \in K^+$ was our choice of uniformizer) and define

$$O(c) := K^+[T_1, \dots, T_r, T_{r+1}^{\pm 1}, \dots, T_{d+1}^{\pm 1}]/(T_1 \cdots T_r - c).$$

Following Faltings, we say that a K^+ -algebra R is *small* if there is an étale map

$$O(c) \rightarrow R.$$

Every smooth (resp. semi-stable) K^+ -scheme has an étale covering by small affines with $c = 1$ (resp. $c = \pi$). Note that with these choices of c , R is a regular ring; in particular, it is a finite product of integrally closed domains, and we shall assume that R is an integrally closed domain. For any c , we endow R with the canonical log-structure (see §2.2.5). If $c = \pi$, there is a natural chart for this log-structure, namely the map

$$\mathbb{N}^r \rightarrow R : (n_1, \dots, n_r) \mapsto \prod_{i=1}^r T_i^{n_i}$$

which makes $\text{Spec}(R) \rightarrow \text{Spec}(K^+)$ into a a log-smooth morphism, where $\text{Spec}(K^+)$ is given the canonical log-structure (see §2.2.5).

2.2.2.

Fix $c \in \{1, \pi\}$. Fix a sequence $(c = c_0, c_1, c_2, \dots)$ of elements of \bar{K} satisfying $c_{n+1}^p = c_n$. If $c = 1$ we take the c_n to be primitive roots of unity. For any $n \in \mathbb{N}$, let K_n^+ be the normalization of K^+ in the extension K_n of K generated by c_n . Define

$$O(c)_n := K_n^+[T_1^{(n)}, T_2^{(n)}, \dots, T_r^{(n)}, T_{r+1}^{(n)\pm 1}, \dots, T_{d+1}^{(n)\pm 1}]/(T_1^{(n)} \cdots T_r^{(n)} - c_n).$$

Of course $O(c)_0 = O(c)$. There is a canonical homomorphism $K_n^+ \rightarrow K_{n+1}^+$ sending c_n to c_{n+1}^p , which extends to a canonical homomorphism

$$O(c)_n \rightarrow O(c)_{n+1} : T_i^{(n)} \mapsto T_i^{(n+1)p}.$$

Also note that there is a canonical isomorphism

$$\begin{aligned} O(c)_n[1/p] &\cong K_n[X_2^{\pm 1}, \dots, X_{d+1}^{\pm 1}] \\ T_1^{(n)} &\mapsto \frac{c_n}{X_2 \cdots X_r} \\ T_i^{(n)} &\mapsto X_i \quad \text{for } i \neq 1 \end{aligned}$$

and hence $O(c)_n[1/p]$ is finite étale over $O(c)[1/p]$. Moreover, since $O(c)_n$ is regular (see Appendix) hence normal, it follows that we may identify $O(c)_n$ with the normalization of $O(c)$ in $K_n[T_2^{\pm p^{-n}}, \dots, T_{d+1}^{\pm p^{-n}}]$ via this map.

If R is a small K^+ -algebra, then we define

$$R_n := R \otimes_{O(c)} O(c)_n.$$

If $K \subset L$ is a finite extension, then write $L_n = L \cdot K_n$ and let L_n^+ be the normalization of K^+ in L_n . Define

$$O(c)_{n,L} := O(c)_n \otimes_{K_n^+} L_n^+.$$

2.2.3.

Define

$$K_\infty^+ := \varinjlim_{n \in \mathbb{N}} K_n^+.$$

We will now consider the almost ring theory of the pair $(K_\infty^+, \mathfrak{m}_\infty)$, where $\mathfrak{m}_\infty \subset K_\infty^+$ is the maximal ideal. Note that there is a sequence of rational numbers ε_n occurring as p -adic valuations of elements of K_∞^+ , and tending to zero with n . If $c = \pi$ then this is clear, and if $c = 1$ then $\varepsilon_n = v_p(\zeta_{p^{n+1}} - 1) = \frac{1}{p^n(p-1)}$, where $\zeta_{p^{n+1}}$ denotes a primitive p^{n+1} th root of unity. Since K_∞^+ is a valuation ring we see that we indeed have $\mathfrak{m}_\infty^2 = \mathfrak{m}_\infty$.

2.2.4.

Let S be a finite integral R -algebra. We say that S is *étale in characteristic zero* if $R_K \rightarrow S_K$ is étale. For all n , let S_∞ be the normalization of $R_\infty \otimes_R S$, where

$$R_\infty := \varinjlim_{n \in \mathbb{N}} R_n.$$

The key result we will need is Faltings' Almost Purity Theorem:

THEOREM 2.11 (Faltings [7]). — *If S is a finite integral normal R -algebra, flat over K^+ and étale in characteristic zero, then the canonical homomorphism*

$$R_\infty \rightarrow S_\infty$$

is an almost étale covering of K_∞^+ -algebras.

We define

$$\tilde{R} := R \otimes_{K^+} \bar{K}^+$$

and

$$\tilde{R}_\infty := R_\infty \otimes_{K_\infty^+} \bar{K}^+.$$

If K^+ is integrally closed in R and p is not a unit in R , then by Proposition 4.5 these are integrally closed domains. If S is a finite integral normal R -algebra which is étale in characteristic zero, then we define \tilde{S}_∞ to be the normalization of $S_\infty \otimes_{K_\infty^+} \bar{K}^+$. Since $R_\infty \rightarrow \tilde{S}_\infty$ factors over $\tilde{R}_\infty \rightarrow \tilde{S}_\infty$, then by Lemma 2.1 we deduce that $\tilde{R}_\infty \rightarrow \tilde{S}_\infty$ is the filtering inductive limit of almost étale coverings, in fact an almost étale covering.

COROLLARY 2.12. — (i) *If $c = \pi$ (resp. $c = 1$), then the Frobenius is surjective on R_∞/pR_∞ (resp. $\tilde{R}_\infty/p\tilde{R}_\infty$).*

(ii) *If $c = \pi$ (resp. $c = 1$) and S is a finite integral normal R -algebra (resp. \tilde{R} -algebra), étale in characteristic zero and flat over K^+ , then the Frobenius is surjective on S_∞/pS_∞ (resp. $\tilde{S}_\infty/p\tilde{S}_\infty$).*

Proof. — We do the case $c = \pi$, the case $c = 1$ being similar. By the Almost Purity Theorem and Corollary 2.5 it suffices to show (i). In this case, by étaleness over

$$O(c)_\infty := \varinjlim_n O(c)_n$$

it suffices to show that the Frobenius is surjective on the latter modulo p . First note that the Frobenius is surjective on K_∞^+/pK_∞^+ : we have $K^+/pK^+ = k[u]/(u^e)$ where e is the ramification index of K and $K_n^+ = K^+[X]/(X^{p^n} - \pi)$, so $K_n^+/pK_n^+ = K^+/pK^+[x]$ with $x^{p^n} = u$ and so K_∞^+/pK_∞^+ has surjective Frobenius. Now every element of $O(c)_\infty/pO(c)_\infty$ has the form

$$\sum v_N T_1^{(m_1)n_1} \dots T_{d+1}^{(m_{d+1})n_{d+1}} = \left(\sum v_N^{1/p} T_1^{(m_1+1)n_1} \dots T_{d+1}^{(m_{d+1}+1)n_{d+1}} \right)^p$$

for some $v_N \in K_\infty^+/pK_\infty^+$ so the claim in this case is clear. □

2.2.5.

Given a flat K^+ -algebra S , we define the *canonical log-structure* \mathcal{L}_S on S as the sheaf which associates to each étale S -algebra S' the monoid

$$(S'[1/p])^* \cap S'.$$

Note that this is an integral and saturated log-structure if S is normal. For example, if R is a *small* K^+ -algebra, then \mathcal{L}_R is the usual fine log-structure of the scheme $\text{Spec}(R)$. Now if S is a finite integral R -algebra,

étale in characteristic zero, let $f : \text{Spec}(S) \rightarrow \text{Spec}(R)$ be the structure morphism and let M_S be the log-structure on $\text{Spec}(S)$ associated to the sheaf of monoids

$$\{x \in \mathcal{L}_S \mid \exists n \in \mathbb{N} : x^{p^n} \in f^* \mathcal{L}_R\}.$$

We call M_S the p -saturation of \mathcal{L}_R in \mathcal{L}_S .

LEMMA 2.13. — *With notation as above.*

- (i) *If $c = \pi$, then there is a saturated integral monoid with surjective Frobenius inducing a chart for the log-structure $M_{S_\infty} := \varinjlim_n M_{S_n}$ on $\text{Spec}(S_\infty)$. In fact, we can take the same monoid as for M_{R_∞} .*
- (ii) *If $c = 1$, then there is a saturated integral monoid with surjective Frobenius inducing a chart for $M_{\tilde{S}_\infty} := \varinjlim_n M_{\tilde{S}_n}$ on $\text{Spec}(\tilde{S}_\infty)$, where $M_{\tilde{S}_n} := \varinjlim_{K_n \subset L} M_{S_n \otimes_{K_n} L^+}$, the limit being over all finite extensions L of K_n in \bar{K} .*

Proof. —

(i). — We first prove the assertion for R_∞ . Consider the map $\mathbb{N}[1/p]^r \rightarrow M_{R_\infty} : (\alpha_1, \dots, \alpha_r) \mapsto \prod_{i=1}^r T_i^{(n_i)m_i}$, where $\alpha_i = \frac{m_i}{p^{n_i}}$. We claim that it is a chart for \mathcal{L}_{R_∞} , which will clearly imply the result in this case. To see this, up to replacing R_∞ by its strict henselization at a point, we may assume that R_∞ is a strictly henselian local ring. Then it is the limit of the strict henselizations of the R_n at the image points, so we can assume that R and R_n are strictly henselian local rings for each $n \in \mathbb{N}$. Since $O(c)_n / (T_i^{(n)})$ is regular, then by étaleness it follows that $R_n / (T_i^{(n)})$ is a regular local ring, if T_i is not a unit in R . So in this case $T_i^{(n)}$ generates a prime ideal of R_n . Moreover, since $\prod_{i=1}^r T_i^{(n)} = c_n$ every prime divisor of c_n in R_n is generated by one of the $T_i^{(n)}$. These are also the prime divisors of p in R_n . It follows from this that for each $n \in \mathbb{N}$ there is a canonical isomorphism

$$\mathcal{L}_{R_n}(R_n) / R_n^* \cong \frac{1}{p^n} \mathbb{N}^s : x \mapsto (v_{T_i}(x))$$

where $v_{T_i}(-)$ denotes the valuation at the prime ideal generated by $T_i^{(n)}$ normalized by $v_{T_i}(T_i) = 1$, assuming T_i is not a unit in R ($0 \leq s \leq r$). These maps are compatible as n varies, so taking the limit over n to get an isomorphism

$$\mathcal{L}_{R_\infty}(R_\infty) / R_\infty^* \cong \mathbb{N}[1/p]^s.$$

Since the natural map $\mathbb{N}[1/p]^r \rightarrow \mathcal{L}_{R_\infty}(R_\infty) / R_\infty^* \cong \mathbb{N}[1/p]^s$ is none other than the projection, this proves the claim.

We now show that this chart induces the log-structure M_{S_∞} . As above for R_∞ , to see this we may assume R , S_n , and S_∞ are a strictly henselian local

rings. Suppose $x \in \mathcal{L}_{S_n}(S_n)$ such that $x^{p^m} \in \mathcal{L}_R(R)$, say $x^{p^m} = u \prod_{i=1}^r T_i^{n_i}$ for some $u \in R^*$. Up to increasing n we may assume that $m \leq n$. Then $y := \frac{x}{\prod_{i=1}^r T_i^{(m)n_i}}$ satisfies $y^{p^m} = u$, whence $y \in S_n^*$, since S_n is an integrally closed domain (being a normal local ring). So x and $\prod_{i=1}^r T_i^{(m)n_i}$ differ by a unit. Since this holds for all x , this implies the assertion.

(ii). — Recall that since $c = 1$, the log-structure for small R can be given by the map

$$\mathbb{N} \rightarrow R : n \mapsto \pi^n.$$

Let us fix a sequence $\{\pi_n\}$ of elements of \bar{K}^+ satisfying $\pi_0 = \pi$ and $\pi_{n+1}^p = \pi_n$ for all $n \in \mathbb{N}$. We claim that the map

$$\mathbb{N}[1/p] \rightarrow \tilde{S}_\infty : \frac{m}{p^n} \mapsto \pi_n^m$$

is a chart for the log-structure $M_{\tilde{S}_\infty}$. To see this we may (as above) reduce to the case R and \tilde{S}_∞ are strictly henselian local ring (and \tilde{S}_∞ is an integrally closed domain, being the limit of normal local rings). If $x \in \mathcal{L}_{\tilde{S}_\infty}(\tilde{S}_\infty)$ is such that $x^{p^m} \in \mathcal{L}_R(R)$, then we may write $x^{p^m} = u\pi^n$ for some $u \in R^*$ and $n \in \mathbb{N}$. Then $y := \frac{x}{\pi_n^{1/p^m}}$ satisfies $y^{p^m} = u$, whence $y \in \tilde{S}_\infty^*$, since \tilde{S}_∞ is an integrally closed domain. So x differs from an element of the image of $\mathbb{N}[1/p]$ up to a unit, and this implies (ii). □

2.3. Almost étale coverings of Fontaine rings

By Corollary 2.12 we may, following Fontaine, construct the Fontaine rings

$$A_\infty^+ := \varprojlim_n H_{\text{crys}}^0(\tilde{R}_\infty/p\tilde{R}_\infty | \Sigma_n, \mathcal{O})$$

$$A_\infty^+(S) := \varprojlim_n H_{\text{crys}}^0(\tilde{S}_\infty/p\tilde{S}_\infty | \Sigma_n, \mathcal{O})$$

where S is a finite integral normal R -algebra, étale in characteristic zero. Our aim in this section is to show that

$$A_\infty^+/p^n A_\infty^+ \rightarrow A_\infty^+(S)/p^n A_\infty^+(S)$$

is an almost étale homomorphism and from this deduce some consequences. We first set up the almost ring theory in this context.

2.3.1.

We will consider the almost ring theory of the ring

$$A_{\text{inf}}(K^+) = W(P(\bar{K}^+/p\bar{K}^+))$$

with respect to the ideal \mathfrak{a} , union of the principal ideal $[p]^\varepsilon$ for all positive rational exponents $\varepsilon > 0$. Note that $[p]^\varepsilon$ is not a zero divisor: this is true modulo p since $P(\bar{K}^+/p\bar{K}^+)$ is a valuation ring (hence an integral domain) and so in general since $A_{\text{inf}}(K^+)$ is p -adically separated and torsion free, being the ring of Witt vectors of a perfect ring of characteristic p . We view $W(\bar{K}^+/p\bar{K}^+)$ as a $A_{\text{inf}}(K^+)$ -algebra under the homomorphism induced by the canonical homomorphism $P(\bar{K}^+/p\bar{K}^+) \rightarrow \bar{K}^+/p\bar{K}^+$. Then the pair

$$(W(\bar{K}^+/p\bar{K}^+), \mathfrak{a} \cdot W(\bar{K}^+/p\bar{K}^+))$$

is an example of the pairs $(W(\Lambda/p\Lambda), [\mathfrak{m}])$ considered in §2.1.2.

2.3.2. Remark.

The ring $A_{\text{cris}}(K^+)/p^n A_{\text{cris}}(K^+)$ can be viewed as an $A_{\text{inf}}(K^+)$ -algebra in 2 different ways, the first is via the canonical map $A_{\text{inf}}(K^+) \rightarrow A_{\text{cris}}(K^+)$ arising from the construction of $A_{\text{cris}}(K^+)$ in the 1st proof of Theorem 1.1, and the second is via the map

$$A_{\text{inf}}(K^+) \rightarrow W(\bar{K}^+/p\bar{K}^+) \rightarrow W_n(\bar{K}^+/p\bar{K}^+) \rightarrow A_{\text{cris}}(K^+)/p^n A_{\text{cris}}(K^+),$$

where the last map is the canonical one arising the construction of

$$A_{\text{cris}}(K^+)/p^n A_{\text{cris}}(K^+)$$

in the 2nd proof of Theorem 1.1. We claim that the image of the ideal \mathfrak{a} under these 2 maps is the same. To see this it suffices to note that by Remark (1.2.2) (d) we have $[p^\varepsilon] = [p^{\varepsilon/p^n}]$ in $A_{\text{cris}}(K^+)/p^n A_{\text{cris}}(K^+)$.

2.3.3.

Let R be a small integral K^+ -algebra and let T be a p -adically complete logarithmic Σ -algebra such that R/pR is a T -algebra. In practice we will have $T = \Sigma$ or T will be a formal Σ -lift of R/pR . For all $n \geq 1$, write $T_n := T/p^n T$.

We assume that the log-structure of T is fine and saturated, defined by a fine saturated monoid Λ_T . Let S be a finite integral normal R -algebra étale in characteristic zero. We endow it with the log-structure M_S of §2.2.5 and

endow $\text{Spec}(S/pS)$ with the inverse image log structure. By taking limits this then defines a log-structure on $\tilde{S}_\infty/p\tilde{S}_\infty$. Define

$$A_{\log,\infty,T}^+ := \varprojlim_n H_{\log\text{-crys}}^0(\tilde{R}_\infty/p\tilde{R}_\infty|T_{n+1}, \mathcal{O})$$

$$A_{\log,\infty,T}^+(S) := \varprojlim_n H_{\log\text{-crys}}^0(\tilde{S}_\infty/p\tilde{S}_\infty|T_{n+1}, \mathcal{O}).$$

PROPOSITION 2.14. — *The canonical homomorphism*

$$A_{\log,\infty,T}^+/p^n A_{\log,\infty,T}^+ \rightarrow A_{\log,\infty,T}^+(S)/p^n A_{\log,\infty,T}^+(S)$$

is an almost étale homomorphism of $A_{\text{inf}}(K^+)$ -algebras.

Proof. — Firstly, by Corollary 2.3 the canonical homomorphism

$$W_n(\tilde{R}_\infty/p\tilde{R}_\infty) \rightarrow W_n(\tilde{S}_\infty/p\tilde{S}_\infty)$$

is almost étale of $W_n(\tilde{K}^+/p\tilde{K}^+)$ -algebras. Now, by Lemma 2.13 there is a saturated integral monoid M with surjective Frobenius giving a chart for the log-structures of both \tilde{R}_∞ and \tilde{S}_∞ , so by Corollary 2.12 we may construct $A_{\log,\infty,T}^+(S)$ as in the second proof of Theorem 1.1. Let M be the saturated integral monoid defining the log-structure on \tilde{R}_∞ , hence also on \tilde{S}_∞ (see Lemma 2.13), and set $Q := M \oplus \Lambda_T$. Let $L := \ker(Q^{\text{gp}} \rightarrow M^{\text{gp}})$, where the map $Q \rightarrow M$ is induced by the maps $M \rightarrow M : m \mapsto m^{p^n}$ and the structure map $\Lambda_T \rightarrow M$. By construction, $A_{\log,\infty,T}^+(S)/p^n A_{\log,\infty,T}^+(S)$ is a quotient of a divided power hull of $(W_n(\tilde{S}_\infty/p\tilde{S}_\infty) \otimes_{\mathbb{Z}} T_n)_{\log}$, where by definition (*cf.* the second proof of Theorem 1.1)

$$(W_n(\tilde{S}_\infty/p\tilde{S}_\infty) \otimes_{\mathbb{Z}} T_n)_{\log} = (W_n(\tilde{S}_\infty/p\tilde{S}_\infty) \otimes_{\mathbb{Z}} T_n) \otimes_{\mathbb{Z}} \mathbb{Z}[L].$$

Now, the canonical homomorphism

$$(W_n(\tilde{R}_\infty/p\tilde{R}_\infty) \otimes_{\mathbb{Z}} T_n) \otimes_{\mathbb{Z}} \mathbb{Z}[L] \rightarrow (W_n(\tilde{S}_\infty/p\tilde{S}_\infty) \otimes_{\mathbb{Z}} T_n) \otimes_{\mathbb{Z}} \mathbb{Z}[L]$$

is almost étale (Cor. 2.3). This implies that we have a commutative diagram

$$\begin{array}{ccc} (W_n(\tilde{R}_\infty/p\tilde{R}_\infty) \otimes_{\mathbb{Z}} T_n) \otimes_{\mathbb{Z}} \mathbb{Z}[L] & \longrightarrow & (W_n(\tilde{S}_\infty/p\tilde{S}_\infty) \otimes_{\mathbb{Z}} T_n) \otimes_{\mathbb{Z}} \mathbb{Z}[L] \\ \Phi^n \otimes 1 \otimes 1 \downarrow & & \Phi^n \otimes 1 \otimes 1 \downarrow \\ (\tilde{R}_\infty/p\tilde{R}_\infty \otimes_{\mathbb{Z}} T_n) \otimes_{\mathbb{Z}} \mathbb{Z}[L] & \longrightarrow & (\tilde{S}_\infty/p\tilde{S}_\infty \otimes_{\mathbb{Z}} T_n) \otimes_{\mathbb{Z}} \mathbb{Z}[L] \\ L \mapsto 1 \downarrow & & L \mapsto 1 \downarrow \\ \tilde{R}_\infty/p\tilde{R}_\infty \otimes_{\mathbb{Z}} T_n & \longrightarrow & \tilde{S}_\infty/p\tilde{S}_\infty \otimes_{\mathbb{Z}} T_n \\ \downarrow & & \downarrow \\ \tilde{R}_\infty/p\tilde{R}_\infty & \longrightarrow & \tilde{S}_\infty/p\tilde{S}_\infty \end{array}$$

in which all three squares are almost cofibred, where we denote by $\Phi^n : W_n(A) \rightarrow A : (a_0, \dots, a_{n-1}) \mapsto a_0^{p^n}$. In particular, the commutative diagram

$$\begin{CD} (W_n(\tilde{R}_\infty/p\tilde{R}_\infty) \otimes_{\mathbb{Z}} T_n) \otimes_{\mathbb{Z}} \mathbb{Z}[L] @>>> (W_n(\tilde{S}_\infty/p\tilde{S}_\infty) \otimes_{\mathbb{Z}} T_n) \otimes_{\mathbb{Z}} \mathbb{Z}[L] \\ @VVV @VVV \\ \tilde{R}_\infty/p\tilde{R}_\infty @>>> \tilde{S}_\infty/p\tilde{S}_\infty \end{CD}$$

is almost cofibred. Taking divided power hulls for the kernels of the vertical maps and then taking the quotient by the ideal generated by all divided powers of elements of the form $q_l \otimes l - p_l \otimes 1$ (where $q_l, p_l \in Q$ satisfy $l = p_l/q_l \in L \subset Q^{\text{gp}}$), it follows from the next lemma that the canonical map

$$\begin{CD} A_{\log, \infty, T}^+ / p^n A_{\log, \infty, T}^+ \otimes_{(W_n(\tilde{R}_\infty/p\tilde{R}_\infty) \otimes_{\mathbb{Z}} T_n) \otimes_{\mathbb{Z}} \mathbb{Z}[L]} (W_n(\tilde{S}_\infty/p\tilde{S}_\infty) \otimes_{\mathbb{Z}} T_n) \otimes_{\mathbb{Z}} \mathbb{Z}[L] \\ @VVV \\ A_{\log, \infty, T}^+(S) / p^n A_{\log, \infty, T}^+(S) \end{CD}$$

is an almost isomorphism of $W_n(\bar{K}^+ / p\bar{K}^+)$ -algebras. This proves that the homomorphism of the statement is an almost étale morphism of $W_n(\bar{K}^+ / p\bar{K}^+)$ -algebras. By Remark (2.3.2), it follows that it is also an almost étale morphism of $A_{\text{inf}}(K^+)$ -algebras. \square

In the proof we have made use of the following almost analogue of a well-known result on divided power hulls.

LEMMA 2.15. — *Suppose there is a homomorphism of rings $A \rightarrow B$ making B into an almost flat A -algebra. Then for any ideal $I \subset A$ the canonical map*

$$D_A(I) \otimes_A B \rightarrow D_B(I \cdot B)$$

is an almost isomorphism, where $D_A(I)$ denotes the divided power hull of A for the ideal I .

Proof. — For any ring A and any module M let $\Gamma_A(M)$ denote the divided power algebra generated by M (see e.g. [3, 3.9]). Since B is an almost flat A -algebra, the canonical map $I \otimes_A B \rightarrow I \cdot B$ is an almost isomorphism, hence $\Gamma_B(I \otimes_A B) \approx \Gamma_B(I \cdot B)$ by [9, Lemma 8.1.13]. But

$$\Gamma_B(I \otimes_A B) \cong \Gamma_A(I) \otimes_A B$$

so $\Gamma_B(I \cdot B)$ is an almost flat $\Gamma_A(I)$ -algebra. Hence if $J \subset \Gamma_A(I)$ denotes the ideal defining $D_A(I)$, then we have $J \otimes_A B \cong J \otimes_{\Gamma_A(I)} \Gamma_B(I \otimes_A B) \approx J \cdot \Gamma_B(I \cdot B)$, and therefore

$$D_A(I) \otimes_A B = \frac{\Gamma_A(I) \otimes_A B}{J \otimes_A B} \approx \frac{\Gamma_B(I \cdot B)}{J \cdot \Gamma_B(I \cdot B)} = D_B(I \cdot B).$$

□

2.3.4.

Assume that K^+ is integrally closed in R , so that \tilde{R}_∞ is an integral domain (Prop. 4.5). Define \bar{R} to be the normalization of R in the maximal profinite connected étale covering of $R[1/p]$ (taking for base point an algebraic closure of $Q(\tilde{R}_\infty)$). Then we have

$$\pi_1(\text{Spec}(\tilde{R}[1/p])) = \text{Gal}(\bar{R}[1/p]/\tilde{R}[1/p])$$

and $\tilde{R}_\infty[1/p] \rightarrow \bar{R}[1/p]$ is the inductive limit of Galois coverings of $\tilde{R}_\infty[1/p]$. Define

$$\Delta := \text{Gal}(\bar{R}[1/p]/\tilde{R}[1/p])$$

and

$$\Delta_\infty := \text{Gal}(\tilde{R}_\infty[1/p]/\tilde{R}[1/p]).$$

Then Δ_∞ is a quotient of Δ and let $\Delta_0 := \ker(\Delta \rightarrow \Delta_\infty)$. With the log-structure of §2.2.5 on \bar{R} , define

$$A_{\log, T}^+ := \varprojlim_n H_{\log\text{-crys}}^0(\bar{R}/p\bar{R}|T_{n+1}, \mathcal{O}).$$

LEMMA 2.16. — *If K^+ is integrally closed in R , then the homomorphism $\tilde{R}_\infty \rightarrow \bar{R}$ is the filtering inductive limit of almost Galois coverings.*

Proof. — Let us first show that every non-zero almost étale covering of \tilde{R}_∞ is almost faithfully flat. Consider the ring $(\tilde{R}_\infty)_* := \text{Hom}_{K_\infty^+}(\mathfrak{m}_\infty, \tilde{R}_\infty)$ defined in [9, §2.2.9]. If $\varphi \in \text{Hom}_{K_\infty^+}(\mathfrak{m}_\infty, \tilde{R}_\infty)$ is an idempotent, then for all $x, y \in \mathfrak{m}_\infty$

$$x\varphi(y) = \varphi(xy) = \varphi(x) \cdot \varphi(y)$$

so $x = \varphi(x)$, whence $\varphi = \text{id}$ in $(\tilde{R}_\infty)_*$. So \tilde{R}_∞ has no almost idempotents, hence every almost projective \tilde{R}_∞ -module of finite rank is either almost zero or everywhere of constant non-zero rank, hence almost faithfully flat.

Now, let S be a finite integral normal R -algebra, such that $R[1/p] \rightarrow S[1/p]$ is a Galois covering. Let L^+ be the integral closure of K^+ in S , and write $L = Q(L^+)$. Then by the almost purity theorem, the homomorphism $R_\infty \rightarrow S_\infty$ is an almost étale covering and factors over the almost étale covering $R_\infty \rightarrow R_{\infty, L}$. By Lemma 2.1 it follows that $R_{\infty, L} \rightarrow S_\infty$ is an almost étale covering. It is easy to see that it must also be an almost Galois covering. Let G be the Galois group of $R_{\infty, L}[1/p] \rightarrow S_\infty[1/p]$. The same

argument with S replaced by $S \otimes_{L^+} E^+$ for E^+ the normalization of L^+ in a finite Galois extension $L \subset E$ proves that

$$R_{\infty,E} \rightarrow S_{\infty,E}$$

is an almost Galois covering, where $S_{\infty,E}$ is the normalization of $R_\infty \otimes_R S \otimes_{L^+} E^+$. Let $\tilde{S}_\infty := \varinjlim_E S_{\infty,E}$. Taking the limit over E we get a homomorphism

$$\tilde{R}_\infty \rightarrow \tilde{S}_\infty$$

which is an almost Galois covering: there is a canonical almost isomorphism

$$\varinjlim_E S_{\infty,E} \otimes_{R_{\infty,E}} S_{\infty,E} \approx \varinjlim_E S_{\infty,E} \times G.$$

□

In particular, $\tilde{R}_\infty \rightarrow \bar{R}$ is almost faithfully flat.

COROLLARY 2.17. — *The canonical homomorphism*

$$A_{\log,\infty,T}^+ / p^n A_{\log,\infty,T}^+ \rightarrow A_{\log,T}^+ / p^n A_{\log,T}^+$$

is the filtering inductive limit of almost Galois coverings.

Proof. — The proof is similar to that of Proposition 2.14, so we refer there for details and point out the differences in this case. First note that since $\tilde{R}_\infty / p\tilde{R}_\infty \rightarrow \bar{R} / p\bar{R}$ is the filtering inductive limit of almost Galois coverings, by Cor. 2.10 so is

$$W_n(\tilde{R}_\infty / p\tilde{R}_\infty) \rightarrow W_n(\bar{R} / p\bar{R}).$$

As in the proof of Proposition 2.14 define $Q := M \oplus \Lambda_T$ and

$$L := \ker(Q^{\text{gp}} \rightarrow M^{\text{gp}}).$$

Looking at the proof of Proposition 2.14 we note that we have an almost cofibred square

$$\begin{array}{ccc} (W_n(\tilde{R}_\infty / p\tilde{R}_\infty) \otimes_{\mathbb{Z}} T_n) \otimes_{\mathbb{Z}} \mathbb{Z}[L] & \longrightarrow & (W_n(\bar{R} / p\bar{R}) \otimes_{\mathbb{Z}} T_n) \otimes_{\mathbb{Z}} \mathbb{Z}[L] \\ \downarrow & & \downarrow \\ \tilde{R}_\infty / p\tilde{R}_\infty & \longrightarrow & \bar{R} / p\bar{R}. \end{array}$$

Hence we deduce that the canonical map

$$\begin{aligned} A_{\log,\infty,T}^+ / p^n A_{\log,\infty,T}^+ \otimes_{(W_n(\tilde{R}_\infty / p\tilde{R}_\infty) \otimes_{\mathbb{Z}} T_n) \otimes_{\mathbb{Z}} \mathbb{Z}[L]} & \\ \rightarrow (W_n(\bar{R} / p\bar{R}) \otimes_{\mathbb{Z}} T_n) \otimes_{\mathbb{Z}} \mathbb{Z}[L] & \rightarrow A_{\log,T}^+ / p^n A_{\log,T}^+ \end{aligned}$$

is an almost isomorphism, and since tensor product commutes with inductive limits, we are done. □

COROLLARY 2.18. — (i) *The canonical map*

$$A_{\log, \infty, T}^+ / p^n A_{\log, \infty, T}^+ \rightarrow \left(A_{\log, T}^+ / p^n A_{\log, T}^+ \right)^{\Delta_0}$$

is an almost isomorphism.

(ii) $A_{\log, T}^+ / p^n A_{\log, T}^+$ *is a discrete Δ -module and for all $i \neq 0$ we have*

$$H^i(\Delta_0, A_{\log, T}^+ / p^n A_{\log, T}^+) \approx 0.$$

Proof. — Part (i) follows from Proposition 2.6 (i) by taking the inductive limit. For (ii), note that $A_{\log, T}^+ / p^n A_{\log, T}^+$ is a quotient of a divided power hull of $(W_n(\bar{R}/p\bar{R}) \otimes_{\mathbb{Z}} T_n) \otimes_{\mathbb{Z}} \mathbb{Z}[L]$. Now, we may view M as a submonoid of \bar{R} . In this way we see that every element of M^{gp} , hence also Q^{gp} , is fixed by an open subgroup of Δ . So every element of $(W_n(\bar{R}/p\bar{R}) \otimes_{\mathbb{Z}} T_n) \otimes_{\mathbb{Z}} \mathbb{Z}[L]$ is also fixed by an open subgroup of Δ . Thus, every element of $A_{\log, T}^+ / p^n A_{\log, T}^+$ is fixed by an open subgroup of Δ , which proves the first assertion of (ii). As in (i), the second assertion of (ii) follows from Proposition 2.6 (iii) by taking the inductive limit. □

In particular, via the Hochschild-Serre spectral sequence of Galois cohomology we deduce canonical almost isomorphisms for all i

$$H^i(\Delta_{\infty}, A_{\log, \infty, T}^+ / p^n A_{\log, \infty, T}^+) \approx H^i(\Delta, A_{\log, T}^+ / p^n A_{\log, T}^+).$$

This almost isomorphism will enable us to express the right-hand side in terms of crystalline cohomology.

3. Geometric Galois cohomology of Fontaine rings

In this section we construct a natural de Rham resolution of the Fontaine rings A_{\log, Σ_n}^+ and then we compute the (geometric) Galois cohomology of its components. Assume throughout that R is an étale localization of a small integral K^+ -algebra and that K^+ is integrally closed in R .

3.1. Canonical de Rham resolutions of Fontaine rings

3.1.1.

If $c = 1$, then let $v = 1$, and if $c = \pi$ then let $v = u$, where $\Sigma_n = W_n(k)[u] \langle u^e \rangle$. Define

$$\Theta(c) := \Sigma[T_1, \dots, T_r, T_{r+1}^{\pm 1}, \dots, T_{d+1}^{\pm 1}] / (T_1 \cdots T_r - v).$$

Since R/pR is an étale $O(c)/pO(c)$ -algebra, for all $n \geq 1$ there exists an étale $\Theta(c)/p^n\Theta(c)$ -algebra \mathcal{R}_n lifting R/pR , unique up to canonical isomorphism. Fix a projective system $\{\mathcal{R}_n\}_{n \in \mathbb{Z}_{>0}}$ of compatible lifts and define $\mathcal{R} := \varprojlim_n \mathcal{R}_n$. We endow \mathcal{R}_n with the log-structure associated to

$$\mathcal{N} : \mathbb{N}^r \rightarrow \mathcal{R}_n : (n_1, \dots, n_r) \mapsto \prod_{i=1}^r T_i^{n_i}$$

in the case $c = \pi$, and

$$\mathcal{N} : \mathbb{N} \rightarrow \mathcal{R}_n : n \mapsto u$$

in the case $c = 1$.

3.1.2.

Consider the canonical map

$$h : \text{Spec}(\bar{R}/p\bar{R}) \rightarrow \text{Spec}(R/pR)$$

and the associated morphism of log-crystalline topoi with respect to the DP-base Σ_n . By Proposition 1.2, $h_*\mathcal{O}$ is a quasi-coherent crystal of \mathcal{O} -modules on $(R/pR|\Sigma_n)_{\text{log-crys}}$. Since $\text{Spec}(\mathcal{R}_n)$ is a log-smooth lift of $\text{Spec}(R/pR)$, by [12, Thm. 6.2] there is an integrable quasi-nilpotent logarithmic connection d on $h_*\mathcal{O}(\mathcal{R}_n)$ whose associated de Rham complex computes the log-crystalline cohomology of $\bar{R}/p\bar{R}$ over the DP-base Σ_n . Since this cohomology vanishes in non-zero degree by Theorem 1.1, it follows that the augmentation

$$A_{\text{log},\Sigma}^+ / p^n A_{\text{log},\Sigma}^+ \cong H_{\text{log-crys}}^0(\bar{R}/p\bar{R}|\Sigma_n, \mathcal{O}) \rightarrow h_*\mathcal{O}(\mathcal{R}_n) \otimes_{\mathcal{R}_n} \omega_{\mathcal{R}_n/\Sigma_n}^\bullet$$

is a quasi-isomorphism, where $\omega_{\mathcal{R}_n/\Sigma_n}^i := \wedge^i_{\mathcal{R}_n} \omega_{\mathcal{R}_n/\Sigma_n}^1$ and $\omega_{\mathcal{R}_n/\Sigma_n}^1$ is the sheaf of logarithmic differentials ([12, 1.7]). This quasi-isomorphism is the analogue of Poincaré’s lemma which we will use in our comparison strategy. Our aim is to (almost) compute the Δ -cohomology of the components of this resolution.

From now on we write

$$A^+ := A_{\text{log},\Sigma}^+$$

and

$$M^+ := \varprojlim_n H_{\text{log-crys}}^0(\bar{R}/p\bar{R}|\mathcal{R}_n, \mathcal{O}).$$

Note that $M^+ / p^n M^+ \cong h_*\mathcal{O}(\mathcal{R}_n)$.

3.1.3.

Let

$$\tilde{O}(c)_\infty := \varinjlim_{n,L} O(c)_{n,L}$$

so that $\tilde{R}_\infty = R \otimes_{O(c)} \tilde{O}(c)_\infty$. The same argument as above can be applied to the morphism

$$h_\infty : \text{Spec}(\tilde{O}(c)_\infty/p\tilde{O}(c)_\infty) \rightarrow \text{Spec}(O(c)/pO(c)).$$

By Proposition 1.2 the sheaf $h_{\infty,*}\mathcal{O}$ is a quasi-coherent crystal of \mathcal{O} -modules on the site $(O(c)/pO(c)|\Sigma_n)_{\log\text{-crys}}$ and hence we have a canonical isomorphism

$$h_{\infty,*}\mathcal{O}(\mathcal{R}_n) \cong h_{\infty,*}\mathcal{O}(\Theta(c)/p^n\Theta(c)) \otimes_{\Theta(c)/p^n\Theta(c)} \mathcal{R}_n.$$

Note that in the notation of the previous section we have

$$h_{\infty,*}\mathcal{O}(\mathcal{R}_n) = A_{\log,\infty,\mathcal{R}}^+ / p^n A_{\log,\infty,\mathcal{R}}^+.$$

Also, there is an integrable quasi-nilpotent logarithmic connection d on $h_{\infty,*}\mathcal{O}(\mathcal{R}_n)$ whose associated de Rham complex is a resolution of

$$A_{\log,\infty,\Sigma}^+ / p^n A_{\log,\infty,\Sigma}^+.$$

Define

$$A_\infty^+ = A_\infty^+(R) := A_{\log,\infty,\Sigma}^+$$

and

$$M_\infty^+ = M_\infty^+(R) := \varprojlim_n h_{\infty,*}\mathcal{O}(\mathcal{R}_n).$$

3.1.4.

Note that, in the notation of §2.3, we have

$$\begin{aligned} M_\infty^+ &= A_{\log,\infty,\mathcal{R}}^+ \\ M^+ &= A_{\log,\mathcal{R}}^+ \end{aligned}$$

in particular, the canonical map $M_\infty^+/p^n M_\infty^+ \rightarrow M^+/p^n M^+$ is the inductive limit of almost Galois coverings, $M_\infty^+/p^n M_\infty^+ \approx (M^+/p^n M^+)^{\Delta_0}$, and we have canonical almost isomorphisms for all i

$$H^i(\Delta_\infty, M_\infty^+/p^n M_\infty^+) \approx H^i(\Delta, M^+/p^n M^+).$$

3.1.5.

Define

$$A_{\text{cris},\infty}(R) := \varprojlim_n H_{\text{crys}}^0(\tilde{R}_\infty/p\tilde{R}_\infty|W_{n+1}(k), \mathcal{O})$$

$$A_{\text{cris}}(R) := \varprojlim_n H_{\text{crys}}^0(\bar{R}/p\bar{R}|W_{n+1}(k), \mathcal{O})$$

This is just the classical crystalline cohomology, i.e. we ignore the log-structures.

LEMMA 3.1. — *Let A be an integrally closed domain of characteristic zero such that the Frobenius is surjective on A/pA . Assume that A contains a sequence $\underline{p} := (p, p^{1/p}, p^{1/p^2}, \dots)$ of p -power roots of p satisfying $(p^{1/p^{n+1}})^p = p^{1/p^n}$ for all n . Let*

$$\theta : W(P(A/pA)) \rightarrow \hat{A}$$

denote the canonical map constructed in the proof of Theorem 1.1. Then $\ker(\theta)$ is a principal ideal generated by $\xi := [\underline{p}] - p$, where we identify \underline{p} with the element it defines in $P(A/pA)$.

Proof. — We assume $x \in \ker(\theta \bmod p)$. Write $x = (x^{(n)})_n$ with $x^{(n+1)p} = x^{(n)}$ for all n . Then we have $x^{(1)p} = 0$. Let $\hat{x}^{(1)} \in A$ be a lift of $x^{(1)}$. Then $\hat{x}^{(1)p} = py$ for some $y \in A$ so $\frac{\hat{x}^{(1)}}{p^{1/p}} \in A$ because A is integrally closed. Hence $x^{(1)} \in p^{1/p} \cdot A/pA$. Continuing in this manner we find that $x \in \underline{p} \cdot P(A/pA)$. This proves the claim modulo p , and since $W(P(A/pA))$ is p -adically complete and \hat{A} is p -torsion free the lemma follows. \square

PROPOSITION 3.2. — *Let A be a flat \bar{K}^+ algebra. Assume that A is an integrally closed domain, and that the Frobenius is surjective on A/pA . Let*

$$A_{\text{cris}}(A/pA) := \varprojlim_n H_{\text{crys}}^0(A/pA|W_n(k), \mathcal{O}).$$

Then

- (i) *There is a canonical isomorphism*

$$A_{\text{cris}}(A/pA)/p \cong (A/pA) \otimes_{\bar{K}^+/p\bar{K}^+} A_{\text{cris}}(K^+)/p.$$

- (ii) *There is a canonical isomorphism*

$$A_{\text{cris}}(A/pA)/p^n \cong W_n(P(A/pA)) \otimes_{W_n(P(\bar{K}^+/p\bar{K}^+))} A_{\text{cris}}(K^+)/p^n.$$

- (iii) $A_{\text{cris}}(A/pA)/p^n$ *is flat over $W_n(k)$.*

Proof. —

(i). — By construction (cf. the second proof of Thm. 1.1), the ring $A_{\text{cris}}(A/pA)/(p)$ is the divided power hull of A/pA for the kernel of the Frobenius, i.e. for the ideal generated by $p^{1/p}$. Since the same is true for $A_{\text{cris}}(K^+)/p$ and A/pA is flat over $\bar{K}^+/p\bar{K}^+$, the result follows from the fact that formation of divided power hulls commutes with flat tensor product.

(ii). — Let $P := P(\bar{K}^+/p\bar{K}^+)$. We first show that $P(A/pA)$ is flat over P . Since P is a valuation ring, every finitely generated ideal is principal. Let $x = (x^{(n)}) \in P$ and $a = (a^{(n)}) \in P(A/pA)$ and consider $a \otimes x \in (a) \otimes_P P(A/pA)$. Then $xa = 0$ if and only if $x^{(n)}a^{(n)} = 0$ for all n , if and only if $a^{(n)} \otimes x^{(n)} = 0$ in $(a^{(n)}) \otimes_{\bar{K}^+/p\bar{K}^+} A/pA$ for all n (A/pA being flat over $\bar{K}^+/p\bar{K}^+$), if and only if $a \otimes x = 0$. This proves the flatness claim. Now since $P(A/pA)$ is a perfect ring, it follows that $W(P(A/pA))$ is p -torsion free, hence by Bourbaki’s flatness criterion we deduce that $W_n(P(A/pA))$ is a flat $W_n(P)$ -algebra for all n . Now by Lemma 3.1, we know that $A_{\text{cris}}(A/pA)/p^n$ is the divided power hull of $W_n(P(A/pA))$ for the ideal generated by ξ , hence since taking divided power hulls commutes with flat base change, (ii) follows.

(iii). — follows easily from these considerations. □

Let $1 \leq i \leq d + 1$. Note that by construction we have an element

$$\underline{T}_i = (T_i^{(0)}, T_i^{(1)}, T_i^{(2)}, \dots) \in P(\tilde{R}_\infty/p\tilde{R}_\infty).$$

If $\alpha = n/p^r$ with $n, r \in \mathbb{N}$, then we define

$$T_i^\alpha := T_i^{(r)n}$$

and

$$\underline{T}_i^\alpha := (T_i^{(r)n}, T_i^{(r+1)n}, \dots) \in P(\tilde{R}_\infty/p\tilde{R}_\infty).$$

We’ll also usually write $[\underline{T}_i]^\alpha := [\underline{T}_i^\alpha]$, where $[\cdot]$ is the Teichmüller lift.

PROPOSITION 3.3. — *In the case $R = O(c)$, there is a canonical isomorphism*

$$M_\infty^+/p^n M_\infty^+ \simeq A_{\text{cris},\infty}(R)/p^n A_{\text{cris},\infty}(R) \langle X, X_2, X_3, \dots, X_{d+1} \rangle$$

where the X, X_2, \dots, X_{d+1} are indeterminates. In particular, $M_\infty^+/p^n M_\infty^+$ is a flat $W_n(k)$ -module.

Proof. — The flatness claim will follow from Proposition 3.2 once we show that $M_\infty^+/p^n M_\infty^+$ has the desired form. For this we check the universal property. Define

$$\underline{T}_i := (T_i, T_i^{(1)}, T_i^{(2)}, \dots) \in P(\tilde{R}_\infty/p\tilde{R}_\infty)$$

and $\underline{c} := (c, c_1, c_2, \dots) = \prod_{i=1}^r \underline{T}_i$. Consider the ring

$$C := W_n(P(\tilde{R}_\infty/p\tilde{R}_\infty)) \left[X, \frac{1}{1+X}, X_2, \frac{1}{1+X_2}, \dots, X_{d+1}, \frac{1}{1+X_{d+1}} \right].$$

Make C into a $W_n[u][T_1, \dots, T_r, T_{r+1}^{\pm 1}, \dots, T_{d+1}^{\pm 1}]/(\prod_{i=1}^r T_i - v)$ -algebra as follows. If $c = \pi$ (resp. $c = 1$), then send u to $[\underline{\pi}] \cdot (1+X)^{-1}$ (resp. u to $[\underline{\pi}] - X$), T_1 to $[\underline{T}_1] \frac{\prod_{i=2}^r (1+X_i)}{1+X}$ (resp. T_1 to $[\underline{T}_1] \frac{\prod_{i=2}^r (1+X_i)}{[\underline{c}]}$), and T_i to $[\underline{T}_i] \cdot (1+X_i)^{-1}$ for $2 \leq i \leq d+1$. For an affine object $(U \hookrightarrow T)$ of the site $(\tilde{R}_\infty/p\tilde{R}_\infty|\Theta(c)/p^n\Theta(c))_{\log\text{-crys}}$ define a map

$$C \rightarrow \mathcal{O}_T(T)$$

extending the canonical map $\theta_T : W_n(P(\tilde{R}_\infty/p\tilde{R}_\infty)) \rightarrow \mathcal{O}_T$ (see the proof of Theorem 1.1) by sending X to $\theta_T([\underline{c}]) \cdot \pi^{-1} - 1$ (resp. X to $\theta_T([\underline{\pi}]) - u$) and X_i to $\theta_T([\underline{T}_i]) \cdot T_i^{-1} - 1$ (note that by Lemma 4.1 these elements exist in $\mathcal{O}_T(T)$). One checks easily that this is the unique map of $W_n[u][T_1, \dots, T_r, T_{r+1}^{\pm 1}, \dots, T_{d+1}^{\pm 1}]/(\prod_{i=1}^r T_i - v)$ -algebras extending θ_T . It follows from Lemma 3.1 that the kernel of the map to $C \rightarrow \tilde{R}_\infty/p\tilde{R}_\infty$ is the ideal generated by p, ξ , and X, X_2, \dots, X_{d+1} . So the divided power hull C^{DP} of C for this ideal is precisely $A_{\text{cris}, \infty}(R) \langle X, X_2, X_3, \dots, X_{d+1} \rangle / (p^n)$. By the uniqueness of θ_T we see that it suffices now to define a log-structure on C^{DP} such that $C^{\text{DP}} \in \text{Ob}(\tilde{R}_\infty/p\tilde{R}_\infty|\Theta(c)/p^n\Theta(c))_{\log\text{-crys}}$, i.e. the surjection $C^{\text{DP}} \rightarrow \tilde{R}_\infty/p\tilde{R}_\infty$ is an exact closed immersion. We show this in the case $c = \pi$, the case $c = 1$ being similar. Recall (§2.2.5) that the log-structure $M_{\tilde{R}_\infty}$ is associated to

$$\begin{aligned} \mu : \mathbb{N}[1/p]^r &\rightarrow \tilde{R}_\infty/p\tilde{R}_\infty \\ (\alpha_1, \dots, \alpha_r) &\mapsto \prod_{i=1}^r T_i^{\alpha_i}. \end{aligned}$$

Let $Q = P(\mathbb{N}[1/p]^r) \oplus \mathbb{N}^r$ and let $\lambda : Q \rightarrow \mathbb{N}[1/p]^r$ the map induced by the natural maps $P(\mathbb{N}[1/p]^r) \rightarrow \mathbb{N}[1/p]^r$ and $\mathbb{N}^r \subset \mathbb{N}[1/p]^r$. Define $L := \ker(\lambda^{\text{gp}} : Q^{\text{gp}} \rightarrow \mathbb{Z}^r)$ and let $QL \subset Q^{\text{gp}}$ be the submonoid consisting of elements of the form ql with $q \in Q$ and $l \in L$. Since

$$P(\mathbb{N}[1/p]^r) \cong \mathbb{N}[1/p]^r : (\alpha \cdot p^{-n})_{n \in \mathbb{N}} \mapsto \alpha$$

we have

$$P(\mathbb{N}[1/p]^r)^{\text{gp}} \cong \mathbb{Z}[1/p]^r.$$

So L is the kernel of the map

$$\begin{aligned} &\mathbb{Z}[1/p]^r \oplus \mathbb{Z}^r \rightarrow \mathbb{Z}[1/p]^r \\ &((\alpha_1, \dots, \alpha_r), (n_1, \dots, n_r)) \mapsto (\alpha_1 + n_1, \dots, \alpha_r + n_r). \end{aligned}$$

That is, L consists of the tuples $((-n_1, \dots, -n_r), (n_1, \dots, n_r))$ with $n_i \in \mathbb{Z}$.

Define $1 + X_1 := \frac{1 + X}{\prod_{i=2}^r (1 + X_i)}$. Note that

$$X, X_1, \dots, X_r \in \ker(C^{\text{DP}} \rightarrow \tilde{R}_\infty/p\tilde{R}_\infty)$$

and hence $1 + X, 1 + X_1, \dots, 1 + X_r$ are units of C^{DP} . We endow C^{DP} with the log-structure associated to

$$QL \rightarrow C^{\text{DP}}$$

induced by the maps

$$\begin{aligned} P(\mathbb{N}[1/p]^{r+1}) &\rightarrow C^{\text{DP}} : (m^{(n)})_{n \in \mathbb{N}} \rightarrow [(\mu(m^{(n)}))_{n \in \mathbb{N}}] \\ \mathbb{N}^r &\rightarrow \Theta(c)/p^n\Theta(c) \rightarrow C^{\text{DP}} \end{aligned}$$

$$L \rightarrow C^{\text{DP}} : ((-n_1, \dots, -n_r), (n_1, \dots, n_r)) \mapsto \prod_{i=1}^r (1 + X_i)^{-n_i}.$$

It is easy to check that these maps are compatible and do indeed define a map $QL \rightarrow C^{\text{DP}}$. With the natural maps $QL \rightarrow \mathbb{N}[1/p]^r$ and $C^{\text{DP}} \rightarrow \tilde{R}_\infty/p\tilde{R}_\infty$, we get a closed immersion of log schemes. Now by the same argument as that of Step 2 of the proof of Theorem 1.1 we see that it is an exact closed immersion, and hence we are done. \square

3.2. Computations in Galois cohomology

3.2.1.

Note that $\Delta_\infty \cong \mathbb{Z}_p(1)^d$. Let $\sigma_2, \dots, \sigma_{d+1}$ be a choice of topological generators of Δ_∞ . For each $2 \leq i \leq d + 1$, we have $\sigma_i(T_i^{(n)}) = \zeta_{p^n} T_i^{(n)}$ for a p^n th root of unity ζ_{p^n} . Define

$$\underline{1} := (1, \zeta_p, \zeta_{p^n}, \dots) \in P(\bar{K}^+/\bar{K}^+)$$

so that $\sigma(T_i) = \underline{1} \cdot T_i$. Furthermore, define

$$t := \log([\underline{1}]).$$

Let us make some remarks on the Galois cohomology of the group Δ_∞ . Let $\mathbb{Z}_p[[\Delta_\infty]]$ be the completion of the group ring $\mathbb{Z}_p[\Delta_\infty]$ in the topology

induced by the profinite topology of Δ_∞ . Then for any discrete p -power torsion Δ_∞ -module N , we have canonical isomorphisms for all i

$$\text{Ext}_{\mathbb{Z}_p[[\Delta_\infty]]}^i(\mathbb{Z}_p, N) \cong H^i(\Delta_\infty, N)$$

where the Ext-group is taken in the category of topological $\mathbb{Z}_p[[\Delta_\infty]]$ -modules. Since $\Delta_\infty \cong \mathbb{Z}_p(1)^d$, we have an isomorphism of rings

$$\mathbb{Z}_p[[\Delta_\infty]] \simeq \mathbb{Z}_p[[\sigma_2 - 1, \dots, \sigma_{d+1} - 1]].$$

This implies that the Koszul complex $L := \otimes_{\mathbb{Z}_p[[\Delta_\infty]]} L_i$, where L_i is the complex defined

$$0 \longrightarrow \mathbb{Z}_p[[\Delta_\infty]] \xrightarrow{\sigma_i - 1} \mathbb{Z}_p[[\Delta_\infty]] \longrightarrow 0,$$

is a homological resolution of \mathbb{Z}_p by free compact $\mathbb{Z}_p[[\Delta_\infty]]$ -modules. Then we have an isomorphism of complexes (up to shifting the degree)

$$\text{Hom}_{\mathbb{Z}_p[[\Delta_\infty]]}(L, N) \cong L \otimes_{\mathbb{Z}_p[[\Delta_\infty]]} N$$

i.e. the Galois cohomology of N can be computed using the Koszul complex $L \otimes_{\mathbb{Z}_p[[\Delta_\infty]]} N$.

3.2.2.

LEMMA 3.4. — Assume $R = O(c)$. Every element of $A_{\text{cris}, \infty}(R)/(p^n)$ can be written as a finite sum of the form

$$\sum_{k \geq 0} x_k \xi^{[k]}$$

with $x_k \in W_n(P(\tilde{R}_\infty/p\tilde{R}_\infty))$ of the form

$$\sum_m p^m v_m \prod_{i=1}^{d+1} [T_i]^{\alpha_{i,m}}$$

where $\alpha_{i,m} \in \mathbb{N}[1/p]$ for $1 \leq i \leq r$ and $\alpha_{i,m} \in \mathbb{Z}[1/p]$ for $r + 1 \leq i \leq d + 1$ and $v_m \in A_{\text{inf}}(K^+)$.

Proof. — The lemma will follow from Lemma 3.1 once we show that we can get the x_k in the desired form. Since the ring of Witt vectors $W_n(A)$ of a perfect ring A of characteristic p is equal to its subring of elements of the form $\sum_m p^m [a_m]$, where $a_m \in A$ and $[\cdot]$ denotes the Teichmüller lift, it suffices to prove the claim modulo p . By definition, we obtain an element of $P(\tilde{R}_\infty/p\tilde{R}_\infty)$ by taking roots of $r \in \tilde{R}_\infty/p\tilde{R}_\infty$. We may write

$$r = \sum_N v_N \prod_{i=1}^{d+1} T_i^{n_i}$$

where $v_N \in \bar{K}^+/p\bar{K}^+$ and $n_i \in \mathbb{N}[1/p]$. Make a choice of p -power roots of elements of $\bar{K}^+/p\bar{K}^+$ (this will not affect the statement of the lemma). If $v \in \bar{K}^+/p\bar{K}^+$, then let $\underline{v} := (v, v^{1/p}, v^{1/p^2}, \dots) \in P(\bar{K}^+/p\bar{K}^+)$ for the made choice of p -power roots of v . Define

$$r := \sum_N \frac{v_N}{N} \prod_{i=1}^{d+1} \frac{T_i^{n_i}}{i}$$

Taking p th roots of r we get

$$\begin{aligned} r^{(1)} &= \sum_N v_N^{1/p} \prod_{i=1}^{d+1} T_i^{(1)n_i} + p^{a_1/p} r_1 \\ r^{(2)} &= \sum_N v_N^{1/p^2} \prod_{i=1}^{d+1} T_i^{(2)n_i} + p^{a_1/p^2} r_1^{p/p^2} + p^{a_2 p/p^2} r_2 \\ &\dots \\ (r^{(n)}) &= \underline{r} + \underline{p}^{a_1} \underline{r}_1 + \underline{p}^{a_2 p} \underline{r}_2 + \dots \end{aligned}$$

where $a_i \in \mathbb{N}$. Now recall that \underline{p} has divided powers in

$$A_{\text{cris},\infty}(R)/pA_{\text{cris},\infty}(R)$$

so we get

$$(r^{(n)}) = \underline{r} + \underline{p}^{a_1} \underline{r}_1.$$

The lemma follows. □

3.2.3.

For all $2 \leq i \leq d$ we define a sub- \bar{K}^+ -algebra $\tilde{O}(c)_\infty^{(i)} \subset \tilde{O}(c)_\infty$ as follows:

- if $2 \leq i \leq r$, then $\tilde{O}(c)_\infty^{(i)}$ is the sub- \bar{K}^+ -algebra of $\tilde{O}(c)_\infty$ generated by $T_j^{\beta_j}$, $j \notin \{1, i\}$, $2 \leq j \leq d + 1$, $\beta_j \in \mathbb{N}[1/p]$, and $(T_i T_1)^\beta$ with $\beta \in \mathbb{N}[1/p]$
- if $r + 1 \leq i \leq d + 1$ then $\tilde{O}(c)_\infty^{(i)}$ is the sub- \bar{K}^+ -algebra of $\tilde{O}(c)_\infty$ generated by $T_j^{\beta_j}$, $j \neq i$, $2 \leq j \leq d + 1$, $\beta_j \in \mathbb{N}[1/p]$.

LEMMA 3.5. — $\tilde{O}(c)_\infty$ is faithfully flat over $\tilde{O}(c)_\infty^{(i)}$.

Proof. — The case $c = 1$ being trivial, we only consider the case $c = \pi$. First assume $2 \leq i \leq r$. Write $\tilde{O}(c)_\infty^{(i)} = \left(\varinjlim_n O(c)_n^{(i)}\right) \otimes_{K_n^+} \bar{K}^+$ where

$$O(c)_n^{(i)} = \frac{K_n^+[Z^{(n)}, T_j^{(n)}]_{2 \leq j \leq d+1, j \neq i}}{(Z^{(n)} \prod_{2 \leq j \leq r, j \neq i} T_j^{(n)} - c_n)}$$

where $Z^{(n)} = T_1^{(n)}T_i^{(n)}$. It suffices to check that $O(c)_n^{(i)} \rightarrow O(c)_n$ is faithfully flat. Note that

$$O(c)_n = O(c)_n^{(i)}[T_1^{(n)}, T_i^{(n)}]/(Z^{(n)} - T_1^{(n)}T_i^{(n)}).$$

Since $O(c)_n$ is an integral domain, it suffices to check that $O(c)_n^{(i)}/(Z^{(n)}) \rightarrow O(c)_n/(Z^{(n)})$ is flat. But $O(c)_n/(Z^{(n)}) = \frac{k[T_1^{(n)}, T_i^{(n)}]}{(T_1^{(n)}T_i^{(n)})} \otimes_k O(c)_n^{(i)}/(Z^{(n)})$ and the claim follows. For $r + 1 \leq i \leq d + 1$ the claim is obvious. \square

LEMMA 3.6. — *Fix some i , $2 \leq i \leq d + 1$, and let $\mathbb{Z}_p(1)$ be the factor of Δ_∞ corresponding to σ_i . There is an isomorphism of $\tilde{O}(c)_\infty^{(i)}[\mathbb{Z}_p(1)]$ -modules*

$$\tilde{O}(c)_\infty = \bigoplus_{\alpha \in \mathbb{Z}[1/p]} E_\alpha$$

where E_α is the σ_i -eigenspace of eigenvalue $\zeta_{p^r}^\alpha$, where $\alpha = n/p^r$ with $(n, p) = 1$. E_α is a free $\tilde{O}(c)_\infty^{(i)}$ -module of rank 1 with generator e_α given by

- if $2 \leq i \leq r$ and $\alpha \geq 0$, then $e_\alpha = T_i^\alpha$
- if $2 \leq i \leq r$ and $\alpha < 0$, then $e_\alpha = T_1^{-\alpha}$
- if $r + 1 \leq i \leq d + 1$, then $e_\alpha = T_i^\alpha$.

Proof. — First assume that $2 \leq i \leq r$. Clearly, every element $f \in \tilde{O}(c)_\infty$ has the form $f = \sum_{\beta, \gamma \in \mathbb{N}[1/p]} x_{\beta, \gamma} T_i^\beta T_1^\gamma$ with $x_{\beta, \gamma}$ in the sub- \bar{K}^+ -algebra of $\tilde{O}(c)_\infty$ generated by $T_j^{\beta_j}$, $j \notin \{1, i\}$, $2 \leq j \leq d + 1$, $\beta_j \in \mathbb{N}[1/p]$. So we can write

$$f = \sum_{\alpha \geq 0} \sum_{\beta - \gamma = \alpha} x_{\beta, \gamma} (T_i T_1)^\gamma T_i^\alpha + \sum_{\alpha < 0} \sum_{\beta - \gamma = \alpha} x_{\beta, \gamma} (T_i T_1)^\beta T_1^{-\alpha}.$$

Hence

$$\tilde{O}(c)_\infty = \sum_{\alpha \in \mathbb{Z}[1/p]} E_\alpha.$$

Let $X_\alpha = \sum_{\beta - \gamma = \alpha} x_{\beta, \gamma} (T_i T_1)^\gamma$ if $\alpha \geq 0$ and $X_\alpha = \sum_{\beta - \gamma = \alpha} x_{\beta, \gamma} (T_i T_1)^\beta$ otherwise. It remains to show that the writing

$$f = \sum_{\alpha \geq 0} X_\alpha T_i^\alpha + \sum_{\alpha < 0} X_\alpha T_1^{-\alpha}$$

is unique. Equivalently it suffices to show that if $f = 0$ then $X_\alpha = 0$ for all α . It suffices to show this for the image of f in $\tilde{O}(c)_\infty[1/p] \cong \varinjlim_n \bar{K}[T_2^{\pm p^{-n}}, \dots, T_{d+1}^{\pm p^{-n}}]$, where it is evident. The case $i > r$ is similar. \square

In the sequel we often write $A_{\text{cris},\infty} := A_{\text{cris},\infty}(O(c))$. For $2 \leq i \leq d$ define a sub- $A_{\text{cris}}(K^+)$ -algebra $A_{\text{cris},\infty}^{(i)}/p^n A_{\text{cris},\infty}^{(i)} \subset A_{\text{cris},\infty}/p^n A_{\text{cris},\infty}$ as follows:

- if $2 \leq i \leq r$, then $A_{\text{cris},\infty}^{(i)}/p^n A_{\text{cris},\infty}^{(i)}$ the sub- $A_{\text{cris}}(K^+)$ -algebra of $A_{\text{cris},\infty}/p^n A_{\text{cris},\infty}$ generated by $[T_j]^{\beta_j}$ with $j \notin \{1, i\}$, $2 \leq j \leq d+1$, $\beta_j \in \mathbb{N}[1/p]$, and $[T_i T_1]^\beta$ with $\beta \in \mathbb{N}[1/p]$
- if $r+1 \leq i \leq d+1$, then $A_{\text{cris},\infty}^{(i)}/p^n A_{\text{cris},\infty}^{(i)}$ the sub- $A_{\text{cris}}(K^+)$ -algebra of $A_{\text{cris},\infty}/p^n A_{\text{cris},\infty}$ generated by $[T_j]^{\beta_j}$, $j \neq i$, $2 \leq j \leq d+1$, $\beta_j \in \mathbb{N}[1/p]$.

PROPOSITION 3.7. — Fix some i , $2 \leq i \leq d+1$, and let $\mathbb{Z}_p(1)$ be the factor of Δ_∞ corresponding to σ_i . There is an isomorphism of $A_{\text{cris},\infty}^{(i)}/p^n A_{\text{cris},\infty}^{(i)}$ $[\mathbb{Z}_p(1)]$ -modules

$$A_{\text{cris},\infty}(O(c))/p^n A_{\text{cris},\infty}(O(c)) \cong \bigoplus_{\alpha \in \mathbb{Z}[1/p]} F_\alpha$$

where F_α is the σ_i -eigenspace of eigenvalue $[1]^\alpha$. F_α is a free $A_{\text{cris},\infty}^{(i)}/p^n A_{\text{cris},\infty}^{(i)}$ -module of rank 1 with generator e_α given by

- if $2 \leq i \leq r$ and $\alpha \geq 0$, then $e_\alpha = [T_i]^\alpha$
- if $2 \leq i \leq r$ and $\alpha < 0$, then $e_\alpha = [T_1]^{-\alpha}$
- if $r+1 \leq i \leq d+1$, then $e_\alpha = [T_i]^\alpha$.

Proof. — First assume $2 \leq i \leq r$. If $n = 1$, then by Proposition 3.2 we have an isomorphism

$$A_{\text{cris},\infty}/pA_{\text{cris},\infty} \cong (\tilde{O}(c)_\infty/p\tilde{O}(c)_\infty) \otimes_{\bar{K}^+/p\bar{K}^+} A_{\text{cris}}(K^+)/pA_{\text{cris}}(K^+)$$

which sends T_i to $T_i^{(1)} \otimes 1$. It follows from Lemma 3.5 that under this isomorphism we have

$$A_{\text{cris},\infty}^{(i)}/pA_{\text{cris},\infty}^{(i)} \cong (\tilde{O}(c)_\infty^{(i)}/p\tilde{O}(c)_\infty^{(i)}) \otimes_{\bar{K}^+/p\bar{K}^+} A_{\text{cris}}(K^+)/pA_{\text{cris}}(K^+)$$

hence in this case the claim follows from the last lemma.

Now we show the result by induction on n . Assume it true for $n - 1$. From Lemma 3.4 we see that we can write any $a \in A_{\text{cris},\infty}/p^n A_{\text{cris},\infty}$ in the form

$$a = \sum_{\beta,\gamma} x_{\beta,\gamma} [T_i]^\beta [T_1]^\gamma = \sum_{\alpha \geq 0} \sum_{\beta-\gamma=\alpha} x_{\beta,\gamma} [T_i T_1]^\gamma [T_i]^\alpha + \sum_{\alpha < 0} \sum_{\beta-\gamma=\alpha} x_{\beta,\gamma} [T_i T_1]^\beta [T_1]^{-\alpha}$$

with $x_{\beta,\gamma}$ in the sub- $A_{\text{cris}}(K^+)$ -algebra of $A_{\text{cris},\infty}/p^n A_{\text{cris},\infty}$ generated by $[T_j]^{\beta_j}$ with $j \notin \{1, i\}$, $2 \leq j \leq d+1$, $\beta_j \in \mathbb{N}[1/p]$. Define $X_\alpha := x_{\beta,\gamma} [T_i T_1]^\gamma$ if $\alpha \geq 0$ and $X_\alpha := x_{\beta,\gamma} [T_i T_1]^\beta$ otherwise. It remains to show that if $a = 0$, then $X_\alpha = 0$ for all α . We have an exact sequence

$$0 \rightarrow p^{n-1} A_{\text{cris},\infty} / p^n A_{\text{cris},\infty} \rightarrow A_{\text{cris},\infty} / p^n A_{\text{cris},\infty} \rightarrow A_{\text{cris},\infty} / p^{n-1} A_{\text{cris},\infty} \rightarrow 0$$

and if $a = 0$, then $X_\alpha \equiv 0 \pmod{p^{n-1}}$ by induction hypothesis. So $X_\alpha = p^{n-1} Y_\alpha$ for all α . Since by flatness over $W_n(k)$ we have

$$p^{n-1} A_{\text{cris},\infty} / p^n A_{\text{cris},\infty} \cong A_{\text{cris},\infty} / p A_{\text{cris},\infty}$$

(Prop. 3.2) we see that $Y_\alpha = 0$ for all α and this completes the proof for $2 \leq i \leq r$. For $i > r$ the proof is similar. □

COROLLARY 3.8. — *Fix some i , $2 \leq i \leq d+1$, and let $\mathbb{Z}_p(1)$ be the factor of Δ_∞ corresponding to σ_i . Then*

$$t \cdot (A_{\text{cris},\infty}(O(c)) / p^n A_{\text{cris},\infty}(O(c)))^{\sigma_i=1} \subset \bigoplus_{\alpha \in \mathbb{Z}, v_p(\alpha) > 0} p^{\max\{0, n - v_p(\alpha)\}} F_\alpha.$$

Proof. — If $f \in F_\alpha$ is invariant under σ_i , then $0 = ([1]^\alpha - 1)f$ and by Proposition 1.4 there exists $b_\alpha \in A_{\text{cris}}(K^+)$ such that $b_\alpha([1]^\alpha - 1)f = p^{v_\alpha} t f$, where $v_\alpha = \max\{0, v_p(\alpha)\}$. So if $v_p(\alpha) \leq 0$, then $t f = 0$. If $v_p(\alpha) \geq 0$, then $\alpha \in \mathbb{Z}$ and so $([1]^\alpha - 1)f = \alpha u_\alpha t f$ for some unit u_α (by *loc. cit.*). So $t f$ is killed by $p^{v_p(\alpha)}$ and hence the result follows from the flatness over $W_n(k)$ (Prop. 3.2). □

COROLLARY 3.9. — *The canonical map*

$$t^d \cdot (A_{\text{cris},\infty}(O(c)) / p^n A_{\text{cris},\infty}(O(c)))^{\Delta_\infty} \rightarrow (M_\infty^+(O(c)) / p^n M_\infty^+(O(c)))^{\Delta_\infty}$$

has image in $B_{\text{log}}^+ / p^n B_{\text{log}}^+ \otimes_{\Sigma_n} \Theta(c) / p^n \Theta(c)$.

Proof. — We have $X_i = [T_i] \otimes T_i^{-1} - 1$. If $\alpha \in \mathbb{N}[1/p]$ with $v_p(\alpha) \geq 0$, then $\alpha \in \mathbb{N}$ and setting $m_\alpha := \max\{0, n - v_p(\alpha)\}$ we have

$$\begin{aligned} p^{m_\alpha} [T_i]^\alpha &= p^{m_\alpha} T_i^\alpha ([T_i] \otimes T_i^{-1} - 1 + 1)^\alpha \\ &= p^{m_\alpha} T_i^\alpha + T_i^\alpha \sum_{r>0} p^{m_\alpha} \alpha(\alpha-1) \cdots (\alpha-r+1) X_i^{[r]} \\ &= p^{m_\alpha} T_i^\alpha \end{aligned}$$

for all $1 \leq i \leq d+1$, so $p^{m_\alpha} [T_i]^\alpha \in \Theta(c) / p^n \Theta(c)$.

Now, let $a \in (A_{\text{cris},\infty}(O(c))/(p^n))^{\Delta_\infty}$. Assume that $2 \leq i \leq r$. Write $a = \sum_{\alpha \geq 0} X_\alpha [T_i]^\alpha + \sum_{\alpha < 0} X_\alpha [T_i]^{-\alpha}$. Then by the last corollary, we have

$$ta = \sum_{\alpha \geq 0, v_p(\alpha) > 0} X'_\alpha p^{m_\alpha} [T_i]^\alpha + \sum_{\alpha < 0, v_p(\alpha) > 0} X'_\alpha p^{m_\alpha} [T_i]^{-\alpha}$$

where $X'_\alpha p^{m_\alpha} = tX_\alpha$. That is,

$$ta = \sum_{\alpha \in \mathbb{N}, v_p(\alpha) > 0} X'_\alpha p^{m_\alpha} T_i^\alpha + \sum_{-\alpha \in \mathbb{N}, v_p(\alpha) > 0} X'_\alpha p^{m_\alpha} T_i^{-\alpha}$$

Now we pick some $j \neq i$, $2 \leq j \leq r$, and apply a similar reasoning to the X'_α to deduce that

$$tX'_\alpha = \sum_{\beta \in \mathbb{N}, v_p(\beta) > 0} Y_\beta p^{m_\beta} T_j^\beta + \sum_{-\beta \in \mathbb{N}, v_p(\beta) > 0} Y_\beta p^{m_\beta} T_j^{-\beta}$$

where the Y_β lie in the sub- $A_{\text{cris}}(K^+)$ -algebra of $A_{\text{cris},\infty}(O(c))/(p^n)$ generated by $[T_j T_i T_1]^\beta$, $\beta \in \mathbb{N}[1/p]$, and $[T_k]^{\beta k}$, $k \notin \{1, i, j\}$, $2 \leq k \leq d + 1$, $\beta_k \in \mathbb{N}[1/p]$. The case for $r + 1 \leq j \leq d + 1$ is similar. Continuing in this way we clearly obtain the result. □

THEOREM 3.10. — *We have*

$$t^d[c] \cdot (M_\infty^+(R)/p^n M_\infty^+(R))^{\Delta_\infty} \subset B_{\log}^+ / p^n B_{\log}^+ \otimes_{\Sigma_n} \mathcal{R}_n.$$

Proof. — We will give the proof in several steps.

Step 0. — Recall that by quasi-coherence of $h_{\infty,*} \mathcal{O}$, we have a canonical isomorphism

$$M_\infty^+(R)/p^n M_\infty^+(R) \cong M_\infty^+(O(c))/p^n M_\infty^+(O(c)) \otimes_{\Theta(c)/p^n \Theta(c)} \mathcal{R}_n.$$

Hence we may assume that $R = O(c)$. In this case, by Proposition 3.3 we have

$$M_\infty^+ / p^n M_\infty^+ \cong A_\infty^+ / p^n A_\infty^+ \langle X_2, \dots, X_{d+1} \rangle.$$

For the proof it will be better for us to replace X resp. X_i by $Y := \frac{-X}{1+X}$ resp. $Y_i := \frac{-X_i}{1+X_i}$, $2 \leq i \leq d + 1$. Then we have a canonical isomorphism

$$M_\infty^+ / p^n M_\infty^+ \cong A_\infty^+ / p^n A_\infty^+ \langle Y_2, \dots, Y_{d+1} \rangle$$

and morally $Y = [\pi]^{-1} \otimes u - 1$ if $c = \pi$ and $Y_i = [T_i]^{-1} \otimes T_i - 1$. Let D_i denote the subalgebra of $M_\infty^+ / p^n M_\infty^+$ consisting of elements of the form $\sum_N a_N \prod_{j,j \neq i} X_j^{[n_j]}$ where $a_N \in A_\infty^+ / p^n A_\infty^+$. That is,

$$D_i = A_\infty^+ / p^n A_\infty^+ \langle Y_2, \dots, \hat{Y}_i, \dots, Y_{d+1} \rangle$$

where \hat{Y}_i means we omit Y_i . We also define

$$D_i^{(i)} := A_{\text{cris},\infty}^{(i)} / p^n A_{\text{cris},\infty}^{(i)} \langle Y, Y_2, \dots, \hat{Y}_i, \dots, Y_{d+1} \rangle$$

Step 1. — Suppose that $m \in M_\infty^+/p^n M_\infty^+$ is t -torsion. Write $m = \sum_n m_n Y_i^{[n]}$. Then m_n is t -torsion for all n . Write

$$m_n = \sum_{\alpha \geq 0} X_{\alpha,n} [T_i]^\alpha + \sum_{\alpha < 0} X_{\alpha,n} [T_i]^{-\alpha}$$

with $X_{\alpha,n} \in D_i^{(i)}$. Then $tX_{\alpha,n} = 0$ for all α (cf. Prop. 3.7). We have

$$\begin{aligned} (\sigma_i - 1)([T_i]^\alpha Y_i^{[n]}) &= [\mathbb{1}]^\alpha [T_i]^\alpha ([\mathbb{1}]^{-1} [T_i]^{-1} \otimes T_i - 1)^{[n]} - [T_i]^\alpha Y_i^{[n]} \\ &= [\mathbb{1}]^{\alpha-n} [T_i]^\alpha ([T_i]^{-1} \otimes T_i - 1 + (1 - [\mathbb{1}]))^{[n]} - [T_i]^\alpha Y_i^{[n]} \\ &= [\mathbb{1}]^{\alpha-n} [T_i]^\alpha \sum_{r=0}^n Y_i^{[r]} (1 - [\mathbb{1}])^{[n-r]} - [T_i]^\alpha Y_i^{[n]} \\ &= ([\mathbb{1}]^{\alpha-n} - 1) [T_i]^\alpha Y_i^{[n]} + [\mathbb{1}]^{\alpha-n} [T_i]^\alpha \sum_{r=0}^{n-1} Y_i^{[r]} (1 - [\mathbb{1}])^{[n-r]}. \end{aligned}$$

Similarly

$$\begin{aligned} (\sigma_i - 1)([T_i]^{-\alpha} Y_i^{[n]}) &= ([\mathbb{1}]^{-\alpha-n} - 1) [T_i]^{-\alpha} Y_i^{[n]} + [\mathbb{1}]^{-\alpha-n} [T_i]^{-\alpha} \sum_{r=0}^{n-1} Y_i^{[r]} (1 - [\mathbb{1}])^{[n-r]}. \end{aligned}$$

Now recall (Prop. 1.4) that $1 - [\mathbb{1}] = tw$ for some unit $w \in A_{\text{cris}}(K^+)$, hence

$$(1 - [\mathbb{1}])^{[n-r]} = t \frac{w^{n-r} t^{n-r-1}}{(n-r)!}$$

is a multiple of t for $0 \leq r < n$. Hence

$$\begin{aligned} (\sigma_i - 1)(m) &= \sum_n Y_i^{[n]} \left(\sum_{\alpha \geq 0} X_{\alpha,n} ([\mathbb{1}]^{\alpha-n} - 1) [T_i]^\alpha + \sum_{\alpha < 0} X_{\alpha,n} ([\mathbb{1}]^{-\alpha-n} - 1) [T_i]^{-\alpha} \right). \end{aligned}$$

We'll need this later.

Step 2. — For $2 \leq i \leq r$, consider the derivation ∂_i of $M_\infty^+/p^n M_\infty^+$ defined

$$\partial_i := [T_i] \frac{\partial}{\partial Y_i}.$$

We have $\partial_i = [T_i T_i] \frac{\partial}{\partial T_i}$, hence ∂_i commutes with σ_i . Moreover, we claim that the kernel of ∂_i lies in D_i up to $[T_i]$ -torsion. Indeed, suppose that

$\partial_i(m) = 0$. Write $m = \sum_k a_k Y_i^{[k]}$ with $a_k \in D_i$. Then $[T_1]a_k = 0$ for $k \neq 0$, as claimed.

On $[T_1] \cdot M_\infty^+ / p^n M_\infty^+$ we define a one-sided inverse \int_i to ∂_i as being the unique D_i -linear map such that

$$\int_i [T_1] Y_i^{[k]} = Y_i^{[k+1]}$$

for all $k \in \mathbb{N}$.

We claim that if $c = \pi$ (resp. $c = 1$)

$$\begin{aligned} (\sigma_i - 1) \int_i [T_1 T_i] T_i^n &= \frac{1 - [1]^{n+1}}{n + 1} [T_i]^{n+1} \\ (\sigma_i - 1) \int_i T_1^n &= \frac{[1]^{1-n} - 1}{n - 1} \frac{[T_1]^{n-1} (1 + Y)}{\prod_{2 \leq j \leq r, j \neq i} (1 + Y_j)} \quad (n > 0) \\ \text{resp. } (\sigma_i - 1) \int_i T_1^n &= \frac{[1]^{1-n} - 1}{n - 1} \frac{[T_1]^{n-1} [c]}{\prod_{2 \leq j \leq r, j \neq i} (1 + Y_j)} \quad (n > 0) \end{aligned}$$

where we recall that for any integer $n \neq 0$ we have $\frac{[1]^n - 1}{n} \in A_{\text{cris}}(K^+)$ (Prop. 1.4).

Let us show this. Since M_∞^+ is flat over $W(k)$ (cf. Prop. 3.3) it suffices to show this in $M_\infty^+[1/p]$. By the binomial theorem we have

$$T_i^n = [T_i]^n (1 + Y_i)^n = [T_i]^n \sum_{r=0}^n \frac{n!}{r!} Y_i^{[n-r]}$$

hence

$$\int_i [T_1 T_i] T_i^n = [T_i]^{n+1} \sum_{r=0}^n \frac{n!}{r!} Y_i^{[n+1-r]}$$

so we have

$$\int_i [T_1 T_i] T_i^n = \frac{T_i^{n+1}}{n + 1} - \frac{[T_i]^{n+1}}{n + 1}.$$

It follows that

$$(\sigma_i - 1) \int_i [T_1 T_i] T_i^n = \frac{1 - [1]^{n+1}}{n + 1} [T_i]^{n+1}.$$

Now for the case of T_1^n ($n > 0$). It is not hard to see that for $c = \pi$

$$T_1 = \frac{[T_1](1 + Y)}{\prod_{2 \leq j \leq r} (1 + Y_j)}$$

hence

$$T_1^n = \frac{[T_1]^{n-1} (1 + Y)^n}{\prod_{2 \leq j \leq r, j \neq i} (1 + Y_j)^n} \frac{[T_1]}{(1 + Y_i)^n}.$$

Also from the binomial theorem we have

$$\frac{1}{(1 + Y_i)^n} = \sum_{m=0}^{\infty} \frac{(n + m - 1)!}{(n - 1)!} (-Y_i)^{[m]}$$

hence

$$\int_i \frac{[T_1]}{(1 + Y_i)^n} = - \sum_{m=0}^{\infty} \frac{(n + m - 1)!}{(n - 1)!} (-Y_i)^{[m+1]}$$

which is the same as

$$\int_i \frac{[T_1]}{(1 + Y_i)^n} = \frac{1}{n - 1} \left(1 - \frac{1}{(1 + Y_i)^{n-1}} \right).$$

So we have

$$\int_i T_1^n = \frac{[T_1]^{n-1} (1 + Y)^n}{(n - 1) \prod_{2 \leq j \leq r, j \neq i} (1 + Y_j)^n} - \frac{T_1^{n-1} (1 + Y)}{(n - 1) \prod_{2 \leq j \leq r, j \neq i} (1 + Y_j)}$$

and hence

$$(\sigma_i - 1) \int_i T_1^n = \frac{[1]^{1-n} - 1}{n - 1} \frac{[T_1]^{n-1} (1 + Y)}{\prod_{2 \leq j \leq r, j \neq i} (1 + Y_j)}.$$

In the case $c = 1$ we have

$$T_1 = \frac{[T_1][c]}{\prod_{2 \leq j \leq r} (1 + Y_j)}$$

and a similar argument gives the desired formula.

Step 3. — Now consider an element $m \in M_{\infty}^+ / p^n M_{\infty}^+$. Assume that $\sigma_i(m) = m$. Write $m = \sum_{k=0}^{k_0} a_k Y_i^{[k]}$ with $a_k \in D_i$. Let us show by induction on k_0 that $t[T_i T_1]m \in t[T_i T_1] \cdot D_i^{\sigma_i=1}[T_i, T_1]$. Here by $D_i^{\sigma_i=1}[T_i, T_1]$ we mean the $D_i^{\sigma_i=1}$ -subalgebra of $M_{\infty}^+ / p^n M_{\infty}^+$ generated by T_i and T_1 (not the polynomial algebra!). For $k_0 = 0$ this is trivial. Since $\partial_i(m) = \sum_{k=1}^{k_0} [T_1] a_k Y_i^{[k-1]}$ we can assume the claim is true for $t[T_i T_1] \partial_i(m)$. That is, we can write $t[T_i T_1] \partial_i(m)$ as a finite sum

$$t[T_i T_1] \partial_i(m) = t[T_i T_1] \sum_n b_n T_i^n + t[T_i T_1] \sum_{n>0} c_n T_1^n$$

where $b_n, c_n \in D_i^{\sigma_i=1}$, $n \in \mathbb{N}$. So we may write

$$[T_i T_1] \partial_i(m) = [T_i T_1] \sum_n b_n T_i^n + [T_i T_1] \sum_{n>0} c_n T_1^n + \tau$$

where τ is t -torsion. Also, if we define

$$a := m - \int_i \partial_i(m)$$

then we have

$$\partial_i(a) = 0$$

hence $[T_1]a \in [T_1] \cdot D_i$.

We now assume that $c = \pi$. The following argument applies to the case $c = 1$ with the formulas obtained in Step 2.

Substituting the expressions for $(\sigma_i - 1) \int_i [T_i T_1] T_i^n$ and $(\sigma_i - 1) \int_i T_1^n$ obtained in Step 2, we find

$$(3.1) \quad \begin{aligned} (1 - \sigma_i)([T_i T_1]a) &= \sum_n b_n \frac{1 - [\mathbb{1}]^{n+1}}{n + 1} [T_i]^{n+1} \\ &+ \sum_{n>0} \frac{[T_i T_1] c_n}{n} \frac{[\mathbb{1}]^{1-n} - 1}{n - 1} \frac{[T_1]^{n-1} (1 + Y)^n}{\prod_{2 \leq j \leq r, j \neq i} (1 + Y_j)^n} + (\sigma_i - 1) \int_i \tau. \end{aligned}$$

Now, since $[T_i T_1]a \in [T_i T_1] \cdot D_i$ we may write

$$[T_i T_1]a = \sum_{\alpha \in \mathbb{N}[1/p]} [T_i T_1] X_\alpha [T_i]^\alpha + \sum_{\alpha \in \mathbb{Z}[1/p], \alpha < 0} [T_i T_1] X_\alpha [T_1]^{-\alpha}$$

with X_α in $D_i^{(i)}$. Hence

$$(3.2) \quad \begin{aligned} (1 - \sigma_i)([T_i T_1]a) &= \sum_{\alpha \in \mathbb{N}[1/p]} [T_i T_1] X_\alpha (1 - [\mathbb{1}]^\alpha) [T_i]^\alpha \\ &+ \sum_{\alpha \in \mathbb{Z}[1/p], \alpha < 0} [T_i T_1] X_\alpha (1 - [\mathbb{1}]^\alpha) [T_1]^{-\alpha}. \end{aligned}$$

Comparing the equations (3.1) and (3.2) we claim that we get an equality of coefficients of $[T_i]^\alpha$ and $[T_1]^{-\alpha}$. Indeed, since the $Y, Y_j, 2 \leq j \leq d + 1, j \neq i$, are independent indeterminates we reduce to the case where the coefficients lie in $A_{\text{cris}, \infty}(O(c))/p^n A_{\text{cris}, \infty}(O(c))$ in which case it follows from Proposition 3.7. Now write

$$\int_i \tau = \sum_{n>0} Y_i^{[n]} \left(\sum_{\alpha \geq 0} X_{\alpha, n} [T_i]^\alpha + \sum_{\alpha < 0} X_{\alpha, n} [T_1]^{-\alpha} \right)$$

with $X_{\alpha, n} \in D_i^{(i)}$ as in Step 1. Since $\int_i \tau$ is t -torsion, by Step 1 we have

$$\begin{aligned} (\sigma_i - 1) \left(\int_i \tau \right) &= \\ &\sum_{n>0} Y_i^{[n]} \left(\sum_{\alpha \geq 0} X_{\alpha, n} ([\mathbb{1}]^{\alpha-n} - 1) [T_i]^\alpha + \sum_{\alpha < 0} X_{\alpha, n} ([\mathbb{1}]^{-\alpha-n} - 1) [T_1]^{-\alpha} \right). \end{aligned}$$

Note that in this sum we have $n > 0$. So comparing with equation (3.2) we deduce that

$$(\sigma_i - 1)\left(\int_i \tau\right) = 0.$$

Hence comparing equations (3.1) and (3.2) we find

$$(3.3) \quad 0 = t[T_i T_1] X_\alpha \quad (v_p(\alpha) < 0)$$

$$(3.4) \quad b_n \frac{1 - [\mathbb{1}]^{n+1}}{n + 1} = [T_i T_1] X_{n+1} (1 - [\mathbb{1}]^{n+1})$$

$$(3.5) \quad [T_i T_1] c_n \frac{([\mathbb{1}]^{1-n} - 1)(1 + Y)^n}{(n - 1) \prod_{2 \leq j \leq r, j \neq i} (1 + Y_j)^n} = [T_i T_1] X_{1-n} (1 - [\mathbb{1}]^{1-n}).$$

In particular, from Prop. 1.4 we deduce that

$$tb_n = [T_i T_1] X_{n+1} t(n + 1)$$

$$tc_n [T_i T_1] = (n - 1) t [T_i T_1] X_{1-n} \frac{\prod_{2 \leq j \leq r, j \neq i} (1 + Y_j)^n}{(1 + Y)^n}.$$

Hence we can write

$$t[T_i T_1] m = t[T_i T_1] a + t[T_i T_1] \sum_n X_{n+1} (T_i^{n+1} - [T_i]^{n+1})$$

$$+ t[T_i T_1] \sum_{n>0} X_{1-n} \left([T_1]^{n-1} - \frac{T_1^{n-1} \prod_{2 \leq j \leq r, j \neq i} (1 + Y_j)^{n-1}}{(1 + Y)^{n-1}} \right)$$

$$= t[T_i T_1] \left\{ a' + \sum_n X_{n+1} T_i^{n+1} - \sum_{n>0} X_{1-n} \frac{T_1^{n-1} \prod_{2 \leq j \leq r, j \neq i} (1 + Y_j)^{n-1}}{(1 + Y)^{n-1}} \right\}$$

where

$$a' := a - \sum_n X_{n+1} [T_i]^{n+1} + \sum_{n>0} X_{1-n} [T_1]^{n-1}$$

is annihilated by $t[T_i T_1]$ by equation (3.3). So we can write

$$t[T_i T_1] m =$$

$$t[T_i T_1] \left\{ \sum_n X_{n+1} T_i^{n+1} - \sum_{n>0} X_{1-n} \frac{T_1^{n-1} \prod_{2 \leq j \leq r, j \neq i} (1 + Y_j)^{n-1}}{(1 + Y)^{n-1}} \right\}$$

and this completes the induction step.

Step 4. — Now we can repeat Step 3 for any other index $j \neq i$, $2 \leq j \leq r$. In more detail, the same argument as that of Step 3 shows that if $m \in (M_\infty^+ / p^n M_\infty^+)^{\sigma_i=1, \sigma_j=1}$, then

$$t^2 [T_j T_i T_1] m \in t^2 [T_j T_i T_1] \cdot D_{i,j}^{\sigma_i=1, \sigma_j=1} [T_j, T_i, T_1]$$

where

$$D_{i,j} := A_\infty^+ / p^n A_\infty^+ \langle Y_2, \dots, \hat{Y}_i, \dots, \hat{Y}_j, \dots, Y_{d+1} \rangle.$$

It is now clear that if $m \in (M_\infty^+ / p^n M_\infty^+)^{\Delta_\infty}$, then

$$t^r [\underline{c}] m \in t^r [\underline{c}] \cdot D_{2,\dots,r}^{\sigma_2=1, \dots, \sigma_r=1} [T_1, T_2, \dots, T_r]$$

where

$$D_{2,\dots,r} := A_\infty^+ / p^n A_\infty^+ \langle Y_{r+1}, \dots, Y_{d+1} \rangle.$$

Step 5. — For an index i , $r + 1 \leq i \leq d + 1$, then we may adapt the previous steps with the following changes. We use $\partial_i := \frac{\partial}{\partial T_i}$ and let f_i be the unique D_i -linear map sending $Y_i^{[n]}$ to $[\underline{T}_i] Y_i^{[n+1]}$. Then one checks easily that we have

$$(\sigma_i - 1) \int_i T_i^n = \frac{1 - [\underline{1}]^{n+1}}{n + 1} [\underline{T}_i]^{n+1}.$$

With these changes a similar but simpler proof as that of Step 3 shows that if $m \in \ker(\sigma_i - 1)$, then $tm \in t \cdot D_i^{\sigma_i=1} [T_i]$.

Step 6. — Combining the previous steps, we deduce that if

$$m \in (M_\infty^+ / p^n M_\infty^+)^{\Delta_\infty},$$

then $t^d [\underline{c}] m \in t^d [\underline{c}] \cdot (A_\infty^+ / p^n A_\infty^+)^{\Delta_\infty} [T_1, \dots, T_{d+1}]$. Hence by Corollary 3.9 the proof is complete. □

3.2.4.

Recall that R is a small integral K^+ -algebra with K^+ integrally closed in R , and \mathcal{R}_n is the étale $\Theta(c)/p^n \Theta(c)$ -algebra lifting R/pR .

THEOREM 3.11. — *For all $i \neq 0$, the B^+ -module*

$$H^i(\Delta_\infty, M_\infty^+ / p^n M_\infty^+)$$

is annihilated by t^d .

Proof. —

Step 0. — Firstly, by Corollary 2.18, $M_\infty^+ / p^n M_\infty^+$ is a discrete p -torsion Δ_∞ -module, so the Koszul complex $L \otimes_{\mathbb{Z}_p[[\Delta_\infty]]} M_\infty^+ / p^n M_\infty^+$ computes the Galois cohomology of $M_\infty^+ / p^n M_\infty^+$ (3.2.1). We first show that for all i and all $m \in M_\infty^+ / p^n M_\infty^+$, $t \cdot m$ lies in the image of the endomorphism $\sigma_i - 1$, and from this we will deduce the statement of the theorem. Since

$$M_\infty^+(R) / p^n M_\infty^+(R) \cong M_\infty^+(O(c)) / p^n M_\infty^+(O(c)) \otimes_{\Theta(c)/p^n \Theta(c)} \mathcal{R}_n$$

we can assume that $R = O(c)$. Fix some $2 \leq i \leq d + 1$. Recall that $X_i = [\underline{T}_i] \otimes T_i^{-1} - 1$. Now every element of $M_\infty^+(O(c))/p^n M_\infty^+(O(c))$ is the sum of monomials of the form

$$\mu_1 = x[\underline{T}_1]^\alpha X_i^{[n]}, \quad \text{or} \quad \mu = x[\underline{T}_i]^\alpha X_i^{[n]}$$

with x invariant under σ_i and $\alpha \in \mathbb{N}[1/p]$. For $m = \mu_1, \mu$, we will show that $tm = (\sigma_i - 1)f_i(m)$ for some $f_i(m)$ by induction on n .

Step 1. — If $n = 0$, then we distinguish three cases: $\alpha = 0$, $v_p(\alpha) \geq 0$, and $v_p(\alpha) < 0$. If $\alpha = 0$ then take

$$f_i(m) = m \log(X_i + 1).$$

Since $\sigma_i \log(X_i + 1) = \log([\underline{1}](X_i + 1)) = t + \log(X_i + 1)$ we obtain the claim in this case. If $v_p(\alpha) \geq 0$ then $\alpha \in \mathbb{N}$ and

$$\begin{aligned} [\underline{T}_i]^\alpha &= T_i^\alpha (1 + X_i)^\alpha \\ &= T_i^\alpha \left(1 + \sum_{r=1}^\alpha \frac{\alpha!}{(\alpha - r)!} X_i^{[r]} \right) \end{aligned}$$

so

$$\begin{aligned} ([\underline{1}]^\alpha - 1)[\underline{T}_i]^\alpha &= (\sigma_i - 1)[\underline{T}_i]^\alpha \\ &= T_i^\alpha (\sigma_i - 1) \left(\sum_{r=1}^\alpha \frac{\alpha!}{(\alpha - r)!} X_i^{[r]} \right) \\ &= (\sigma_i - 1) \left(\alpha \cdot T_i^\alpha \left(\sum_{r=1}^\alpha \frac{(\alpha - 1)!}{(\alpha - r)!} X_i^{[r]} \right) \right) \end{aligned}$$

and hence

$$t[\underline{T}_i]^\alpha = (\sigma_i - 1) \left(\frac{t\alpha}{[\underline{1}]^{\alpha - 1}} T_i^\alpha \left(\sum_{r=1}^\alpha \frac{(\alpha - 1)!}{(\alpha - r)!} ([\underline{\pi}\underline{T}_i^{-1}] \otimes u^{-1}T_i - 1)^{[r]} \right) \right)$$

(recall that $\frac{t\alpha}{[\underline{1}]^{\alpha - 1}} \in A_{\text{cris}}(K^+)$ by Proposition 1.4). Similarly we have

$$t[\underline{T}_1]^\alpha = (\sigma_i - 1) \left(\frac{t\alpha}{[\underline{1}]^{\alpha - 1}} T_1^\alpha \left(\sum_{r=1}^\alpha \frac{(\alpha - 1)!}{(\alpha - r)!} X_1^{[r]} \right) \right)$$

where $X_1 = [\underline{T}_1] \otimes T_1^{-1} - 1$. If $v_p(\alpha) < 0$, then

$$t[\underline{T}_i]^\alpha = (\sigma_i - 1) \left(\frac{t}{[\underline{1}]^{\alpha - 1}} [\underline{T}_i]^\alpha \right).$$

This begins the induction.

Step 2. — Fix some $2 \leq i \leq d + 1$. Let $\partial_i := T_i \frac{\partial}{\partial T_i}$. We claim that if $m \in M_\infty^+$ and $\partial_i(m) = 0$, then $tm = (\sigma_i - 1)m'$ for some $m' \in M_\infty^+$. Indeed, write

$m = \sum_j y_j X_i^{[j]}$ with $\partial_i(y_j) = 0$ and $y_j = \sum_{\alpha \geq 0} x_\alpha [T_i]^\alpha + \sum_{\alpha < 0} x_\alpha [T_i]^{-\alpha}$ with x_α invariant under σ_i . Then

$$\partial_i(m) = \left(\sum_j y_j X_i^{[j-1]} \right) (-1)(1 + X_i) = 0$$

and since $1 + X_i$ is a unit we deduce that $y_j = 0$ for all $j \neq 0$, hence $m = y_0$. So by Step 1 above this proves the claim.

Step 3. — We claim that ∂_i is surjective. It suffices to show that $X_i^{[j]}$ lies in the image of ∂_i for all $j \in \mathbb{N}$. We have

$$\partial_i(-X_i^{[j+1]}) = X_i^{[j]} + (j + 1)X_i^{[j+1]}$$

so by induction we deduce the formula

$$X_i^{[j]} = \sum_{k=1}^{\infty} (-1)^k \frac{(j + k - 1)!}{j!} \partial_i(X_i^{[j+k]})$$

thereby proving the claim.

Step 4. — We claim that for all $\alpha \in \mathbb{N}[1/p]$ and all $n \in \mathbb{N}$, the elements

$$nt[T_1]^\alpha X_i^{[n]}, \quad nt[T_i]^\alpha X_i^{[n]}$$

are in the image of $\sigma_i - 1$. Let us show this for $nt[T_i]^\alpha X_i^{[n]}$, the argument for $nt[T_1]^\alpha X_i^{[n]}$ being the same. First of all, if $v_p(\alpha) \geq 0$ then

$$[T_i]^\alpha = T_i^\alpha ([T_i] \otimes T_i^{-1} - 1 + 1)^\alpha = T_i^\alpha \sum_{r \geq 0} \frac{\alpha!}{(\alpha - r)!} X_i^{[r]}$$

so by considering each summand individually we may assume that either $\alpha = 0$ or $v_p(\alpha) < 0$. Now, we have

$$\begin{aligned} & (\sigma_i - 1)([T_i]^\alpha X_i^{[n]}) \\ &= [\mathbb{1}]^\alpha [T_i]^\alpha ([\mathbb{1}][T_i] \otimes T_i^{-1} - 1)^{[n]} - [T_i]^\alpha X_i^{[n]} \\ &= [\mathbb{1}]^{\alpha+n} [T_i]^\alpha ([T_i] \otimes T_i^{-1} - 1 + (1 - [\mathbb{1}]^{-1}))^{[n]} - [T_i]^\alpha X_i^{[n]} \\ &= [\mathbb{1}]^{\alpha+n} [T_i]^\alpha \sum_{r=0}^n X_i^{[r]} (1 - [\mathbb{1}]^{-1})^{[n-r]} - [T_i]^\alpha X_i^{[n]} \\ &= ([\mathbb{1}]^{\alpha+n} - 1)[T_i]^\alpha X_i^{[n]} + [\mathbb{1}]^{\alpha+n} [T_i]^\alpha \sum_{r=0}^{n-1} X_i^{[r]} (1 - [\mathbb{1}]^{-1})^{[n-r]}. \end{aligned}$$

Now recall (Prop. 1.4) that $1 - [\mathbb{1}]^{-1} = tw$ for some unit $w \in A_{\text{cris}}(K^+)$, hence

$$(1 - [\mathbb{1}]^{-1})^{[n-r]} = t \frac{w^{n-r} t^{n-r-1}}{(n-r)!}$$

is a multiple of t for $0 \leq r < n$. So by induction on n we may assume that

$$[\mathbb{1}]^{\alpha+n} [T_i]^\alpha \sum_{r=0}^{n-1} X_i^{[r]} (1 - [\mathbb{1}]^{-1})^{[n-r]}$$

lies in the image of $\sigma_i - 1$. Hence

$$([\mathbb{1}]^{\alpha+n} - 1) [T_i]^\alpha X_i^{[n]}$$

lies in the image of $\sigma_i - 1$. Recall (Prop. 1.4) that we have

$$tp^{\max(v_p(\alpha+n), 0)} = b([\mathbb{1}]^{\alpha+n} - 1)$$

for some $b \in A_{\text{cris}}(K^+)$. If $\alpha = 0$, then this immediately proves the claim. If $v_p(\alpha) < 0$, then $v_p(\alpha + n) = v_p(\alpha) < 0$, and so the claim also follows.

Step 5. — Consider an element of the form

$$t[T_1]^\alpha X_i^{[n]}, \quad t[T_i]^\alpha X_i^{[n]}.$$

We have

$$\partial_i(-t[T_i]^\alpha X_i^{[n]}) = t[T_i]^\alpha X_i^{[n-1]} + nt[T_i]^\alpha X_i^{[n]}.$$

So by induction hypothesis and by Step 4 we know that

$$\partial_i(t[T_i]^\alpha X_i^{[n]}) = (\sigma_i - 1)m$$

for some $m \in M_\infty^+$. By Step 3 we know that $m = \partial_i(m_1)$ for some $m_1 \in M_\infty^+$, hence

$$\partial_i(t[T_i]^\alpha X_i^{[n]}) = \partial_i((\sigma_i - 1)(m_1)).$$

So we deduce that

$$t[T_i]^\alpha X_i^{[n]} = (\sigma_i - 1)(m_1) + m_2$$

with $\partial_i(m_2) = 0$. We claim that $m_2 = (\sigma_i - 1)(m_3)$. Since $n > 0$ and $\partial_i(m_2) = 0$, it suffices to show that the coefficient of $X_i^{[0]}$ in $(\sigma_i - 1)(m_1)$ lies in the image of $\sigma_i - 1$. To see this, write $m_1 = \sum_\alpha c_\alpha [T_i]^\alpha X_i^{[n_\alpha]}$ for some $c_\alpha \in (M_\infty^+/p^n)^{\partial_i=0, \sigma_i=1}$. Then by a computation above we have

$$(\sigma_i - 1)([T_i]^\alpha X_i^{[n_\alpha]}) = ([\mathbb{1}]^{\alpha+n_\alpha} - 1) [T_i]^\alpha X_i^{[n_\alpha]} + [\mathbb{1}]^{\alpha+n_\alpha} [T_i]^\alpha \sum_{r=0}^{n_\alpha-1} X_i^{[r]} (1 - [\mathbb{1}]^{-1})^{[n_\alpha-r]}$$

and in this sum the coefficient of $X_i^{[0]}$ is a multiple of t if $n_\alpha \neq 0$ and otherwise it is $([\mathbb{1}]^\alpha - 1)[T_i]^\alpha$ which is in the image of $\sigma_i - 1$. Hence the coefficient of $X_i^{[0]}$ in $(\sigma_i - 1)(m_1)$ lies in the image of $\sigma_i - 1$. Hence m_2 lies in the image of $\sigma_i - 1$.

Since the same argument also works for $t[T_1]^\alpha X_i^{[n]}$ this completes the induction, and proves that for all i

$$H_0(L_i \otimes_{\mathbb{Z}_p[[\Delta_\infty]]} M_\infty^+ / p^n M_\infty^+)$$

is annihilated by t , where L_i is the complex defined in (3.2.1).

Step 6. — Now, consider the Koszul complex $L = \otimes_{\mathbb{Z}_p[[\Delta_\infty]]} L_i$. For any complex K of $\mathbb{Z}_p[[\Delta_\infty]]$ -modules we have short exact sequences ([15, Ch. IV, Prop. 1])

$$0 \rightarrow H_0(L_i \otimes_{\mathbb{Z}_p[[\Delta_\infty]]} H_p(K)) \rightarrow H_p(L_i \otimes_{\mathbb{Z}_p[[\Delta_\infty]]} K) \rightarrow H_1(L_i \otimes_{\mathbb{Z}_p[[\Delta_\infty]]} H_{p-1}(K)) \rightarrow 0.$$

Applying this inductively to $K = L_{\leq e} \otimes_{\mathbb{Z}_p[[\Delta_\infty]]} M_\infty^+ / p^n M_\infty^+$, with $L_{\leq e} := \otimes_{i \leq e} L_i$ the theorem follows. □

3.2.5.

We can now tie everything together. Denote

$$C^\bullet(\Delta, -) = \text{Hom}_{\text{cont.,}\Delta}(\Delta^{\times \bullet}, -)$$

the usual functorial complex computing continuous group cohomology of Δ .

COROLLARY 3.12. — *Suppose that R is a small K^+ -algebra. Then there is a canonical morphism of the derived category*

$$B_{\log}^+ \otimes_\Sigma \omega_{\mathcal{R}_n/\Sigma_n}^\bullet \rightarrow C^\bullet(\Delta, A_{\log,\Sigma}^+ / p^n A_{\log,\Sigma}^+)$$

which is an almost quasi-isomorphism up to $t^d[\mathcal{C}]$ -torsion.

Proof. — The morphism in the derived category comes from the morphisms of complexes

$$\begin{aligned} C^*(\Delta, A^+ / p^n A^+) &\xrightarrow{\sim} C^*(\Delta, M^+ / p^n M^+ \otimes_{\mathcal{R}_n} \omega_{\mathcal{R}_n/\Sigma_n}^\bullet) \\ &\longleftarrow B_{\log}^+ / p^n B_{\log}^+ \otimes_{\Sigma_n} \omega_{\mathcal{R}_n/\Sigma_n}^\bullet \end{aligned}$$

where the morphism of the left is a quasi-isomorphism by (3.1.2). Now by Corollary 2.17, the result follows from Theorems 3.10 and 3.11. □

3.3. Kummer sequence compatibility

Let R be a small integral K^+ -algebra and let $f \in R^*$. Let \mathcal{R}_n be the ind-étale $\Theta(c)/p^n\Theta(c)$ -algebra lifting $R \otimes \mathbb{F}_p$ and let \hat{f} be a lift of $f \pmod p$ to \mathcal{R}_n . The element f defines, via the Kummer sequence, a 1-cocycle

$$\left(\delta \mapsto \frac{\delta(f^{p^{-n}})}{f^{p^{-n}}} \right) \in C^1(\Delta, \mathbb{Z}/p^n\mathbb{Z}(1)).$$

Under the logarithm $\log : \mathbb{Z}/p^n\mathbb{Z}(1) \rightarrow A^+/p^n A^+$ it maps to

$$\log \left(f^{p^{-n}} \right) := \left(\delta \mapsto \log \left(\frac{\delta(f^{p^{-n}})}{f^{p^{-n}}} \right) \right) \in C^1(\Delta, A^+/p^n A^+).$$

PROPOSITION 3.13. — We have

$$d\log(\hat{f}) = -\log \left(f^{p^{-n}} \right) \quad \text{in } H^1(\Delta, A^+/p^n A^+).$$

Proof. — We have morphisms of complexes

$$\begin{aligned} C^*(\Delta, A^+/p^n A^+) &\xrightarrow{\sim} C^*(\Delta, M^+/p^n M^+ \otimes_{\mathcal{R}_n} \omega_{\mathcal{R}_n/\Sigma_n}^\bullet) \\ &\longleftarrow B_{\log}^+/p^n B_{\log}^+ \otimes_{\Sigma_n} \omega_{\mathcal{R}_n/\Sigma_n}^\bullet \end{aligned}$$

and $d\log(\hat{f})$ is the image under d of

$$\log \left([f^{p^{-n}}]^{-1} \otimes \hat{f} \right) \in C^0(\Delta, M^+/p^n M^+)$$

and the latter has image $\log \left(f^{-p^{-n}} \right) = -\log \left(f^{p^{-n}} \right) \in C^1(\Delta, M^+/p^n M^+)$. □

4. Appendix: results from commutative algebra

4.1. Complement on log structures

4.1.1.

Part (i) of the next lemma is implicit in [12].

LEMMA 4.1. — Let $i : (X_0, M_0) \hookrightarrow (X, M)$ be a nilpotent closed immersion of log-schemes of ideal \mathcal{I} . Then

- (i) $i^*M = i^{-1}M/(1 + \mathcal{I})$; in particular, i is exact if and only if the map $i^{-1}M \rightarrow M_0$ is surjective and locally for all sections m, m' of $i^{-1}M$ with same image in M_0 , there exists $u \in 1 + \mathcal{I}$ such that $m = um'$

- (1) if X is affine, i exact, and M integral, then $\Gamma(X, M)/\Gamma(X, 1 + \mathcal{I}) = \Gamma(X_0, M_0)$.
- (ii) If i is exact, M_0 fine and saturated, and M integral, then M is fine and saturated and is given in a neighbourhood of a geometric point $\bar{x} \rightarrow X_0$ by the fine saturated monoid $P := M_{0, \bar{x}}/\mathcal{O}_{X_0, \bar{x}}^*$.

Proof. —

(i). — One first shows easily that $\mathcal{O}_{X_0}^* \cong i^{-1}\mathcal{O}_X^*/(1 + \mathcal{I})$. If $L \rightarrow \mathcal{O}_{X_0}$ is a log-structure on X_0 together with a morphism of pre-log-structures

$$i^{-1}M \rightarrow L$$

then $1 + \mathcal{I} \subset i^{-1}M$ maps to $1 \in L$, so the map factors (necessarily uniquely)

$$i^{-1}M/(1 + \mathcal{I}) \rightarrow L.$$

So the claim will follow if we can show that $i^{-1}M/(1 + \mathcal{I})$ is a log-structure. If $\alpha : i^{-1}M \rightarrow i^{-1}\mathcal{O}_X$ is the inverse image by i of the map defining M as a log-structure on X , then α induces an isomorphism $\alpha^{-1}i^{-1}\mathcal{O}_X^* \cong i^{-1}\mathcal{O}_X^*$, whence an isomorphism

$$\alpha^{-1}i^{-1}\mathcal{O}_X^*/(1 + \mathcal{I}) \cong i^{-1}\mathcal{O}_X^*/(1 + \mathcal{I})$$

i.e. $i^{-1}M/(1 + \mathcal{I}) \rightarrow i^{-1}\mathcal{O}_X/\mathcal{I} \cong \mathcal{O}_{X_0}$ is a log-structure on X_0 .

(ii). — I thank the referee for this argument. Let $s \in \Gamma(X_0, i^*M)$. Choose an étale covering $\mathcal{U} = \{U_j \rightarrow X_0\}_{j \in J}$ such that $s|_{U_j}$ is the image of $t_j \in i^{-1}(M)(U_j)$. For all $j, k \in J$, let $U_{jk} := U_j \times_{X_0} U_k$. Then since M is integral, $\{t_j/t_k \in (1 + \mathcal{I})(U_{jk})\}_{(j,k) \in J^2}$ is a one-cocycle with values in $1 + \mathcal{I}$. We claim that, up to refining \mathcal{U} , it is a coboundary. It suffices to show that $H_{\text{ét}}^1(X, 1 + \mathcal{I}) = 0$. To see this, note we have a finite filtration of $1 + \mathcal{I}$ by subgroups of the form $1 + \mathcal{I}^n$, $n > 1$, whose graded is isomorphic to $\mathcal{I}^n/\mathcal{I}^{n+1}$. Since the latter are quasi-coherent sheaves, their cohomology vanishes in degree at least 1, and the claim follows from this. So, after refining \mathcal{U} , we may assume that $t_j/t_k = u_j/u_k$ for some $u_j \in (1 + \mathcal{I})(U_j)$, $j \in J$. Then $\{u_j^{-1}t_j \in i^{-1}(M)(U_j)\}_{j \in J}$ glues to give an element of $\Gamma(X_0, i^{-1}M)$ which maps to s . So the map $\Gamma(X, M) \rightarrow \Gamma(X_0, i^*M)$ is surjective. Now the result follows from (i).

(iii). — Since M_0 is fine and saturated, P^{gp} is a finite free \mathbb{Z} -module, so there is a section $P^{\text{gp}} \rightarrow M_{0, \bar{x}}^{\text{gp}}$. This extends to a map $P \rightarrow M_{0, \bar{x}} \rightarrow \mathcal{O}_{X_0, \bar{x}}$ which is a chart of the log-structure M_0 in a neighbourhood of \bar{x} . By (i) it follows that $P = M_{\bar{x}}/\mathcal{O}_{X, \bar{x}}^*$, so similarly we get a map $s : P \rightarrow M_{\bar{x}} \rightarrow \mathcal{O}_{X, \bar{x}}$ which we claim defines a chart. Let $\mathcal{P} \rightarrow \mathcal{O}_X$ be the log-structure associated to s in a neighbourhood of \bar{x} . It comes equipped with a morphism of log-structures $\phi : \mathcal{P} \rightarrow M$ and by construction $i^*\mathcal{P} \cong M_0 = i^*M$. So if $m \in M$

is a section, there is $p \in P$ and $u \in \mathcal{O}_X^*$ such that $\phi(pu)$ and m have same image in M_0 . By (i) there is $v \in 1 + \mathcal{I}$ such that $\phi(pu) = mv$, hence $m = \phi(puv^{-1})$ so ϕ is surjective. Similarly, if $\phi(p_1u_1) = \phi(p_2u_2)$, then there is $w \in 1 + \mathcal{I}$ such that $p_1u_1 = wp_2u_2$ in \mathcal{P} . So $\phi(p_1u_1) = w\phi(p_2u_2)$ in M , whence $w = 1$ since M is integral. So $p_1u_1 = p_2u_2$, and ϕ is injective. \square

4.1.2.

Let R be a strictly henselian ring, Q and integral monoid together with a map $\alpha_Q : Q \rightarrow R$. Let $Q^a \rightarrow R$ be the log structure on $\text{Spec}(R)$ associated to α_Q , i.e. $Q^a = Q \oplus R^*/ \sim$, where $(q_1, u_1) \sim (q_2, u_2)$ ($(q_i, u_i) \in Q \oplus R^*$ for $i = 1, 2$) if and only if there are $h_1, h_2 \in \alpha_Q^{-1}(R^*)$ such that $h_1q_1 = h_2q_2$ and $h_2u_1 = h_1u_2$.

LEMMA 4.2. — (i) *The natural map $Q^{\text{gp}} \rightarrow (Q^a)^{\text{gp}}/R^*$ is surjective.*

(ii) *The natural map $Q^{\text{gp}} \rightarrow (Q^a)^{\text{gp}}$ is injective.*

Proof. —

(i). — We first note that if $f : M \rightarrow N$ is a surjective map of integral monoids, then $M^{\text{gp}} \rightarrow N^{\text{gp}}$ is surjective. Indeed, if $n_1/n_2 \in N^{\text{gp}}$ with $n_1, n_2 \in N$, then we can find $m_i \in M$ such that $f(m_i) = n_i$ for $i = 1, 2$, and then n_1/n_2 is the image of m_1/m_2 .

Consider the monoid Q^a/R^* . Since Q^a is an integral monoid (since Q is) it follows easily that Q^a/R^* is also integral, where $Q^a/R^* := Q^a/ \sim$, where $x \sim y$ if and only if there is $u \in R^*$ such that $xu = y$. Write $\bar{Q} := Q^a/R^*$. Then the natural map $Q \rightarrow \bar{Q}$ is surjective: if $q \in \bar{Q}$ then it is the image of $(q', u) \in Q^a$ for $q' \in Q$ and $u \in R^*$; since $(q', u) = (q', 1)(1, u)$ it follows the image of $(q', 1)$ in \bar{Q} is q , and so $q' \in Q$ maps to $q \in \bar{Q}$. So the map $Q^{\text{gp}} \rightarrow (\bar{Q})^{\text{gp}}$ is surjective.

Also, the map $(Q^a)^{\text{gp}} \rightarrow (\bar{Q})^{\text{gp}}$ is surjective. If $x \in (Q^a)^{\text{gp}}$ lies in its kernel, then write $x = q_1/q_2$ for $q_i \in Q^a$ for $i = 1, 2$. Then q_1 and q_2 have the same image in \bar{Q} , so $q_1 = uq_2$ for some $u \in R^*$. Hence $(Q^a)^{\text{gp}}/R^* \cong (\bar{Q})^{\text{gp}}$, and this proves (i).

(ii). — Let $x \in \ker(Q^{\text{gp}} \rightarrow (Q^a)^{\text{gp}})$. Write $x = m_1/m_2$ with $m_i \in Q$ for $i = 1, 2$. Then m_1 and m_2 have same image in Q^a , i.e. $(m_1, 1) \sim (m_2, 1)$ for the equivalence relation described before the statement of the lemma. So there are $h_1, h_2 \in \alpha_Q^{-1}(R^*)$ such that $h_1m_1 = h_2m_2$ and $h_2 \cdot 1 = h_1 \cdot 1$. So $h_1 = h_2$ and $h_1m_1 = h_1m_2$. Since Q is integral we must have $m_1 = m_2$. \square

Now let $I \subset R$ a nilpotent ideal, and $\bar{R} := R/I$. Suppose that $Q \rightarrow M$ is a surjection of integral monoids which fits in a commutative diagram of

(multiplicative) monoids

$$\begin{array}{ccc}
 Q & \longrightarrow & M \\
 \alpha_Q \downarrow & & \alpha_M \downarrow \\
 R & \longrightarrow & \bar{R}
 \end{array}$$

where the map $R \rightarrow R/I$ is the quotient map. We write Q^a for the log structure on $\text{Spec}(R)$ associated to the map α_Q and write M^a for the log structure on $\text{Spec}(\bar{R})$ associated to α_M .

LEMMA 4.3. — *With notation and hypothesis as above. Suppose that the group $L := \ker(Q^{\text{gp}} \rightarrow M^{\text{gp}})$ consists of 1-units, i.e. the image of L in $(Q^a)^{\text{gp}}$ lies in the subgroup $1 + I \subset R^* \subset Q^a \subset (Q^a)^{\text{gp}}$ (this makes sense because Q^a is a log structure). If the canonical map*

$$Q^{\text{gp}} \cap R^* \rightarrow M^{\text{gp}} \cap \bar{R}^*$$

is surjective, then we have an exact sequence

$$1 \rightarrow 1 + I \rightarrow (Q^a)^{\text{gp}} \rightarrow (M^a)^{\text{gp}} \rightarrow 1.$$

Proof. — Let $\alpha_{Q^a} : Q^a \rightarrow R$ (resp. $\alpha_{M^a} : M^a \rightarrow \bar{R}$) be the map deduced from α_Q (resp. α_M). Then the map $\alpha_{Q^a}^{-1}(R^*) \rightarrow R^*$ is an isomorphism and we simply write R^* for $\alpha_{Q^a}^{-1}(R^*)$ when this doesn't lead to confusion. Then we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 1 & \longrightarrow & 1 + I & \longrightarrow & R^* & \longrightarrow & \bar{R}^* & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 1 & \longrightarrow & K & \longrightarrow & (Q^a)^{\text{gp}} & \longrightarrow & (M^a)^{\text{gp}} & \longrightarrow & 1
 \end{array}$$

in which the vertical maps are injective, so by the snake lemma we get an exact sequence

$$1 \rightarrow K/(1 + I) \rightarrow (Q^a)^{\text{gp}}/R^* \rightarrow (M^a)^{\text{gp}}/\bar{R}^* \rightarrow 1.$$

It fits in a commutative diagram

$$\begin{array}{ccccccccc}
 1 & \longrightarrow & L & \longrightarrow & Q^{\text{gp}} & \longrightarrow & M^{\text{gp}} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 1 & \longrightarrow & K/(1 + I) & \longrightarrow & (Q^a)^{\text{gp}}/R^* & \longrightarrow & (M^a)^{\text{gp}}/\bar{R}^* & \longrightarrow & 1
 \end{array}$$

where the maps $Q^{\text{gp}} \rightarrow (Q^a)^{\text{gp}}/R^*$, $M^{\text{gp}} \rightarrow (M^a)^{\text{gp}}/\bar{R}^*$ are surjective by Lemma 4.2 (i). By Lemma 4.2 (ii) we have

$$\ker(Q^{\text{gp}} \rightarrow (Q^a)^{\text{gp}}/R^*) = Q^{\text{gp}} \cap R^*$$

(intersection taken in $(Q^a)^{\text{gp}}$) and similarly

$$M^{\text{gp}} \cap \bar{R}^* = \ker(M^{\text{gp}} \rightarrow (M^a)^{\text{gp}}/\bar{R}^*).$$

Moreover, $\ker(L \rightarrow K/(1 + I)) = L$ since $L \subset 1 + I$ by assumption. So by the snake lemma we get an exact sequence

$$1 \rightarrow L \rightarrow Q^{\text{gp}} \cap R^* \rightarrow M^{\text{gp}} \cap \bar{R}^* \rightarrow K/(1 + I) \rightarrow 1.$$

Since the map $Q^{\text{gp}} \cap R^* \rightarrow M^{\text{gp}} \cap \bar{R}^*$ is surjective by assumption, we have $K/(1 + I) = 1$, and this completes the proof. \square

4.2. Integrality results

Notation as in (2.2.2).

LEMMA 4.4. — [(i)]

- (1) $O(c)_n$ is a regular integral domain.
- (2) $O(c)_{n,L}$ is an integrally closed domain.

Proof. —

(i). — This is clear since $O(c)_n$ has either good ($c = 1$) or semi-stable ($c = \pi$) reduction.

(ii). — Since $O(c)_{n,L}$ is an integral domain, it suffices to show that it is normal. Since L_n is a finite extension of K_n and normality is stable by étale localization, up to making a finite unramified extension of K_n we may assume that L_n is a totally ramified extension of K_n , in particular $L_n^+ \cong K_n^+[X]/(f)$, where f is an Eisenstein polynomial ([14, Ch. I, §6, Prop. 18]). Since $O(c)_n$ is a Krull ring, by the conjunction of [4, Ch. 7, §4, no. 2, Thm. 2] and [4, Ch. 5, §1, no. 2, Prop. 8], we may localize at height 1 prime ideals to reduce to the case $O(c)_n$ is a discrete valuation ring. If $O(c)_n$ is of equal characteristic zero, then $O(c)_{n,L}$ is étale over $O(c)_n$, hence normal. If $O(c)_n$ is of mixed characteristic and $c = \pi$, then c_n is a uniformizer for $O(c)_n$ and $O(c)_{n,L} \cong O(c)_n[X]/(f)$. So $O(c)_{n,L}$ is a discrete valuation ring ([14, Ch. I, §6, Prop. 17]), in particular a normal ring. If $c = 1$, then it is easy to see that the special fibre of $\text{Spec}(O(c)_n) \rightarrow \text{Spec}(K_n^+)$ is integral, hence a uniformizer of K_n^+ is a uniformizer of $O(c)_n$ and f is an Eisenstein polynomial, so we can conclude. \square

Note that the normalization of K_n^+ in R_n is unramified over K_n^+ because R is small. Define $R_{n,L} := R \otimes_{O(c)} O(c)_{n,L}$.

PROPOSITION 4.5. — Assume $K \cap R = K^+$. If R is an integral domain and K^+ is integrally closed in R , then $R_{n,L}$ is an integrally closed domain for all n and L .

Proof (Ramero). — Since $R_{n,L}$ is étale over $O(c)_{n,L}$ it follows that $R_{n,L}$ is a normal ring. So it suffices to show that $R_{n,L}$ is an integral domain.

We first show that $S := R \otimes_{K^+} K_n^+$ is an integral domain. Note that since K^+ is integrally closed in R we have $Q(R) \cap K_n = K$. Let $Q(R) \cdot K_n$ be the composite of $Q(R)$ and K_n . Now, $S \subset Q(R) \otimes_{K^+} K_n^+$ and the latter is a field: we have a surjective map

$$Q(R) \otimes_{K^+} K_n^+ \rightarrow Q(R) \cdot K_n$$

and $Q(R) \otimes_{K^+} K_n^+$ is a $Q(R)$ vector space of dimension $[K_n : K]$; since $Q(R) \cap K_n = K$, $\dim_{Q(R)} Q(R) \cdot K_n = [K_n : K]$, hence the map is an isomorphism. This proves that S is an integral domain.

We now show that $R_n = R_{n,K}$ is an integral domain. It suffices to show that its spectrum is connected. Since R_n is finite flat over S the image of a connected component of R_n under the morphism $f : \text{Spec}(R_n) \rightarrow \text{Spec}(S)$ is both open and closed, hence equal to S because f is generically finite étale (in particular every connected component of R_n dominates S). So it suffices to show that f has a single connected fibre. Let \mathfrak{q} be a generic point of $\text{Spec}(S/\pi S)$, considered as a prime ideal of S , and let $\mathfrak{p} = \mathfrak{q} \cap O$, where $O := O(c) \otimes_{K^+} K_n^+$. Let $S_{\mathfrak{q}}^h$ resp. $O_{\mathfrak{p}}^h$ denote the henselization of S at \mathfrak{q} resp. O at \mathfrak{p} . Since the prime ideals \mathfrak{p} and \mathfrak{q} have height one, these are discrete valuation rings. Let $O_n := O(c)_n$. We claim that

$$O_{\mathfrak{p}}^h \otimes_O O_n$$

is a noetherian henselian local ring. It suffices to show that

$$k(\mathfrak{p}) \otimes_O O_n$$

is an integral domain. In the case $c = \pi$ we have $\mathfrak{p} = (c_n, T_i)$ for some $1 \leq i \leq r$, and otherwise we have $\mathfrak{p} = (\pi_n)$ where π_n denotes a uniformizer of K_n^+ . First consider the case $c = \pi$. Then $k(\mathfrak{p})$ is the fraction field of $k[T_1^{\pm 1}, \dots, \widehat{T_i^{\pm 1}}, \dots, T_{d+1}^{\pm 1}]$ and $k(\mathfrak{p}) \otimes_O O_n$ is a localization of

$$k[T_1^{\pm 1}, \dots, \widehat{T_i^{\pm 1}}, \dots, T_{d+1}^{\pm 1}] \otimes_O O_n \cong k \left[T_1^{(n)}, \frac{1}{T_1^{(n)}}, \dots, \widehat{T_i^{(n)}}, \frac{1}{T_i^{(n)}}, \dots, T_{d+1}^{(n)}, \frac{1}{T_{d+1}^{(n)}} \right]$$

hence is an integral domain. As usual, here the hat over a symbol means that we omit it. This proves the claim in the case $c = \pi$ and the case $c = 1$ is simpler. Note also that $O_{\mathfrak{p}}^h \otimes_O O_n \cong (O_{\mathfrak{p}} \otimes_O O_n)^h$ is normal since $O_{\mathfrak{p}} \otimes_O O_n$ is ([10, IV₄, Thm. 18.6.9]). Hence it is a discrete valuation ring.

Now, the extension

$$O_{\mathfrak{p}}^h \rightarrow S_{\mathfrak{q}}^h$$

is finite étale of degree f (say) and the extension

$$O_{\mathfrak{p}}^h \rightarrow O_{\mathfrak{p}}^h \otimes_O O_n$$

is totally ramified of degree e (say). So the composite $S_{\mathfrak{q}}^h \cdot O_n$ is an extension of $O_{\mathfrak{p}}^h$ of degree $d = e \cdot f$. Hence the canonical map

$$S_{\mathfrak{q}}^h \otimes_{O_{\mathfrak{p}}^h} O_{\mathfrak{p}}^h \otimes_O O_n \rightarrow S_{\mathfrak{q}}^h \cdot O_n$$

is a surjective map of free $O_{\mathfrak{p}}^h$ -modules of same rank, so it is an isomorphism. Thus, $S_{\mathfrak{q}}^h \otimes_O O_n \cong S_{\mathfrak{q}}^h \cdot O_n$ is connected, and so (since $S_{\mathfrak{q}}^h$ is a henselian noetherian local ring) has connected special fibre

$$(S_{\mathfrak{q}}^h \otimes_O O_n)/\mathfrak{q} \cdot (S_{\mathfrak{q}}^h \otimes_O O_n) \cong (S_{\mathfrak{q}}/\mathfrak{q}) \otimes_O O_n.$$

This implies that the fibre of $R_n = S \otimes_O O_n$ over \mathfrak{q} is connected, hence so is R_n .

Finally, $R_{n,L}$ is a domain, by the same dimension counting argument as for S . \square

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