ANNALES

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L'INSTITUT FOURIER

Kenichi ITO & Shu NAKAMURA

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Tome 62, nº 3 (2012), p. 1091-1121.

<http://aif.cedram.org/item?id=AIF_2012__62_3_1091_0>

Ann. Inst. Fourier, Grenoble
62, 3 (2012) 1091-1121

REMARKS ON THE FUNDAMENTAL SOLUTION TO SCHRÖDINGER EQUATION WITH VARIABLE COEFFICIENTS

by Kenichi ITO & Shu NAKAMURA (*)

Abstract. — We consider Schrödinger operators H on $\mathbb{R}^n$ with variable coefficients. Let $H_0 = -\frac{1}{2}\triangle$ be the free Schrödinger operator and we suppose H is a "short-range" perturbation of H_0. Then, under the nontrapping condition, we show that the time evolution operator: e^{-itH} can be written as a product of the free evolution operator e^{-itH_0} and a Fourier integral operator $W(t)$ which is associated to the canonical relation given by the classical mechanical scattering. We also prove a similar result for the wave operators. These results are analogous to results by Hassell and Wunsch, but the assumptions, the proof and the formulation of results are considerably different. The proof employs an Egorov-type theorem similar to those used in previous works by the authors combined with a Beals-type characterization of Fourier integral operators.

Résumé. — Nous considérons des opérateurs de Schrödinger H à coefficients variables sur $\mathbb{R}^n$, qui sont des perturbations "à courte portée" de l'opérateur de Schrödinger libre $H_0 = -\frac{1}{2}\triangle$. Dans le cas non captant, nous montrons que l'opérateur d'évolution temporelle e^{-itH} s'écrit comme le produit de l'opérateur d'évolution libre e^{-itH_0} et d'un opérateur intégral de Fourier $W(t)$, qui est associé à la relation canonique donnée par la diffusion classique. Nous établissons aussi un résultat similaire pour les opérateurs d'onde. Ces résultats sont analogues à ceux obtenus par Hassell et Wunsch, mais leurs hypothèses, leur preuve et leur formulation sont nettement différents. La preuve repose sur un théorème de type Egorov semblable à ceux utilisés dans les travaux précédents des auteurs, et qui est combiné ici à une caractérisation de type Beals des opérateurs intégraux de Fourier.

Keywords: Schrödinger equation, fundamental solutions, scattering theory.
Math. classification: 35Q40, 35A17, 35A21.
(*) KI is supported partially by JSPS Research Fellowship (2006-2007); JSPS Wakate (B) 21740090 (2009–2012).
SN is supported partially by JSPS Grant Kiban (B) 17340033 (2005–2008); JSPS Kiban (A) 21244008 (2009–2013).

1. Introduction

We consider Schrödinger equations on $\mathbb{R}^n$ with $n \geqslant 1$ of the following form:

$$\begin{aligned} &i\frac{\partial}{\partial t}\psi(t,x) = H\psi(t,x), \quad t \in \mathbb{R}, x \in \mathbb{R}^n, \\ &\psi(0,x) = \psi_0(x) \in L^2(\mathbb{R}^n), \\ &H = -\frac{1}{2}\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} a_{jk}(x) \frac{\partial}{\partial x_k} + V(x), \end{aligned}$$

where $a_{jk}(x)$ and $V(x)$ are real-valued C^∞-functions on $\mathbb{R}^n$.

Assumption A. — *There exists $\mu > 0$ such that for any $\alpha \in \mathbb{Z}_+^n$*

$$\left|\partial_x^\alpha \left(a_{jk}(x) - \delta_{jk}\right|\right) \leqslant C_\alpha \langle x\rangle^{-\mu-|\alpha|}, \quad \left|\partial_x^\alpha V(x)\right| \leqslant C_\alpha \langle x\rangle^{2-\mu-|\alpha|}$$

for $x \in \mathbb{R}^n$ with some $C_\alpha > 0$.

It is well-known that under our assumptions H is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$. We denote the unique self-adjoint extension by the same symbol H. Then the solution to the Schrödinger equation is given by $\psi(t) = e^{-itH}\psi_0 \in L^2(\mathbb{R}^n)$ by Stone's theorem.

We are interested in the microlocal structure of the evolution operator e^{-itH}. If $a_{jk}(x) = \delta_{jk}$, *i.e.*, if the metric is flat, then e^{-itH} is represented by an oscillatory integral similar to Fourier integral operators (Fujiwara [2]), though it is not a Fourier integral operator (FIO) in the sense of Hörmander ([5, 6]). For general H, it is difficult to show similar representations because of the existence of the caustics. In this paper, we discuss different representation of the evolution operator, namely, we show

$$e^{-itH} = e^{-itH_0}W(t)$$

where $H_0 = -\frac{1}{2}\triangle$ is the free Schrödinger operator, and $W(t)$ is possibly an FIO. In the following, we show $W(t)$ is in fact an FIO under suitable conditions. We note

$$W(t) = e^{itH_0}e^{-itH},$$

and we study the microlocal structure of $W(t)$ defined as above.

In order to state the condition, we consider the classical mechanics associated to our Hamiltonian. We set

$$p(x,\xi) = k(x,\xi) + V(x), \quad k(x,\xi) = \frac{1}{2}\sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k, \quad p_0(\xi) = \frac{1}{2}|\xi|^2$$

on $T^*\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$. $k(x,\xi)$ is the kinetic energy, $p(x,\xi)$ is the classical Hamilton function, and $p_0(\xi)$ is the free energy function. We denote the corresponding Hamilton vector fields by H_p and H_k, H_{p_0}, respectively, *i.e.*,

$$H_p = \sum_{j=1}^n \frac{\partial p}{\partial \xi_j}\frac{\partial}{\partial x_j} - \sum_{j=1}^n \frac{\partial p}{\partial x_j}\frac{\partial}{\partial \xi_j}, \quad H_k = \sum_{j=1}^n \frac{\partial k}{\partial \xi_j}\frac{\partial}{\partial x_j} - \sum_{j=1}^n \frac{\partial k}{\partial x_j}\frac{\partial}{\partial \xi_j},$$

etc. and we denote the Hamilton flows on $T^*\mathbb{R}^n$ by $\exp(tH_p)$, $\exp(tH_k)$ and $\exp(tH_{p_0})$ $(t \in \mathbb{R})$, respectively. We write

$$T^*M \setminus 0 = \big\{(x,\xi) \bigm| (x,\xi) \in T^*M, \xi \neq 0\big\}.$$

Definition 1.1. — *Let $(x_0,\xi_0) \in T^*\mathbb{R}^n \setminus 0$, and we denote*

$$(y(t),\eta(t)) = \exp(tH_k)(x_0,\xi_0), \quad \text{for } t \in \mathbb{R}.$$

(x_0,ξ_0) is said to be forward (backward, resp.) nontrapping if

$$|y(t)| \to +\infty \quad \text{as } t \to \pm\infty.$$

If (x_0,ξ_0) is forward/backward nontrapping, then it is well-known that

$$\xi_\pm = \lim_{t\to\pm\infty} \eta(t)$$

exist under Assumption A. Moreover, if $\mu > 1$, then

$$z_\pm = \lim_{t\to\pm\infty} (y(t) - t\eta(t))$$

are also well-known to exist (see, *e.g.*, [14]). These imply $y(t) \sim z_\pm + t\xi_\pm$ as $t \to \pm\infty$. We call $(z_\pm,\xi_\pm)$ the scattering data of (x_0,ξ_0), and we denote

$$w_\pm(x_0,\xi_0) = (z_\pm,\xi_\pm) = \lim_{t\to\pm\infty} \exp(-tH_{p_0}) \circ \exp(tH_k)(x_0,\xi_0).$$

We note that $z_\pm(x,\xi)$ and $\xi_\pm(x,\xi)$ are homogeneous of order 0 and 1 with respect to ξ, respectively, since both $k(x,\xi)$ and $p_0(\xi)$ are homogeneous of order 2 in ξ. Moreover, $w_\pm$ are canonical transform on the domain where $w_\pm$ are defined.

Theorem 1.2. — *Suppose Assumption A with $\mu = 2$, and suppose the global nontrapping condition, i.e., every $(x_0,\xi_0) \in T^*\mathbb{R}^n \setminus 0$ is nontrapping. Then $W(t)$ is an FIO associated to $w_\pm$ for each $t \in \mathbb{R}_\pm$.*

Remark 1.3. — We suppose the global nontrapping condition for the sake of simplicity. If we suppose (x_0,ξ_0) is forward nontrapping and $t > 0$, then we can find a symbol $a(x,\xi) \in S^0_{cl}(\mathbb{R}^n)$ such that $a_0(x_0,\xi_0) \neq 0$ and $W(t)a(x,D_x)$ is an FIO associated to w_+ defined in a conic neighborhood of (x_0,ξ_0). Here we have denoted the principal symbol of a by a_0. The same generalization applies to the following theorems, but we do not discuss in detail. The proof of the above statement is same as that of the theorem. In fact, we need only to prove the theorem microlocally, and we prove the above claim to conclude the main result by using the microlocal partition of unity.

Remark 1.4. — Theorem 1.2 actually implies a propagation of singularity result. Namely, if we set

$$\Lambda_\pm = \big\{(y,\eta,x,-\xi) \bigm| (y,\eta) = w_\pm(x,\xi)\big\} \subset T^*\mathbb{R}^n \times T^*\mathbb{R}^n,$$

then Theorem 1.2 implies $\mathrm{WF}(W(t)) \subset \Lambda_\pm$ for $t \in \mathbb{R}_\pm$, where $\mathrm{WF}(W(t))$ denotes the wave front set of the distribution kernel of $W(t)$. Thus, in turn, it implies

$$\mathrm{WF}(W(t)u) \subset w_\pm(\mathrm{WF}(u))$$

for $u \in L^2(\mathbb{R}^n)$ or $u \in \mathcal{E}'(\mathbb{R}^n)$. In fact, we have the equality in the above inclusion ([14, 13, 8]).

We now consider the general short-range case, *i.e.*, the case when $1 < \mu < 2$. Then we learn that Theorem 1.2 does not hold as it is, and we need to modify the definition of the FIOs. In [6], an FIO is defined as an operator of which the distribution kernel is a Lagrangian distribution associated to a *conic* Lagrangian submanifold. We need to employ a Lagrangian distribution associated to an *asymptotically conic* Lagrangian manifold. Such Lagrangian submanifold is associated to an *asymptotically homogeneous* canonical transform. We discuss these definitions in Section 4.

We set

$$w(t) = \exp(-tH_{p_0}) \circ \exp(tH_p) \ : \ T^*\mathbb{R}^n \to T^*\mathbb{R}^n.$$

We can show $w(t)(x,\xi) = w_\pm(x,\xi) + O(|\xi|^{2-\mu})$ as $|\xi| \to \infty$ for $\pm t > 0$, and hence $w(t)$ is asymptotically close to the homogeneous canonical transform $w_\pm$ if $\mu > 1$ (Appendix, Lemma A.1).

Theorem 1.5. — *Suppose Assumption A with $1 < \mu < 2$, and suppose the global nontrapping condition. Then $W(t)$ is an FIO associated to $w(t)$.*

Remarks 1.3 and 1.4 also apply to Theorem 1.5.

Next we consider the wave operators. Here we suppose

Assumption B. — *There exists $\mu > 1$ such that for any $\alpha \in \mathbb{Z}_+^n$*

$$\big|\partial_x^\alpha\big(a_{jk}(x) - \delta_{jk}\big)\big| \leqslant C_\alpha \langle x\rangle^{-\mu-|\alpha|}, \quad \big|\partial_x^\alpha V(x)\big| \leqslant C_\alpha \langle x\rangle^{-\mu-|\alpha|}$$

for $x \in \mathbb{R}^n$ with some $C_\alpha > 0$.

Under Assumption B, it is well-known that the wave operators:

$$W_\pm = \operatorname*{s-lim}_{t\to\pm\infty} e^{itH}e^{-itH_0}$$

exist on $L^2(\mathbb{R}^n)$.

Theorem 1.6. — *Suppose Assumption B, and suppose the global nontrapping condition. Then $W_\pm$ are FIOs associated to $w_\pm^{-1}$.*

Note that $w_\pm$ is homogeneous in ξ, and $W_\pm$ are FIOs in the sense of Hörmander even if $\mu < 2$.

We prove our main results combining a Beals-type characterization of FIOs and Egorov-type theorems, which are variations of corresponding theorems proved in [14, 13, 8] to characterize the wave front set of solutions to Schrödinger equations. We discuss the Beals-type characterization theorem in Section 2, and then its generalization to FIOs associated to asymptotically homogeneous canonical transforms in Section 4. Using these, we prove Theorem 1.2 and Theorem 1.5 in Section 3 and Section 4, respectively. Application of these ideas to wave operators (Theorem 1.6) is discussed in Section 5. Several technical lemmas are proved in Appendix A.

The fundamental solution to Schrödinger equation with the flat Laplacian as the principal terms has been studied by many authors, for example Fujiwara [2] and Yajima [18]. In this case, a global construction of the fundamental solution is known, and it was applied to various estimates, for example dispersive estimates of the Schrödinger evolution group.

On the other hand, not much has been known about the fundamental solution to the Schrödinger equation with variable coefficients. The local regularity properties of the fundamental solution under nontrapping condition is known for some time, but it is not enough to characterize the singularities of solutions to the Schrödinger equation (*cf. e.g.,* Kapitanski-Safarov [9]), because the equation has infinite propagation speed. The first step of the analysis of microlocal singularity for the equation was carried out by Craig, Kappeler and Strauss [1]. They proved microlocal smoothing property of the equation, and thus gave a sufficient condition for the microlocal regularity of solutions. The result has been improved or generalized by Wunsch [17], Robbiano-Zuily [15], Nakamura [12], Ito [7], Martinez-Nakamura-Sordoni [10], etc. Then a complete characterization of the microlocal singularities of solutions was given by Hassell and Wunsch [3, 4] by constructing a parametrix as a Legendre distribution on scattering manifolds. The result was later generalized by Nakamura [14, 13], Ito-Nakamura [8] and Martinez-Nakamura-Sordoni [11]. In these papers, the authors do not construct a parametrix, but instead use Egorov-type theorems to obtain the characterization of singularities of solutions.

In this paper, we show that these Egorov-type theorems actually imply that the fundamental solution is written in terms of an FIO and the free evolution operator. Our result is analogous to results by Hassell and Wunsch [3, 4], but the formulation is quite different, and we work in the standard framework of FIOs and Lagrangian distributions, combined with classical mechanical scattering theory. Our model is restricted to Schrödinger operators on the Euclidean space, but our assumption on the perturbation is weaker than theirs. In particular, we consider general *short-range* perturbations, not necessarily having an asymptotic expansion starting from $O(|x|^{-2})$ terms. We may also include operators with unbounded potential terms. Our method can be applied also to operators on scattering manifolds, and we discuss them in a forthcoming paper.

We use the following notation throughout this paper: For function spaces X, Y, $\mathcal{L}(X,Y)$ denotes the linear space of the continuous linear maps from X to Y. We write $\mathbb{R}_+=(0,\infty)$ and $\mathbb{R}_-=(-\infty,0)$. Also we write $\mathbb{Z}_+=\{0,1,2,\dots\}$. For $u\in\mathcal{S}(\mathbb{R}^n)$, we denote the Fourier transform by

$$\hat{u}(\xi)=\mathcal{F}u(\xi)=(2\pi)^{-n/2}\int_{\mathbb{R}^n}e^{-ix\cdot\xi}f(x)\,dx.$$

$\mathcal{F}$ is extended to a map from $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$. We denote the standard symbol class of pseudodifferentail operators as follows: We write $a\in S^m_{\rho,\delta}(\mathbb{R}^n)$ if $a(x,\xi)\in C^\infty(\mathbb{R}^n\times\mathbb{R}^n)$ and for any $\alpha,\beta\in\mathbb{Z}^n_+$,

$$\left|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)\right|\leqslant C_{\alpha\beta}\langle\xi\rangle^{m+\delta|\alpha|-\rho|\beta|},\quad x,\xi\in\mathbb{R}^n,$$

with some $C_{\alpha\beta}>0$. We write $a\in S^m_{cl}(\mathbb{R}^n)$ if $a\in S^m_{1,0}$ and a has an asymptotic expansion:

$$a(x,\xi)\sim\sum_{j=0}^\infty a_j(x,\xi),\quad\text{as } |\xi|\to\infty,$$

where $a_j(x,\xi)$ are homogeneous of order $(m-j)$ in ξ. For a symbol $a(x,\xi)$, the pseudodifferential operator $a(x,D_x)$ is defined by

$$a(x,D_x)u(x)=(2\pi)^{-n/2}\int_{\mathbb{R}^n}e^{ix\cdot\xi}a(x,\xi)\hat{u}(\xi)\,d\xi,\quad u\in\mathcal{S}(\mathbb{R}^n),$$

and the Weyl quantization is defined by

$$a^W(x,D_x)u(x)=(2\pi)^{-n}\int_{\mathbb{R}^{2n}}e^{i(x-y)\cdot\xi}a\Big(\frac{x+y}{2},\xi\Big)u(y)\,dy\,d\xi.$$

For $(x_0,\xi_0)\in T^*M$, $\Omega\subset T^*M$ is called a *conic neighborhood* of (x_0,ξ_0) if Ω is a neighborhood such that if $(x,\xi)\in\Omega$ then $(x,\lambda\xi)\in\Omega$ for any $\lambda\in[1,\infty)$.

A conic set is called *narrow* if for any $(x,\xi),(x,\eta)\in\Omega$, $(\xi,\eta\neq 0)$, $\hat{\xi}\cdot\hat{\eta}>a$ with fixed $a>0$, where $\hat{\xi}=\xi/|\xi|$.

2. Beals-type characterization of Fourier integral operators

In this section we prove a Beals-type characterization theorem for Fourier integral operators. At first we recall several standard definitions.

Let $\Lambda\subset T^*\mathbb{R}^m$ be a smooth m-dimensional submanifold. L is called *Lagrangian* if the pull back of the standard canonical form vanishes on Λ, *i.e.*, $i^*(d\xi\wedge dx)=0$ on $T^*\Lambda$. Λ is called *conic* if $(x,\xi)\in\Lambda$ implies $(x,\lambda\xi)\in\Lambda$ for $\lambda>0$.

Definition 2.1 (Besov space $B_2^{\sigma,\infty}(\mathbb{R}^m)$). — *Let $\sigma\in\mathbb{R}$ and let $u\in\mathcal{S}'(\mathbb{R}^m)$ such that $\hat{u}\in L^2_{\mathrm{loc}}(\mathbb{R}^m)$. Then we set*

$$\|u\|_{B_2^{\sigma,\infty}}=\left(\int_{|\xi|\leqslant 1}|\hat{u}(\xi)|^2d\xi\right)^{1/2}+\sup_{j\geqslant 0}\left(\int_{2^j\leqslant|\xi|\leqslant 2^{j+1}}\left|2^{\sigma j}\hat{u}(\xi)\right|^2d\xi\right)^{1/2}.$$

Then $B_2^{\sigma,\infty}(\mathbb{R}^m)$ is defined by

$$B_2^{\sigma,\infty}(\mathbb{R}^m)=\left\{u\in\mathcal{S}'(\mathbb{R}^m)\;\middle|\;\|u\|_{B_2^{\sigma,\infty}}<\infty\right\}$$

and

$$B_{2,\mathrm{loc}}^{\sigma,\infty}(\mathbb{R}^m)=\left\{u\in\mathcal{D}'(\mathbb{R}^m)\;\middle|\;\varphi u\in B_2^{\sigma,\infty}(\mathbb{R}^m)\text{ for any }\varphi\in C_0^\infty(\mathbb{R}^m)\right\}.$$

According to Hörmander [6] (see also Sogge [16]), the Lagrangian distribution is defined as follows:

Definition 2.2 (Lagrangian distribution). — *Let $\Lambda\subset T^*\mathbb{R}^m\setminus 0$ be a conic Lagrangian submanifold, $u\in\mathcal{S}'(\mathbb{R}^m)$ and $\sigma\in\mathbb{R}$. u is called Lagrangian distribution associated to Λ of order σ if for any $p_1,\dots,p_N\in S^1_{cl}(\mathbb{R}^m)$ such that the principal symbols of p_j vanish on Λ ($j=1,2,\dots,N$),*

$$p_1(x,D_x)p_2(x,D_x)\cdots p_N(x,D_x)u\in B_{2,\mathrm{loc}}^{-\sigma-m/4,\infty}(\mathbb{R}^m),$$

and we denote $u\in I^\sigma(\mathbb{R}^m,\Lambda)$.

Definition 2.3 (Fourier integral operators). — *Let S be a canonical transform from $T^*\mathbb{R}^n$ to $T^*\mathbb{R}^n$, and suppose S is homogeneous of order 1 with respect to ξ. Let*

$$\Lambda_S=\left\{(y,x,\eta,-\xi)\;\middle|\;(y,\eta)=S(x,\xi),(x,\xi)\in T^*\mathbb{R}^n\setminus 0\right\}\subset T^*\mathbb{R}^{2n}\setminus 0.$$

Let $U\in\mathcal{L}(\mathcal{S}(\mathbb{R}^n),\mathcal{S}'(\mathbb{R}^n))$ and let $u\in\mathcal{S}'(\mathbb{R}^{2n})$ be its distribution kernel. Then U is called a Fourier integral operator of order σ associated to S if $u\in I^\sigma(\mathbb{R}^{2n},\Lambda_S)$.

Note Λ_S is a conic Lagrangian submanifold since S is a homogeneous canonical transform.

Remark 2.4. — If U is a Fourier integral operator, it is known that there is $m \leqslant 2n$, a phase function $\Psi(x,\theta,y)$ ($x,y \in \mathbb{R}^n$, $\theta \in \mathbb{R}^m$), which is homogeneous of order 1 in θ, and a symbol $a(x,\theta,y) \in S^{\sigma+n/2-m/2}_{1,0}$ such that

$$U\varphi(x) = (2\pi)^{-n/2+m/2} \int_{\mathbb{R}^m\times\mathbb{R}^n} e^{i\Psi(x,\theta,y)} a(x,\theta,y)\varphi(y)dyd\theta$$

and we have a familiar representation of an FIO (see [6] or [16] for the detail).

We give a characterization of FIOs in terms of conjugation of operators. Let S be a canonical transform and $U \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n))$ as above. Let $a \in S^m_{1,0}(\mathbb{R}^n)$ ($m \in \mathbb{R}$) such that

$$\text{(2.1)}\quad \big\{x \bigm| a(x,\xi) \neq 0 \text{ for some } \xi \in \mathbb{R}^n\big\} \Subset \mathbb{R}^n, \quad \operatorname{supp} a \cap (\mathbb{R}^n \times \{0\}) = \emptyset.$$

For such a, we set

$$Ad_S(a)U = (a\circ S^{-1})(x,D_x)U - Ua(x,D_x) \ : \ \mathcal{S} \to \mathcal{S}'.$$

Theorem 2.5. — *Let S be as above, and let $U \in \mathcal{L}\big(L^2_{\text{cpt}}(\mathbb{R}^n), L^2_{\text{loc}}(\mathbb{R}^n)\big)$. U is an FIO of order 0 associated to S if and only if for any $a_1, a_2, \dots, a_N \in S^1_{cl}(\mathbb{R}^n)$ satisfying* (2.1),

$$\text{(2.2)}\qquad Ad_S(a_1)Ad_S(a_2)\cdots Ad_S(a_N)U \in \mathcal{L}\big(L^2_{\text{cpt}}(\mathbb{R}^n), L^2_{\text{loc}}(\mathbb{R}^n)\big).$$

The next corollary gives convenient sufficient conditions.

Corollary 2.6.

(i) *Let S and U be as in Theorem 2.5. If for any $a \in S^1_{cl}(\mathbb{R}^n)$ satisfying* (2.1), *there is $b \in S^0_{1,0}(\mathbb{R}^n)$ such that*

$$Ad_S(a)U = b(x,D_x)U + R$$

with a smoothing operator R, then U is an FIO associated to S.

(ii) *Let S and U as above, and suppose U is invertible. If for any $a \in S^1_{cl}(\mathbb{R}^n)$ satisfying* (2.1) *there is $b \in S^0_{cl}(\mathbb{R}^n)$ such that*

$$Ua(x,D_x)U^{-1} = (a\circ S^{-1})(x,D_x) + b(x,D_x),$$

then U is an FIO of order 0 associated to S.

Proof. — (i) The condition (2.2) with $N=1$ follows immediately from the assumption of (i). Let $N=2$, and let $Ad_S(a_j)U = b_j(x,D_x)U + R_j$, $j=1,2$. Then we have

$$\begin{aligned} Ad_S(a_1)Ad_S(a_2)U &= (a_1\circ S^{-1})(x,D_x)b_2(x,D_x)U - b_2(x,D_x)Ua_1(x,D_x) \\ &\qquad + Ad_S(a_1)R_2 \\ &= \big[(a_1\circ S^{-1})(x,D_x), b_2(x,D_x)\big]U \\ &\quad - b_2(x,D_x)b_1(x,D_x)U + Ad_S(a_1)R_2 + b_2(x,D_x)R_1 \\ &= b_{12}(x,D_x)U + R_{12} \end{aligned}$$

where $b_{12}\in S^0_{1,0}(\mathbb{R}^n)$ and R_{12} is a smoothing operator. Repeating this procedure, we conclude (2.2) for any N. Now (ii) follows easily from (i). $\square$

In order to prove Theorem 2.5, we first notice that the L^2_{cpt}-L^2_{loc} boundedness implies the distribution kernel is locally $B_2^{-n/2,\infty}$.

Lemma 2.7. — *Suppose $U\in\mathcal{L}(L^2_{\mathrm{cpt}}(\mathbb{R}^n), L^2_{\mathrm{loc}}(\mathbb{R}^n))$, and let $u\in\mathcal{S}'(\mathbb{R}^{2n})$ be its distribution kernel. Then $u\in B^{-n/2,\infty}_{2,\mathrm{loc}}(\mathbb{R}^{2n})$.*

Proof. — Let u be the distribution kernel of U. Let $\chi,\psi\in C_0^\infty(\mathbb{R}^n;\mathbb{R})$, and we suppose χ is an even function. We consider

$$I = \int_{\mathbb{R}^{2n}} \big|\psi(\xi)\psi(\eta)\mathcal{F}_{2n}[\chi(x)\chi(y)u(x,y)](\xi,\eta)\big|^2 d\xi\, d\eta.$$

Here we denote the Fourier transform on $\mathbb{R}^{2n}$ by $\mathcal{F}_{2n}$. In the following, $\mathcal{F}$ denotes the Fourier transform on $\mathbb{R}^n$. We choose $\chi_1\in C_0^\infty(\mathbb{R}^n)$ so that $\chi_1(x)=1$ on $\operatorname{supp}\chi$. We note I can be expressed in terms of the Hilbert-Schmidt norm:

$$I = \big\|\psi\mathcal{F}\chi U\chi\mathcal{F}^{-1}\psi\big\|^2_{HS}$$

where $\psi=\psi(\xi)$ and $\chi=\chi(x)$ denote the multiplication operators on $L^2(\mathbb{R}^n_\xi)$ and $L^2(\mathbb{R}^n_x)$, respectively. $\|\cdot\|_{HS}$ denotes the Hilbert-Schmidt norm. Then we represent the Hilbert-Schmidt norm by a trace:

$$\begin{aligned} I &= \mathrm{Tr}\big[(\psi\mathcal{F}\chi U\chi\mathcal{F}^{-1}\psi)^*(\psi\mathcal{F}\chi U\chi\mathcal{F}^{-1}\psi)\big] \\ &= \mathrm{Tr}\big[\psi\mathcal{F}\chi U^*\chi\mathcal{F}^{-1}\psi^2\mathcal{F}\chi U\chi\mathcal{F}^{-1}\psi\big] \\ &= \mathrm{Tr}\big[((\chi_1U\chi_1)^*\chi\mathcal{F}^{-1}\psi^2\mathcal{F}\chi)((\chi_1U\chi_1)\chi\mathcal{F}^{-1}\psi^2\mathcal{F}\chi)\big] \\ &= \mathrm{Tr}\big[((\chi_1U\chi_1)^*\chi\psi^2(D_x)\chi)((\chi_1U\chi_1)\chi\psi^2(D_x)\chi)\big]. \end{aligned}$$

We use the Schwarz inequality for the trace to obtain

$$\begin{aligned} I &\leqslant \big\|(\chi_1 U\chi_1)^*\chi\psi^2(D_x)\chi\big\|_{HS}\big\|(\chi_1 U\chi_1)\chi\psi^2(D_x)\chi\big\|_{HS} \\ &\leqslant \|\chi_1 U\chi_1\|^2_{\mathcal{L}(L^2)}\big\|\chi\psi^2(D_x)\chi\big\|^2_{HS} \\ &= (2\pi)^{-n}\|\chi_1 U\chi_1\|^2_{\mathcal{L}(L^2)}\|\psi\|^4_{L^4}\|\chi\|^2_{L^2}\|\chi\|^2_{L^\infty}. \end{aligned}$$

Here we have used the well-known properties: $\|AB\|_{HS} \leqslant \|A\|\,\|B\|_{HS}$ and $\|a(x)b(D_x)\|_{HS} = (2\pi)^{-n/2}\|a\|_{L^2}\|b\|_{L^2}$.

Now we choose $\psi \in C_0^\infty(\mathbb{R}^n)$ so that $\psi(\xi) = 1$ for $1 \leqslant |\xi| \leqslant 2$, and we set

$$\psi_N(\xi) = \psi(2^{-N}\xi), \quad \text{for } N = 1, 2, \dots, \text{ and } \xi \in \mathbb{R}^n.$$

We note $\|\psi_N\|^4_{L^4} = 2^{nN}\|\psi\|^4_{L^4}$. Then, by the above estimate, we have

$$\begin{aligned} &\iint_{2^N \leqslant |\xi|,|\eta| \leqslant 2^{N+1}} \big|\mathcal{F}_{2n}[\chi(x)\chi(y)u(x,y)](\xi,\eta)\big|^2 d\xi d\eta \\ &\qquad \leqslant \iint \big|\psi_N(\xi)\psi_N(\eta)\mathcal{F}_{2n}[\chi(x)\chi(y)u(x,y)](\xi,\eta)\big|^2 d\xi d\eta \\ &\qquad \leqslant (2\pi)^{-n}\|\chi_1 U\chi_1\|^2_{\mathcal{L}(L^2)}\|\psi\|^4_{L^4}\|\chi\|^2_{L^2}\|\chi\|^2_{L^\infty} \times 2^{nN}, \end{aligned}$$

and this implies $\chi(x)\chi(y)u(x,y) \in B_2^{-n/2,\infty}(\mathbb{R}^{2n})$ for any $\chi \in C_0^\infty(\mathbb{R}^n)$. □

We set

$$\tilde{\Lambda}_S = \big\{(y,\eta,x,\xi) \bigm| (y,\eta) = S(x,\xi)\big\} \subset T^*\mathbb{R}^n \times T^*\mathbb{R}^n.$$

Lemma 2.8. — *Let $p \in S^1_{cl}(\mathbb{R}^{2n})$ such that the principal symbol of p vanishes on $\tilde{\Lambda}_S$, and suppose p is supported in a narrow convex conic neighborhood of $(S(x_0,\xi_0),x_0,\xi_0) \in \tilde{\Lambda}_S$. Then there exist $b_j \in S^0_{cl}(\mathbb{R}^{2n})$, $f_j \in S^1_{cl}(\mathbb{R}^n)$ $(j = 1, 2, \dots 2n)$, and $r \in S^0_{cl}(\mathbb{R}^{2n})$ such that*

$$p(y,\eta,x,\xi) = \sum_{j=1}^{2n} b_j(y,\eta,x,\xi)\big((f_j \circ S^{-1})(y,\eta) - f_j(x,\xi)\big) + r(y,\eta,x,\xi).$$

Proof. — We may assume p is homogeneous of order 1 without loss of generality. We denote

$$(z,\zeta) = S^{-1}(y,\eta)$$

so that $p(y,\eta,z,\zeta) = 0$. We let Γ_2, Γ_3 be convex conic neighborhoods of (x_0,ξ_0) such that $\overline{\Gamma_2} \subset \Gamma_3$, and let Γ_0, Γ_1 be convex conic neighborhoods of (y_0,η_0,x_0,ξ_0) such that

$$\operatorname{supp} p \subset \Gamma_0 \subset \overline{\Gamma_0} \subset \Gamma_1 \subset (S\Gamma_2) \times \Gamma_2.$$

We choose $\chi \in S^0_{cl}(\mathbb{R}^{2n})$ so that $\chi = 1$ on Γ_0 and $\operatorname{supp}\chi \subset \Gamma_1$. We also choose $\rho \in S^0_{cl}(\mathbb{R}^n)$ so that $\rho = 1$ on Γ_2 and $\operatorname{supp}\rho \subset \Gamma_3$. Then we compute

$$\begin{aligned} p(y,\eta,x,\xi) &= p(y,\eta,x,\xi) - p(y,\eta,z,\zeta) \\ &= \int_0^1 \frac{d}{dt}\big(p(y,\eta,tx+(1-t)z,t\xi+(1-t)\zeta)\big)dt \\ &= \sum_{j=1}^n (x_j - z_j)\int_0^1 \frac{\partial p}{\partial x_j}(y,\eta,tx+(1-t)z,t\xi+(1-t)\zeta)dt \\ &\quad + \sum_{j=1}^n (\xi_j - \zeta_j)\int_0^1 \frac{\partial p}{\partial \xi_j}(y,\eta,tx+(1-t)z,t\xi+(1-t)\zeta)dt. \end{aligned}$$

We now set

$$\begin{aligned} g_j(y,\eta,x,\xi) &= \chi(y,\eta,x,\xi)\int_0^1 \frac{\partial p}{\partial x_j}(y,\eta,tx+(1-t)z,t\xi+(1-t)\zeta)dt \\ g_{n+j}(y,\eta,x,\xi) &= \chi(y,\eta,x,\xi)\int_0^1 \frac{\partial p}{\partial \xi_j}(y,\eta,tx+(1-t)z,t\xi+(1-t)\zeta)dt \end{aligned}$$

for $j = 1,2,\dots,n$. We note $g_j \in S^1_{cl}(\mathbb{R}^{2n})$ for $j = 1,2,\dots,n$, and $g_j \in S^0_{cl}(\mathbb{R}^{2n})$ for $j = n+1,\dots,2n$. By the choice of χ, we have

$$p(y,\eta,x,\xi) = \sum_{j=1}^n (x_j - z_j)g_j(y,\eta,x,\xi) + \sum_{j=1}^n (\xi_j - \zeta_j)g_{n+j}(y,\eta,x,\xi). \tag{2.3}$$

We also set

$$f_j(x,\xi) = x_j|\xi|\rho(x,\xi), \quad f_{n+j}(x,\xi) = \xi_j\rho(x,\xi)$$

for $j = 1,2,\dots,n$. Then, as well as the computation above, we have

$$\begin{aligned} (f_j \circ S^{-1})&(y,\eta) - f_j(x,\xi) = f_j(z,\zeta) - f(x,\xi) \\ &= -\int_0^1 \frac{d}{dt} f_j(tx+(1-t)z,t\xi+(1-t)\zeta)dt \\ &= -\sum_{k=1}^n (x_k - z_k)\int_0^1 \frac{\partial f_j}{\partial x_k}(tx+(1-t)z,t\xi+(1-t)\zeta)dt \\ &\quad - \sum_{k=1}^n (\xi_k - \zeta_k)\int_0^1 \frac{\partial f_j}{\partial \xi_k}(tx+(1-t)z,t\xi+(1-t)\zeta)dt. \end{aligned}$$

It is easy to see that on $(S\Gamma_2)\times\Gamma_2$, we have

$$\begin{aligned}
\frac{\partial f_j}{\partial x_k}(tx+(1-t)z,t\xi+(1-t)\zeta) &= \delta_{jk}|t\xi+(1-t)\zeta|,\\
\frac{\partial f_j}{\partial \xi_k}(tx+(1-t)z,t\xi+(1-t)\zeta) &= r_{jk}(y,\eta,x,\xi),\\
\frac{\partial f_{n+j}}{\partial x_k}(tx+(1-t)z,t\xi+(1-t)\zeta) &= 0,\\
\frac{\partial f_{n+j}}{\partial \xi_k}(tx+(1-t)z,t\xi+(1-t)\zeta) &= \delta_{jk},
\end{aligned}$$

where $r_{jk}\in S^0_{cl}(\mathbb{R}^{2n})$, $j,k=1,2,\dots,n$. Thus we have

$$(f_j\circ S^{-1})(y,\eta)-f_j(x,\xi) = -(x_j-z_j)\int_0^1|t\xi+(1-t)\zeta|dt+r_j(y,\eta,x,\xi)$$

$$(f_{n+j}\circ S^{-1})(y,\xi)-f_{n+j}(x,\xi) = -(\xi_j-\zeta_j)$$

on $(S\Gamma_2)\times\Gamma_2$ for $j=1,2,\dots,n$, where $r_j\in S^0_{cl}(\mathbb{R}^{2n})$. Since g_j are supported in $\Gamma_1\subset(S\Gamma_2)\times\Gamma_2$, we can find $b_j\in S^0_{cl}(\mathbb{R}^{2n})$ such that

$$(x_j-z_j)g_j(y,\eta,x,\xi) = b_j(y,\eta,x,\xi)((f_j\circ S^{-1})(y,\eta)-f_j(x,\xi))+r'_j,$$

$$(\xi_j-\zeta_j)g_{n+j}(y,\eta,x,\xi) = b_{n+j}(y,\eta,x,\xi)((f_{n+j}\circ S^{-1})(y,\eta)-f_{n+j}(x,\xi))$$

with $r'_j\in S^0_{cl}(\mathbb{R}^{2n})$, $j=1,2,\dots,n$. The assertion now follows from these and (2.3). □

Proof of Theorem 2.1. — The "only if " part is straightforward: If $a_j\in S^1_{cl}(\mathbb{R}^n)$ satisfying (2.1), then $p_j(y,\eta,x,\xi)=(a_j\circ S^{-1})(y,\eta)-a_j(x,-\xi)$ vanish on Λ_S, and hence (2.2) follows from the definition of the FIOs and the L^2-boundedness theorem of FIOs (see, *e.g.*, [6] Theorem 25.3.1 or [16] Theorem 6.2.1).

We suppose (2.2) and show the "if" part. At first, we note $\mathrm{WF}(u)\subset\Lambda_S$: If $(y_0,x_0,\eta_0,-\xi_0)\notin\Lambda_S$, then we can find $a\in S^0_{cl}(\mathbb{R}^n)$ and $b\in S^1_{cl}(\mathbb{R}^n)$ such that $a_0(y_0,\eta_0)\neq 0$, $b_0(x_0,\xi_0)\neq 0$ and that a and b are supported in small conic neighborhoods of (y_0,η_0) and (x_0,ξ_0), respectively, so that $a(y,\eta)\cdot(b\circ S^{-1})(y,\eta)=0$. By (2.2) with Lemma 2.7, we learn

$$\begin{aligned}
&a(y,D_y)((b\circ S^{-1})(y,D_y)-b(x,-D_x))u\\
&\qquad\qquad = (-a(y,D_y)b(x,-D_x)+R(y,D_y))u\in B_2^{-n/2,\infty}(\mathbb{R}^{2n})
\end{aligned}$$

with $R\in S^0_{1,0}(\mathbb{R}^n)$. This implies $a(y,D_y)b(x,-D_x)u\in B_2^{-n/2,\infty}(\mathbb{R}^{2n})$ by the boundedness of R in $B_2^{-n/2,\infty}(\mathbb{R}^{2n})$ (see [6] Corollary B.1.6). Iterating this procedure, we learn

$$[a(y,D_y)b(x,-D_x)]^N u\in B_2^{-n/2,\infty}(\mathbb{R}^{2n})$$

for any N, and this implies $(y_0, \eta_0, x_0, -\xi_0) \notin \mathrm{WF}(u)$.

We now let $p_1, p_2, \dots, p_N \in S^1_{cl}(\mathbb{R}^{2n})$ such that p_j vanish on $\tilde{\Lambda}_S$, and we show

$$p_1(y, D_y, x, -D_x) p_2(y, D_y, x, -D_x) \cdots p_N(y, D_y, x, -D_x) u \in B^{-n/2,\infty}_{2,\mathrm{loc}}(\mathbb{R}^{2n}).$$

By the above observation, we may assume u is essentially supported in an arbitrarily small conic neighborhood of Λ_S. Moreover, by partition of unity, we may also assume p_j are supported in a small convex conic neighborhood of $(y_0, \eta_0, x_0, \xi_0) \in \tilde{\Lambda}_S$, where $(y_0, \eta_0) = S(x_0, \xi_0)$. Then by Lemma 2.8, we have

$$\begin{aligned} p_j(y, D_y, x, -D_x) = \sum_{k=1}^{2n} & b_{jk}(y, D_y, x, -D_x)((f_k \circ S^{-1})(y, D_y) - f_k(x, -D_x)) \\ & + r_j(y, D_y, x, -D_x) \end{aligned}$$

for each $j = 1, \dots, N$, where $b_{jk} \in S^0_{cl}(\mathbb{R}^{2n})$ and $f_j \in S^1_{cl}(\mathbb{R}^{2n})$ are those given in Lemma 2.8, and $r_j \in S(1, dy^2 + \frac{d\eta^2}{\langle\eta\rangle^2} + dx^2 + \frac{d\xi^2}{\langle\xi\rangle^2})$, where $S(m, g)$ denotes the symbol class defined in [6], Section 18.5 (Weyl calculus). Then by simple symbol calculus, we can show

$$\begin{aligned} \prod_{j=1}^{N} p_j(y, D_y, x, -D_x) u = & \sum_{k_1=1}^{2n} \cdots \sum_{k_N=1}^{2n} \prod_{j=1}^{N} b_{jk_j}(y, D_y, x, -D_x) \\ & \times \prod_{j=1}^{N} \big((f_{k_j} \circ S^{-1})(y, D_y) - f_{k_j}(x, -D_x)\big) u + R(y, D_y, x, -D_x) u \end{aligned} \tag{2.4}$$

with some $R \in S(1, dy^2 + \frac{d\eta^2}{\langle\eta\rangle^2} + dx^2 + \frac{d\xi^2}{\langle\xi\rangle^2})$. Note R is bounded in $B^{-n/2,\infty}_2$ $(\mathbb{R}^{2n})$. Each term in the RHS has the form

$$B(y, D_y, x, -D_x) \ker\big[Ad_S(f_{k_1}) \cdots Ad_S(f_{k_N}) U\big]$$

with $B \in S^0_{cl}(\mathbb{R}^{2n})$ except for R, where $\ker[A]$ denotes the distribution kernel of an operator A. Now the claim follows from the assumption and Lemma 2.7. □

3. Proof of Theorem 1.2

Here we prove that, under Assumption A with $\mu = 2$, $W(t) = e^{itH_0} e^{-itH}$ satisfies the condition of Corollary 2.6-(ii) with $S = w_\pm$, where $t \in \mathbb{R}_\pm$. The condition is an Egorov-type theorem, and it was essentially proved in [14] (see also [13, 8]) in the semiclassical formalism. We modify the argument to prove the Egorov theorem in $S^1_{1,0}(\mathbb{R}^n)$ symbol class. Namely, we prove the following:

Theorem 3.1. — *Suppose Assumption A with $\mu = 2$, and suppose the global nontrapping condition. Let $\pm t > 0$. Then for any $a \in S^1_{1,0}(\mathbb{R}^n)$ satisfying* (2.1)*, there is $b(t) \in S^0_{1,0}(\mathbb{R}^n)$ such that*

$$W(t)a(x,D_x)W(t)^{-1} = (a \circ w_\pm^{-1})(x,D_x) + b(t,x,D_x).$$

We first sketch the outline following [14]. We set

$$A(t) = W(t)a(x,D_x)W(t)^{-1}, \quad t \in \mathbb{R}.$$

If we consider $W(t)$ as an evolution operator, we can compute the generator as follows: For $\psi \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\begin{aligned}\frac{d}{dt}W(t)\psi &= e^{itH_0}(iH_0 - iH)e^{-itH_0}W(t)\psi \\ &= -i\big(e^{itH_0}He^{-itH_0} - H_0\big)W(t)\psi = -iL(t)W(t)\psi.\end{aligned}$$

Since

$$e^{itH_0}D_xe^{-itH_0} = D_x, \quad e^{itH_0}xe^{-itH_0} = x - tD_x,$$

we learn that the principal symbol of $L(t)$ is given by

$$\ell(t,x,\xi) = \frac{1}{2}\sum_{j,k=1}^{n}(a_{jk}(x-t\xi) - \delta_{jk})\xi_j\xi_k + V(x-t\xi).$$

In fact, if we use the Weyl calculus, which we do, the symbol of $L(t)$ is given by $\ell(t,x,\xi)$ modulo $S^0_{1,0}$-terms. The classical flow generated by $\ell(t,x,\xi)$ is

$$w(t) = \exp(-tH_{p_0}) \circ \exp(tH_p).$$

Here we use, however, the flow:

$$w_0(t) = \exp(-tH_{p_0}) \circ \exp(tH_k)$$

which is generated by

$$\ell_0(t,x,\xi) = \frac{1}{2}\sum_{j,k=1}^{n}(a_{jk}(x-t\xi) - \delta_{jk})\xi_j\xi_k.$$

Analogously to the usual Egorov theorem, we expect the principal symbol of $A(t)$ is given by $(a_0 \circ w_0(t)^{-1})(x,\xi)$, where a_0 is the principal symbol of a. We construct an asymptotic expansion of $A(t)$ by solving transport equations iteratively. We set

$$\psi_0(t,x,\xi) = (a \circ w_0(t)^{-1})(x,\xi)$$

for $a \in S^1_{1,0}(\mathbb{R}^n)$. We note that by Lemma A.1, $\psi_0(t,\cdot,\cdot) \in S^1_{1,0}(\mathbb{R}^n)$, uniformly in t. For a symbol $q(x,\xi) \in S^m_{1,0}(\mathbb{R}^n)$, we define a family of seminorms by

$$|q|_{m,L,K} = \sum_{|\alpha|+|\beta|\leqslant L} \sup_{x\in K, \xi\in\mathbb{R}^n} \left|\langle\xi\rangle^{-m+|\beta|}\partial_x^\alpha\partial_\xi^\beta q(x,\xi)\right|$$

for $L \in \mathbb{N}$, $K \Subset \mathbb{R}^n$. For $T>0$, we write $I_T = [-T,T]$.

Lemma 3.2. — *Let $a \in S^1_{1,0}(\mathbb{R}^n)$ satisfying (2.1). Then there exists $\psi(t,x,\xi)$ such that*

(i) $\psi(0,x,\xi) = a(x,\xi)$.

(ii) $\psi(t,\cdot,\cdot) \in S^1_{1,0}(\mathbb{R}^n)$ *and for any $L, T>0$ and $K \Subset \mathbb{R}^n$,*

$$|\psi(t,\cdot,\cdot)|_{1,L,K} \leqslant C_{L,T,K}, \quad t \in I_T.$$

(iii) $\psi(t,\cdot,\cdot) - \psi_0(t,\cdot,\cdot) \in S^0_{1,0}(\mathbb{R}^n)$, *and for any $L, T>0$ and $K \Subset \mathbb{R}^n$,*

$$|\psi(t,\cdot,\cdot) - \psi_0(t,\cdot,\cdot)|_{0,L,K} \leqslant C_{L,TK}, \quad t \in I_T.$$

(iv) *Let $G(t) = \psi^W(t,x,D_x)$, and set*

$$R(t) = \frac{d}{dt}G(t) + i[L(t),G(t)].$$

Then $R(t)$ is a smoothing operator, and $\|\langle D_x\rangle^N R(t)\langle D_x\rangle^N\|_{\mathcal{L}(L^2)} \leqslant C_{T,N}$, for any N, $t \in I_T$.

Proof. — We can find $K \Subset \mathbb{R}^n$ such that $\psi_0(t,x,\xi) = 0$ if $x \notin K$, since $w_0(t)(x,\xi)$ has limits as $t \to \pm\infty$. We note $\ell_0(t,\cdot,\cdot) \in S^2_{1,0}(\mathbb{R}^n)$ and for any L, $|\ell_0(t,\cdot,\cdot)|_{2,L,K}$ is uniformly bounded. By the construction, ψ_0 satisfies

$$\frac{\partial}{\partial t}\psi_0(t,x,\xi) = -\{\ell_0,\psi_0\}(t,x,\xi),$$

where $\{\cdot,\cdot\}$ denotes the Poisson bracket. Then by virtue of the Weyl calculus, we learn

$$\frac{\partial}{\partial t}\psi_0^W(t,x,D_x) + i[L(t),\psi_0^W(t,x,D_x)] = r_0^W(t,x,D_x)$$

with $r_0 \in S^0_{1,0}(\mathbb{R}^n)$, and the seminorms of r_0 are locally uniformly bounded in t. Then we solve the transport equation:

$$\frac{\partial}{\partial t}\psi_1(t,x,\xi) + \{\ell_0,\psi_1\}(t,x,\xi) = -r_0(t,x,\xi) \tag{3.1}$$

with $\psi_1(0,x,\xi)=0$. It is easy to see that $\psi_1 \in S^0_{1,0}(\mathbb{R}^n)$ and the seminorms are locally uniformly bounded in t. Iterating this procedure, we obtain

$\psi_j \in S^{1-j}_{1,0}(\mathbb{R}^n)$, $j = 1, 2, \dots$, and we set the asymptotic sum as ψ:

$$\psi \sim \sum_{j=0}^{\infty} \psi_j \quad \text{in } S^1_{1,0}(\mathbb{R}^n).$$

Now it follows from the above construction that

$$\frac{\partial}{\partial t}\psi^W(t,x,D_x) + i[L(t), \psi^W(t,x,D_x)] = r(t,x,D_x) \tag{3.2}$$

with $r \in S^{-\infty}_{1,0}(\mathbb{R}^n)$. Thus our ψ satisfies the required properties. □

Proof of Theorem 3.1. — Let $\psi(t,x,\xi)$ be as in Lemma 3.2 and let $G(t) = \psi^W(t,x,D_x)$. By the lemma, we have

$$\frac{d}{dt}(W(t)^{-1}G(t)W(t)) = W(t)^{-1}R(t)W(t)$$

and the RHS is a smoothing operator, and its seminorms are uniformly bounded. Since $W(0)^{-1}G(0)W(0) = a^W(x,D_x)$, we learn

$$W(t)a^W(x,D_x)W(t)^{-1} - G(t) = R_2(t) \tag{3.3}$$

is a smoothing operator. Thus, the principal symbol of $A(t)$ is $a \circ w_0(t)^{-1}$. It remains to compare $a \circ w_0(t)^{-1}$ with $a \circ w_\pm^{-1}$.

We denote

$$(\tilde{x}(t,z,\eta), \tilde{\xi}(t,z,\eta)) = w_0(t)^{-1}(z,\eta), \quad (\tilde{x}_\pm(z,\eta), \tilde{\xi}_\pm(z,\eta)) = w_\pm^{-1}(z,\eta).$$

Then by Lemma A.1, we learn

$$\begin{aligned}\left|\partial_z^\alpha \partial_\eta^\beta (\tilde{x}(t,z,\eta) - \tilde{x}_\pm(z,\eta))\right| &\leqslant C_{\alpha\beta}\langle\eta\rangle^{-|\beta|}\langle t\eta\rangle^{-\mu+1},\\ \left|\partial_z^\alpha \partial_\eta^\beta (\tilde{\xi}(t,z,\eta) - \tilde{\xi}_\pm(z,\eta))\right| &\leqslant C_{\alpha\beta}\langle\eta\rangle^{1-|\beta|}\langle t\eta\rangle^{-\mu+1}\end{aligned}$$

for $\pm t > 0$. We then compute

$$\begin{aligned}&(a \circ w_0(t)^{-1})(z,\eta) - (a \circ w_\pm^{-1})(z,\eta)\\ &\quad = a(\tilde{x}(t,z,\eta), \tilde{\xi}(t,z,\eta)) - a(\tilde{x}_\pm(z,\eta), \tilde{\xi}_\pm(z,\eta))\\ &\quad = \int_0^1 \frac{\partial}{\partial s} a(s\tilde{x}(t,z,\eta) + (1-s)\tilde{x}_\pm(z,\eta), s\tilde{\xi}(t,z,\eta) + (1-s)\tilde{\xi}_\pm(z,\eta))ds\\ &\quad = (\tilde{x}(t,z,\eta) - \tilde{x}_\pm(z,\eta)) \int_0^1 (\partial_x a)(s\tilde{x} + (1-s)\tilde{x}_\pm, s\tilde{\xi} + (1-s)\tilde{\xi}_\pm)ds\\ &\qquad + (\tilde{\xi}(t,z,\eta) - \tilde{\xi}_\pm(z,\eta)) \int_0^1 (\partial_\xi a)(s\tilde{x} + (1-s)\tilde{x}_\pm, s\tilde{\xi} + (1-s)\tilde{\xi}_\pm)ds\\ &\quad = (\tilde{x}(t) - \tilde{x}_\pm)\cdot A(z,\eta) + (\tilde{\xi}(t) - \tilde{\xi}_\pm)\cdot B(z,\eta),\end{aligned}$$

and it is easy to see

$$\left|\partial_z^\alpha \partial_\eta^\beta A(z,\eta)\right| \leqslant C_{\alpha\beta}\langle\eta\rangle^{1-|\beta|}, \quad \left|\partial_z^\alpha \partial_\eta^\beta B(z,\eta)\right| \leqslant C_{\alpha\beta}\langle\eta\rangle^{-|\beta|}.$$

Combining these, we now have

$$\left|\partial_z^\alpha\partial_\eta^\beta(a\circ w_0(t)^{-1}-a\circ w_\pm^{-1})\right|\leqslant C_{\alpha\beta}\langle\eta\rangle^{1-|\beta|}\langle t\eta\rangle^{-\mu+1}$$

for $\pm t>0$. For fixed $t\neq 0$, this implies $a\circ w_0(t)^{-1}-a\circ w_\pm^{-1}\in S^{2-\mu}_{1,0}=S^0_{1,0}$ since $\mu=2$. Combining this with $\psi(t,\cdot,\cdot)-a\circ w_0(t)^{-1}\in S^0_{1,0}$, we learn $\psi(t,\cdot,\cdot)-a\circ w_\pm\in S^0_{1,0}$. The assertion follows from this and (3.3). □

Theorem 1.2 now follows immediately from Theorem 3.1 and Corollary 2.6-(ii).

4. Fourier integral operators associated to asymptotically homogeneous canonical transform

Here we discuss FIOs associated to asymptotically homogeneous canonical transform, e.g., $w(t)$ and $w_0(t)$, and we prove Theorem 1.5. We start with several definitions.

Definition 4.1. — *Let $\Lambda\subset T^*\mathbb{R}^m\backslash 0$ be a d-dimensional conic submanifold, and let $(x_0,\xi_0)\in\Lambda$. Suppose Ω be a conic neighborhood of (x_0,ξ_0) and let $\Phi:\Omega\to\mathbb{R}^{2m}$ be a local coordinate system on Ω. Φ is called an admissible conic local coordinate system (associated to Λ) if Φ satisfies the following conditions:*

(i) *Φ is expressed as*

$$\Phi(x,\xi)=(|\xi|,\sigma(x,\hat\xi),\tau(x,\hat\xi)),\quad \sigma(x,\hat\xi)\in\mathbb{R}^{d-1},\tau(x,\hat\xi)\in\mathbb{R}^{2m-d}$$

for $(x,\xi)\in T^*\mathbb{R}^{2m}$, where $\hat\xi=\xi/|\xi|$, i.e., $\sigma(x,\xi)$ and $\tau(x,\xi)$ are independent of $|\xi|$.

(ii) *$\Lambda\cap\Omega=\big\{(x,\xi)\in\Omega\bigm|\tau(x,\xi)=0\big\}$.*

Definition 4.2. — *Let $\Lambda\subset T^*\mathbb{R}^m$ be a d-dimensional submanifold. Λ is called asymptotically conic if Λ satisfies the following conditions:*

(i) *There exists a d-dimensional conic submanifold $\Lambda_c\subset T^*\mathbb{R}^m$ such that for any $K\Subset\mathbb{R}^m$ and $\Omega\subset T^*\mathbb{R}^m$: a conic neighborhood of $\Lambda_c\cap(K\times\mathbb{R}^m)$, there is $R>0$ such that*

$$\Lambda\cap\big\{(x,\xi)\bigm|x\in K,|\xi|\geqslant R\big\}\subset\Omega.$$

(ii) *Let Ω be a conic neighborhood of $(x_0,\xi_0)\in\Lambda_c$, and let Φ be an admissible conic local coordinate system on Ω. Then there are $R>0$, an $\mathbb{R}^{2m-d}$-valued function $\varphi(\lambda,\sigma)$ such that*

$$\Lambda\cap\big\{(x,\xi)\in\Omega\bigm||\xi|\geqslant R\big\}$$
$$=\big\{(x,\xi)\in\Omega\bigm|\tau(x,\xi)=\varphi(|\xi|,\sigma(x,\xi)),|\xi|\geqslant R\big\},$$

and $\varepsilon > 0$ such that for any $k \in \mathbb{Z}_+$ and $\alpha \in \mathbb{Z}_+^{d-1}$,

$$\left|\partial_\lambda^k \partial_\sigma^\alpha \varphi(\lambda, \sigma)\right| \leqslant C_{k\alpha} \langle\lambda\rangle^{-\varepsilon-k}, \quad \lambda \geqslant R.$$

Definition 4.3 (Lagrangian distribution). — *Let $\Lambda \subset T^*\mathbb{R}^m$ be an asymptotically conic Lagrangian submanifold, and let $u \in \mathcal{S}'(\mathbb{R}^m)$. Let $\nu \in \mathbb{R}$. u is called a Lagrangian distribution associated to Λ of order ν, if for any $p_1, \dots, p_N \in S^1_{1,0}(\mathbb{R}^m)$ such that $p_j = 0$ on Λ $(j = 1, 2, \dots, N)$,*

$$p_1(x, D_x) p_2(x, D_x) \cdots p_N(x, D_x) u \in B^{-\nu-m/4,\infty}_{2,\mathrm{loc}}(\mathbb{R}^m).$$

We then write $u \in I^\nu(\Lambda, \mathbb{R}^m)$.

Definition 4.4. — *Let $S : T^*\mathbb{R}^n \to T^*\mathbb{R}^n$ be a diffeomorphism. S is called asymptotically homogeneous (of order one) if S satisfies the following conditions: We write $S(x, \xi) = (y(x, \xi), \eta(x, \xi))$. There exists $S_c : T^*\mathbb{R}^n \to T^*\mathbb{R}^n$, a homogeneous canonical map (of order one) such that*

$$y(x, \xi) - y_c(x, \xi) \in (S^{-\varepsilon}_{1,0}(\mathbb{R}^n))^n, \quad \eta(x, \xi) - \eta_c(x, \xi) \in (S^{1-\varepsilon}_{1,0}(\mathbb{R}^n))^n,$$

with some $\varepsilon > 0$, where we denote $S_c(x, \xi) = (y_c(x, \xi), \eta_c(x, \xi))$.

Remark 4.5. — By Lemmas A.1 and A.2, we learn $w_0(t)$ and $w(t)$ are asymptotically homogeneous for $t \neq 0$, and they are associated to homogeneous canonical transforms $w_\pm$ for $t \in \mathbb{R}_\pm$, respectively.

Lemma 4.6. — *Suppose $S : T^*\mathbb{R}^n \to T^*\mathbb{R}^n$ is an asymptotically homogeneous canonical transform. Then*

$$\Lambda_S = \left\{(y, x, \eta, -\xi) \in T^*\mathbb{R}^{2n} \;\middle|\; (y, \eta) = S(x, \xi)\right\}$$

is an asymptotically conic Lagrangian submanifold of $T^*\mathbb{R}^{2n}$.

Proof. — Let S_c be the associated homogeneous canonical transform, and let Λ_{S_c} be the corresponding Lagrangian manifold. Let $(x_0, \xi_0) \in T^*\mathbb{R}^n$ and let $(y_0, x_0, \eta_0, -\xi_0) \in \Lambda_{S_c}$ with $(y_0, \eta_0) = S_c(x_0, x_0)$. Let $\Omega_1 \times \Omega_2 \subset \mathbb{R}^n \times S^{n-1}$ be a small neighborhood of $(x_0, \hat{\xi}_0) \in \Lambda_{S_c}$ and let

$$\psi : \Omega_1 \times \Omega_2 \to \mathbb{R}^n \times \mathbb{R}^{n-1}, \quad (x, \omega) \mapsto (x - x_0, \sigma(\omega))$$

be a local coordinate system on $\Omega_1 \times \Omega_2$. We then set

$$\Psi : \mathbb{R}_+ \times \psi(\Omega_1 \times \Omega_2) \times B_\varepsilon(0) \times B_\varepsilon(0) \to T^*\mathbb{R}^{2n}$$

with

$$\Psi : (\lambda, \alpha, \beta, \tau, \tau') \mapsto$$
$$(y_c(\psi^{-1}(\alpha, \beta)) + \tau, x_0 + \alpha, \lambda(\eta_c(\psi^{-1}(\alpha, \beta)) + \tau'), -\lambda\sigma^{-1}(\beta)),$$

where $B_\varepsilon(0) = \{x \in \mathbb{R}^n \,|\, |x| < \varepsilon\}$ with sufficiently small $\varepsilon > 0$. Then $\operatorname{Ran}\Psi$ is a conic neighborhood of $(y_0, x_0, \eta_0, -\xi_0)$ in $T^*\mathbb{R}^{2n}$, and Ψ^{-1} is an admissible conic local coordinate system. We note if R is sufficiently large,

$$\begin{aligned}&\Lambda_S \cap (\operatorname{Ran}\Psi(\{\lambda > R\}))\\ &\quad = \big\{\big(y(x_0+\alpha, \lambda\sigma^{-1}(\beta)), x_0+\alpha, \eta(x_0+\alpha, \lambda\sigma^{-1}(\beta)), -\lambda\sigma^{-1}(\beta)\big) \;\big|\\ &\qquad\qquad\qquad \lambda > R, (\alpha,\beta) \in \psi(\Omega_1 \times \Omega_2)\big\}\end{aligned}$$

and hence $(y, x, \eta, -\xi) \in \Lambda_S \cap (\operatorname{Ran}\Psi(\{\lambda > R\}))$ if and only if

$$\begin{aligned}\tau &= y(x_0+\alpha, \lambda\sigma^{-1}(\beta)) - y_c(\psi^{-1}(\alpha,\beta)),\\ \tau' &= \lambda^{-1}(\eta(x_0+\alpha, \lambda\sigma^{-1}(\beta)) - \eta_c(x_0+\alpha, \lambda\sigma^{-1}(\beta))),\end{aligned}$$

where $(\lambda, \alpha, \beta, \tau, \tau') = \Psi^{-1}(y, x, \eta, -\xi)$. Now it is easy to check Λ_S satisfies conditions of Definition 4.2 if we set $\sigma \to (\alpha, \beta)$, $\tau \to (\tau, \tau')$,

$$\varphi_j(\lambda, \alpha, \beta) = y_j(x_0+\alpha, \lambda\sigma^{-1}(\beta)) - y_{c,j}(x_0+\alpha, \lambda\sigma^{-1}(\beta)),$$

and

$$\varphi_{n+j}(\lambda, \alpha, \beta) = \lambda^{-1}\big(\eta_j(x_0+\alpha, \lambda\sigma^{-1}(\beta)) - \eta_{c,j}(x_0+\alpha, \lambda\sigma^{-1}(\beta))\big)$$

for $j = 1, 2, \dots, n$. □

Definition 4.7. — *Let S be an asymptotically homogeneous canonical transform from $T^*\mathbb{R}^n$ to $T^*\mathbb{R}^n$, and let Λ_S be the associated Lagrangian manifold in $T^*\mathbb{R}^{2n}$, which is asymptotically conic by Lemma 4.6. Let $U \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n))$ and let $u \in \mathcal{S}'(\mathbb{R}^{2n})$ be its distribution kernel. Then U is called a Fourier integral operator associated to S of order $\sigma \in \mathbb{R}$ if $u \in I^\sigma(\Lambda_S, \mathbb{R}^{2n})$.*

Given Definition 4.7, we now have the exact meaning of Theorem 1.5, and we prove Theorem 1.5 in the remaining of this section. We note Theorem 2.5 holds with little modification for FIOs associated to asymptotically homogeneous canonical transforms:

Theorem 4.8. — *Let $S : T^*\mathbb{R}^n \to T^*\mathbb{R}^n$ be an asymptotically homogeneous canonical transform, and let $U \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n))$. U is an FIO associated to S if and only if for any $a_1, a_2, \dots, a_N \in S^1_{1,0}(\mathbb{R}^n)$ satisfying* (2.1),

$$Ad_S(a_1)Ad_S(a_2)\cdots Ad_S(a_N)U \in \mathcal{L}(L^2_{\mathrm{cpt}}(\mathbb{R}^n), L^2_{\mathrm{loc}}(\mathbb{R}^n)).$$

The proof of Theorem 4.8 is almost the same as that of Theorem 2.5. In the "only if" part, we use the fact that L^2-boundedness theorem holds for FIOs associated to asymptotically homogeneous canonical transforms. An analogue of Lemma 2.8 is given as follows:

Lemma 4.9. — *Let S be an asymptotically homogeneous canonical transform as above. Let $p \in S^1_{1,0}(\mathbb{R}^{2n})$ such that p vanishes on*

$$\tilde{\Lambda}_S = \big\{(y,\eta,x,\xi) \bigm| (y,\eta) = S(x,\xi)\big\}.$$

Then there exist $b_j \in S^0_{1,0}(\mathbb{R}^{2n})$, $f_j \in S^1_{1,0}(\mathbb{R}^n)$, $(j = 1,2,\dots,2n)$, and $r \in S^0_{1,0}(\mathbb{R}^{2n})$ such that

$$p(y,\eta,x,\xi) = \sum_{j=1}^{2n} b_j(y,\eta,x,\xi)\big((f_j \circ S^{-1})(y,\eta) - f_j(x,\xi)\big) + r(y,\eta,x,\xi).$$

Lemma 4.9 is proved in almost the same manner as Lemma 2.8. Then the rest of the proof of Theorem 4.8 follows from the argument of Theorem 2.5, and we omit the detail.

Proof of Theorem 1.5. — Given the above formulation, the proof of Theorem 4.8 is similar to that of Theorem 1.2. Here we explain only the necessary modifications. When we construct the asymptotic solution to the Heisenberg equation: $\partial_t G(t) = i[L(t), G(t)]$, we use

$$\psi_0(t,x,\xi) = (a \circ w(t)^{-1})(x,\xi)$$

instead of $(a \circ w_0(t)^{-1})(x,\xi)$ in Section 3. By virtue of Lemma A.2, we learn $\psi_0 \in S^1_{1,0}(\mathbb{R}^n)$, and we can carry out the symbol calculus as in Section 3 with no difficulty. Then the remainder terms of the asymptotic expansion (*e.g.*, r_0 in the proof of Lemma 3.2) is in $S^0_{1,0}(\mathbb{R}^n)$ even if $1 < \mu < 2$, since $w(t)$ includes the influence of the potential function $V(x)$. The rest of the proof is almost identical. □

5. Microlocal structure of wave operators

Throughout this section, we suppose Assumption B with $1 < \mu < 2$, and we prove Theorem 1.6. We use an argument analogous to Lemma 3.2, but we need to examine the t-dependence of seminorms more carefully. We note

$$\big|\partial_x^\alpha \partial_\xi^\beta \ell_0(t,x,\xi)\big| \leqslant C_{\alpha\beta K}\langle t\xi\rangle^{-\mu-|\alpha|}\langle\xi\rangle^{2-|\beta|}, \tag{5.1}$$

$$\big|\partial_x^\alpha \partial_\xi^\beta \tilde{V}(t,x,\xi)\big| \leqslant C_{\alpha\beta K}\langle t\xi\rangle^{-\mu-|\alpha|}\langle\xi\rangle^{-|\beta|} \tag{5.2}$$

for any $\alpha, \beta \in \mathbb{Z}^n_+$, $K \Subset \mathbb{R}^n$, where $\tilde{V}(t,x,\xi) = V(x - t\xi)$. For $a \in S^1_{cl}(\mathbb{R}^n)$ satisfying (2.1), we set $\psi_0(t,x,\xi) = (a \circ w_0(t)^{-1})(x,\xi)$ as in Section 3. Then we have

$$\big|\psi_0(t,\cdot,\cdot)\big|_{1,L,K} \leqslant C_{L,K}, \quad t \in \mathbb{R},$$

for any $L>0$, $K\Subset\mathbb{R}^n$. Moreover, by (5.1) and (5.2), we also have

$$\big|r_0(t,\cdot,\cdot)\big|_{0,L,K}\leqslant C_{L,K}\langle t\rangle^{-\mu},\quad t\in\mathbb{R},$$

where r_0 is defined as in the proof of Lemma 3.2. Here we have used the fact $|\xi|\geqslant c>0$ for all t on the support of ψ_0. Thus, since the solution to the transport equation (3.1) is uniformly bounded, we have

$$\big|\psi_1(t,\cdot,\cdot)\big|_{0,L,K}\leqslant C_{L,K},\quad t\in\mathbb{R}.$$

We repeat this procedure. We set ψ_j be the solution to the transport equations:

$$\frac{\partial}{\partial t}\psi_j(t,x,\xi)+\{\ell_0,\psi_j\}(t,x,\xi)=-r_{j-1}(t,x,\xi)$$

with $\psi_j(0,x,\xi)=0$ (as in the proof of Lemma 3.2), and we set $r_j\in S^{-j}_{1,0}$ such that

$$r_j^W(t,x,D_x)=\frac{\partial}{\partial t}\psi_j^W(t,x,D_x)+i[L(t),\psi_j^W(t,x,D_x)]+r_{j-1}^W(t,x,D_x)$$

given $\psi_j\in S^{1-j}_{1,0}(\mathbb{R}^n)$. Then we learn, similarly as above,

$$\big|r_j(t,\cdot,\cdot)\big|_{-j,L,K}\leqslant C_{jLK}\langle t\rangle^{-\mu},\quad \big|\psi_j(t,\cdot,\cdot)\big|_{1-j,L,K}\leqslant C_{jLK},\tag{5.3}$$

uniformly in $t\in\mathbb{R}$ with any $L>0$, $K\Subset\mathbb{R}^n$, for each j. By (5.3) and the transport equations, we learn

$$\psi_{j,\pm}(x,\xi)=\lim_{t\to\pm\infty}\psi_j(t,x,\xi)\in S^{1-j}_{1,0}(\mathbb{R}^n),\quad j=0,1,2,\dots,$$

exist, and they converge with respect to the seminorms in $S^{1-j}_{1,0}(\mathbb{R}^n)$ by virtue of Lemma A.1. More precisely, we have

$$\big|\psi_j(t,\cdot,\cdot)-\psi_{j,\pm}(\cdot,\cdot)\big|_{1-j,L,K}\leqslant C_{jLK}\langle t\rangle^{1-\mu},\quad t\in\mathbb{R}\tag{5.4}$$

for all j. We note

$$\psi_{0,\pm}(x,\xi)=(a\circ w_\pm^{-1})(x,\xi)$$

by our construction.

We now construct the asymptotic sum: $\psi\sim\sum_{j=0}^\infty\psi_j$ as follows. We choose $K\Subset\mathbb{R}^n$ so large that all symbols in the above construction are supported in $K\times\mathbb{R}^n$ for all t. We choose $\varepsilon_j>0$ so that

$$\sup\big\{\langle\xi\rangle^{j-2}|\partial_x^\alpha\partial_\xi^\beta\psi_j(t,x,\xi)|\;\big|\;|\alpha|+|\beta|\leqslant j,x\in K,|\xi|\geqslant\varepsilon_j^{-1},t\in\mathbb{R}\big\}\leqslant 2^{-j},$$

which is possible since $|\psi_j(t,\cdot,\cdot)|_{1-j,L,K}$ is uniformly bounded in t. We let $\chi\in C^\infty(\mathbb{R}^n)$ such that $\chi(\xi)=0$ for $|\xi|\leqslant 1$, and $\chi(\xi)=1$ for $|\xi|\geqslant 2$. Then we set

$$\psi(t,x,\xi)=\sum_{j=0}^\infty\chi(\varepsilon_j\xi)\psi_j(t,x,\xi).$$

By the standard argument, we learn $\psi(t,\cdot,\cdot)\in S^1_{1,0}(\mathbb{R}^n)$ for all $t\in\mathbb{R}$. Moreover, by the same argument with (5.4), we learn

$$\psi_\pm(x,\xi)=\lim_{t\to\pm\infty}\psi(t,x,\xi)\in S^1_{1,0}(\mathbb{R}^n)$$

exist, and they converge with respect to seminorms of $S^1_{1,0}(\mathbb{R}^n)$.

Lemma 5.1. — *Let ψ as above, and let $r\in S^1_{1,0}(\mathbb{R}^n)$ such that*

$$r^W(t,x,D_x)=\frac{\partial}{\partial t}\psi^W(t,x,D_x)+i[L(t),\psi^W(t,x,D_x)].$$

Then $r(t,\cdot,\cdot)\in S^{-\infty}_{1,0}(\mathbb{R}^n)$ for each $t\in\mathbb{R}$, and for any N,

$$\big|r(t,\cdot,\cdot)\big|_{-N,L,K}\leqslant C_{NLK}\langle t\rangle^{-\mu}.$$

Proof. — We write $\tilde\psi_j(t,x,\xi)=\chi(\varepsilon_j\xi)\psi_j(t,x,\xi)$ and set $\tilde r_j(t,x,\xi)\in S^{-j}_{1,0}$ so that

$$\frac{\partial}{\partial t}\tilde\psi_j^W(t,x,D_x)+i[L(t),\tilde\psi_j^W(t,x,D_x)]=\tilde r_j^W(t,x,D_x).$$

By estimating $\partial_t\tilde\psi_j$ and $[L(t),\tilde\psi_j^W(t,x,D_x)]$ separately, we obtain rather crude estimates:

$$\big|\tilde r_j(t,\cdot,\cdot)\big|_{2-j,L,K}\leqslant C_{LK}\langle t\rangle^{-\mu}2^{-j},\quad t\in\mathbb{R}, \tag{5.5}$$

if $j\geqslant L+1$, where C_{LK} is independent of t and j. Similarly we have

$$\big|\tilde r_j(t,\cdot,\cdot)-r_j(t,\cdot,\cdot)\big|_{-N,L,K}\leqslant C_{jNLK}\langle t\rangle^{-\mu} \tag{5.6}$$

for each j. Now we compute

$$\begin{aligned}r^W(t,x,D_x)&=\sum_{j=0}^\infty\Big(\frac{\partial}{\partial t}\tilde\psi_j^W(t,x,D_x)+i\big[L(t),\tilde\psi_j^W(t,x,D_x)\big]\Big)\\&=\mathrm{I}+\mathrm{II}+\mathrm{III},\end{aligned}$$

where

$$\begin{aligned}
\mathrm{I} &= \sum_{j=0}^{M}\Big(\frac{\partial}{\partial t}\psi_j^W(t,x,D_x) + i\big[L(t),\psi_j^W(t,x,D_x)\big]\Big)\\
&= r_M^W(t,x,D_x),\\
\mathrm{II} &= \sum_{j=0}^{M}\Big(\frac{\partial}{\partial t}(\tilde\psi_j-\psi_j)^W(t,x,D_x) + i\big[L(t),(\tilde\psi_j-\psi_j)^W(t,x,D_x)\big]\Big)\\
&= \sum_{j=0}^{M}\big((\tilde r_j - r_j)^W(t,x,D_x)\big),\\
\mathrm{III} &= \sum_{j=M+1}^{\infty}\Big(\frac{\partial}{\partial t}\tilde\psi_j^W(t,x,D_x) + i\big[L(t),\tilde\psi_j^W(t,x,D_x)\big]\Big)\\
&= \sum_{j=M+1}^{\infty}\tilde r_j^W(t,x,D_x),
\end{aligned}$$

where $M = \max(N+2, L)$. Here we denote the symbol of an operator A by $\sigma(A)$. Then we have $|\sigma(\mathrm{I})|_{-N,L,K} \leqslant C\langle t\rangle^{-\mu}$ by (5.3). Using (5.6), we also have $|\sigma(\mathrm{II})|_{-N,L,K} \leqslant C\langle t\rangle^{-\mu}$. Finally we have

$$|\sigma(\mathrm{III})|_{-N,L,K} \leqslant C\sum_{j=M+1}^{\infty}\langle t\rangle^{-\mu}2^{-j} \leqslant C'\langle t\rangle^{-\mu}$$

by (5.5), and the claim follows from these inequalities. □

Proof of Theorem 1.3. — Let $R(t) = r^W(t,x,D_x)$. Lemma 5.1 implies

$$\|R(t)\|_{\mathcal{L}(H^{-N},H^N)} \leqslant C_N\langle t\rangle^{-\mu}, \quad t\in\mathbb{R}$$

for any N. By our construction, we have

$$W(t)^{-1}G(t)W(t) - a^W(x,D_x) = \int_0^t W(s)^{-1}R(s)W(s)ds,$$

where $G(t) = \psi^W(t,x,D_x)$. Hence we have

$$W(t)^{-1}G(t) - a^W(x,D_x)W(t)^{-1} = \int_0^t W(s)^{-1}R(s)W(s)W(t)^{-1}ds. \tag{5.7}$$

We note

$$\begin{aligned}
\big\|W(t)\big\|_{\mathcal{L}(H^N,H^N)} &= \big\|\langle D_x\rangle^N e^{itH_0}e^{-itH}\langle D_x\rangle^{-N}\big\|_{\mathcal{L}(L^2)}\\
&\leqslant \big\|\langle D_x\rangle^N\langle H\rangle^{-N/2}\big\|_{\mathcal{L}(L^2)}\big\|\langle H\rangle^{N/2}\langle D_x\rangle^{-N}\big\|_{\mathcal{L}(L^2)}
\end{aligned}$$

is bounded uniformly in $t \in \mathbb{R}$ with any $N \in \mathbb{R}$. Hence we learn

$$\left\| W(s)^{-1} R(t) W(s) W(t)^{-1} \right\|_{\mathcal{L}(H^{-N}, H^N)} \leqslant C_N \langle t \rangle^{-\mu}$$

and then the RHS of (5.7) converges absolutely in $\mathcal{L}(H^{-N}, H^N)$ as $t \to \pm\infty$. On the other hand, $G(t)$ converges to $\psi_\pm^W(x, D_x)$ in $OPS^1_{1,0}(\mathbb{R}^n)$, and hence in $\mathcal{L}(H^1, L^2)$ as $t \to \pm\infty$. By the definition of wave operators, we then have

$$W(t)^{-1} G(t) \to W_\pm \psi_\pm^W(x, D_x),$$
$$a^W(x, D_x) W(t)^{-1} \to a^W(x, D_x) W_\pm$$

strongly in $\mathcal{L}(H^1, H^{-1})$ as $t \to \pm\infty$. Thus we learn

$$W_\pm \psi_\pm^W(x, D_x) - a^W(x, D_x) W_\pm = R \in \mathcal{L}(H^{-N}, H^N)$$

with any N. Since $\psi_\pm - a \circ w_\pm^{-1} \in S^0_{1,0}(\mathbb{R}^n)$, *i.e.*, $a - \psi_\pm \circ w_\pm \in S^0_{1,0}(\mathbb{R}^n)$, Theorem 1.6 now follows from Corollary 2.6. □

Appendix A. Classical trajectories

Here we prove several technical inequalities.

Lemma A.1. — *Suppose Assumption A with $\mu > 0$, and assume the global nontrapping condition. Let*

$$(z(t,x,\xi), \eta(t,x,\xi)) = w_0(t)(x,\xi) = \exp(-tH_{p_0}) \circ \exp(tH_k)(x,\xi),$$
$$(z_\pm(x,\xi), \xi_\pm(x,\xi)) = w_\pm(x,\xi) = \lim_{t \to \pm\infty} w_0(t)(x,\xi).$$

Then for any $\alpha, \beta \in \mathbb{Z}_+^n$ and $K \Subset \mathbb{R}^n$ there is $C_{\alpha\beta K} > 0$ such that

$$\left|\partial_x^\alpha \partial_\xi^\beta z(t,x,\xi)\right| \leqslant C_{\alpha\beta K} \langle \xi \rangle^{-|\beta|}, \quad \left|\partial_x^\alpha \partial_\xi^\beta \eta(t,x,\xi)\right| \leqslant C_{\alpha\beta K} \langle \xi \rangle^{1-|\beta|} \tag{A.1}$$

and, moreover,

$$\left|\partial_x^\alpha \partial_\xi^\beta (z(t,x,\xi) - z_\pm(x,\xi))\right| \leqslant C_{\alpha\beta K} \langle \xi \rangle^{-|\beta|} \langle t\xi \rangle^{-\mu+1}, \tag{A.2}$$

$$\left|\partial_x^\alpha \partial_\xi^\beta (\eta(t,x,\xi) - \xi_\pm(x,\xi))\right| \leqslant C_{\alpha\beta K} \langle \xi \rangle^{1-|\beta|} \langle t\xi \rangle^{-\mu} \tag{A.3}$$

for $x \in K$, $\xi \in \mathbb{R}^n$.

Proof. — We note

$$\frac{\partial}{\partial t} z_j = \frac{\partial \ell_0}{\partial \eta_j}(z,\eta), \quad \frac{\partial}{\partial t} \eta_j = -\frac{\partial \ell_0}{\partial z_j}(z,\eta) \tag{A.4}$$

and ℓ_0 satisfies

$$\left|\partial_z^\alpha \partial_\eta^\beta \ell_0(z,\eta)\right| \leqslant C_{\alpha\beta K} \langle t\eta \rangle^{-\mu-|\alpha|} \langle \eta \rangle^{2-|\beta|}$$

for $z \in K \Subset \mathbb{R}^n$. We first show (A.1) by induction in $|\alpha| + |\beta| = m$. If $\alpha = \beta = 0$, (A.1) is well-known (see, *e.g.*, [14], Lemma 3.). Suppose (A.1) holds for $|\alpha| + |\beta| < m$, and let $|\alpha| + |\beta| = m$. By differentiating (A.4), we have

$$\frac{\partial}{\partial t}\big(\partial_x^\alpha \partial_\xi^\beta z_j\big) = \sum_{k=1}^n \big(\partial_x^\alpha \partial_\xi^\beta z_k\big)\frac{\partial^2 \ell_0}{\partial z_k \partial \eta_j} + \sum_{k=1}^n \big(\partial_x^\alpha \partial_\xi^\beta \eta_k\big)\frac{\partial^2 \ell_0}{\partial \eta_k \partial \eta_j} + r_1, \tag{A.5}$$

$$\frac{\partial}{\partial t}\big(\partial_x^\alpha \partial_\xi^\beta \eta_j\big) = \sum_{k=1}^n \big(\partial_x^\alpha \partial_\xi^\beta z_k\big)\frac{\partial^2 \ell_0}{\partial z_k \partial z_j} - \sum_{k=1}^n \big(\partial_x^\alpha \partial_\xi^\beta \eta_k\big)\frac{\partial^2 \ell_0}{\partial \eta_k \partial z_j} + r_2, \tag{A.6}$$

where

$$r_1 = O\big(\langle t\eta\rangle^{-\mu}\langle \eta\rangle^{1-|\beta|}\big), \quad r_2 = O\big(\langle t\eta\rangle^{-1-\mu}\langle \eta\rangle^{2-|\beta|}\big)$$

by the induction hypothesis. We also note

$$\frac{\partial^2 \ell_0}{\partial z \partial \eta} = O\big(\langle t\eta\rangle^{-1-\mu}\langle\eta\rangle\big),\ \frac{\partial^2 \ell_0}{\partial \eta^2} = O\big(\langle t\eta\rangle^{-\mu}\big),\ \frac{\partial^2 \ell_0}{\partial z^2} = O\big(\langle t\eta\rangle^{-2-\mu}\langle\eta\rangle^2\big).$$

We consider the case: $t > 0$. The case: $t < 0$ is handled similarly. We now let $R \gg 0$, and $R \leqslant |\xi| \leqslant 2R$. This also implies $R/C \leqslant |\eta| \leqslant CR$ with some $C > 0$. Then we have

$$\left|\frac{\partial}{\partial t}\big(|\partial_x^\alpha \partial_\xi^\beta z| + R^{-1}|\partial_x^\alpha \partial_\xi^\beta \eta|\big)\right| \leqslant C\langle tR\rangle^{-\mu}R\big(|\partial_x^\alpha \partial_\xi^\beta z| + R^{-1}|\partial_x^\alpha \partial_\xi^\beta \eta|\big) + C\langle tR\rangle^{-\mu}R^{1-|\beta|}.$$

By the Duhamel formula and the estimate on the initial condition:

$$\big(|\partial_x^\alpha \partial_\xi^\beta z| + R^{-1}|\partial_x^\alpha \partial_\xi^\beta \eta|\big)\Big|_{t=0} \leqslant CR^{-|\beta|},$$

we learn

$$\begin{aligned}|\partial_x^\alpha \partial_\xi^\beta z| + R^{-1}|\partial_x^\alpha \partial_\xi^\beta \eta| &\leqslant C\exp\left(C\int_0^\infty \langle tR\rangle^{-\mu}R dt\right)\times \\ &\qquad \times\left(CR^{-|\beta|} + C\int_0^\infty \langle tR\rangle^{-\mu}R^{1-|\beta|}dt\right) \\ &\leqslant C'R^{-|\beta|}\end{aligned}$$

since $\mu > 1$. This implies (A.1).

Now we use (A.5)–(A.6) again with (A.1) to learn

$$\begin{aligned}\left|\frac{\partial}{\partial t}\left(|\partial_x^\alpha\partial_\xi^\beta z|\right)\right| &\leqslant C\langle tR\rangle^{-1-\mu}R^{1-|\beta|}+C\langle tR\rangle^{-\mu}R^{1-|\beta|}+C\langle tR\rangle^{-\mu}R^{1-|\beta|}\\ &\leqslant C'\langle tR\rangle^{-\mu}R^{1-|\beta|},\\ \left|\frac{\partial}{\partial t}\left(|\partial_x^\alpha\partial_\xi^\beta \eta|\right)\right| &\leqslant C\langle tR\rangle^{-2-\mu}R^{2-|\beta|}+C\langle tR\rangle^{-1-\mu}R^{2-|\beta|}+C\langle tR\rangle^{-1-\mu}R^{2-|\beta|}\\ &\leqslant C'\langle tR\rangle^{-1-\mu}R^{2-|\beta|}.\end{aligned}$$

Hence, by integrating these on $[t,\infty)$, we learn

$$\begin{aligned}\left|\partial_x^\alpha\partial_\xi^\beta(z(t,x,\xi)-z_+(x,\xi))\right| &\leqslant C\int_t^\infty \langle tR\rangle^{-\mu}R^{1-|\beta|}dt\\ &= CR^{-|\beta|}\int_{Rt}^\infty\langle s\rangle^{-\mu}ds\leqslant CR^{-|\beta|}\langle tR\rangle^{1-\mu},\\ \left|\partial_x^\alpha\partial_\xi^\beta(\eta(t,x,\xi)-\xi_+(x,\xi))\right| &\leqslant C\int_t^\infty \langle tR\rangle^{-1-\mu}R^{2-|\beta|}dt\\ &= CR^{1-|\beta|}\int_{Rt}^\infty\langle s\rangle^{-1-\mu}ds\leqslant CR^{1-|\beta|}\langle tR\rangle^{-\mu},\end{aligned}$$

For the case: $t<0$, we integrate these inequalities on $(-\infty,t]$ to obtain corresponding estimates. (A.2) and (A.3) follows immediately from these estimates. □

Next we consider the evolution:

$$(z_1(t,x,\xi),\eta_1,t,x,\xi))=w(t)(x,\xi)=\exp(-tH_{p_0})\circ\exp(tH_p)(x,\xi)$$

and compare it with $w_0(t)(x,\xi)$ as $|\xi|\to\infty$ for t in a fixed bounded interval. We denote $I_T=[-T,T]$, and we consider the case $t>0$ only in the proof.

Lemma A.2. — *Suppose Assumption A with $1<\mu<2$, and assume the global nontrapping condition. Let $T>0$. Then for any $\alpha,\beta\in\mathbb{Z}_+^n$ and $K\Subset\mathbb{R}^n$, there is $C=C(\alpha,\beta,K,T)>0$ such that*

$$\left|\partial_x^\alpha\partial_\xi^\beta z_1(t,x,\xi)\right|\leqslant C\langle\xi\rangle^{-|\beta|},\quad \left|\partial_x^\alpha\partial_\xi^\beta \eta_1(t,x,\xi)\right|\leqslant C\langle\xi\rangle^{1-|\beta|}\tag{A.7}$$

and

$$\left|\partial_x^\alpha\partial_\xi^\beta(z_1(t,x,\xi)-z(t,x,\xi))\right|\leqslant C\langle\xi\rangle^{1-\mu-|\beta|},\tag{A.8}$$

$$\left|\partial_x^\alpha\partial_\xi^\beta(\eta_1(t,x,\xi)-\eta(t,x,\xi))\right|\leqslant C\langle\xi\rangle^{1-\mu-|\beta|}\tag{A.9}$$

for $t\in I_T$, $x\in K$ and $\xi\in\mathbb{R}^n$.

Proof. — **Step 1:** We first show (A.8) and (A.9) with $\alpha = \beta = 0$. We denote $\tilde{V}(t, z, \eta) = V(z - t\eta)$ so that $\ell(t, z, \eta) = \ell_0(t, z, \eta) + \tilde{V}(t, z, \eta)$. We note

$$\frac{\partial}{\partial t}(z_1 - z) = \frac{\partial \ell_0}{\partial \eta}(z_1, \eta_1) - \frac{\partial \ell_0}{\partial \eta}(z, \eta) + \frac{\partial \tilde{V}}{\partial \eta}(z_1, \eta_1), \tag{A.10}$$

$$\frac{\partial}{\partial t}(\eta_1 - \eta) = -\frac{\partial \ell_0}{\partial z}(z_1, \eta_1) + \frac{\partial \ell_0}{\partial z}(z, \eta) - \frac{\partial \tilde{V}}{\partial z}(z_1, \eta_1). \tag{A.11}$$

We now suppose

$$|z_1 - z| \leqslant \varepsilon_0 |z|, \quad |\eta_1 - \eta| \leqslant \varepsilon_0 |\eta| \tag{A.12}$$

for $x \in K \Subset \mathbb{R}^n$, $\xi \in \mathbb{R}^n$ and $t \in I_{T'}$ with $T' > 0$. We suppose $t > 0$ and we have

(A.13)
$$\begin{aligned}\frac{\partial}{\partial t}(z_1 - z) = (z_1 - z) \cdot &\int_0^1 \frac{\partial^2 \ell_0}{\partial z \partial \eta}(s z_1 + (1-s)z, s\eta_1 + (1-s)\eta) ds \\ &+ (\eta_1 - \eta) \cdot \int_0^1 \frac{\partial^2 \ell_0}{\partial \eta \partial \eta}(s z_1 + (1-s)z, s\eta_1 + (1-s)\eta) ds \\ &+ \frac{\partial \tilde{V}}{\partial \eta}(z_1, \eta_1),\end{aligned}$$

(A.14)
$$\begin{aligned}\frac{\partial}{\partial t}(\eta_1 - \eta) = -(z_1 - z) \cdot &\int_0^1 \frac{\partial^2 \ell_0}{\partial z \partial z}(s z_1 + (1-s)z, s\eta_1 + (1-s)\eta) ds \\ &- (\eta_1 - \eta) \cdot \int_0^1 \frac{\partial^2 \ell_0}{\partial \eta \partial z}(s z_1 + (1-s)z, s\eta_1 + (1-s)\eta) ds \\ &- \frac{\partial \tilde{V}}{\partial z}(z_1, \eta_1),\end{aligned}$$

for $t \in [0, T']$. We again assume $R \leqslant |\xi| \leqslant 2R$ with $R \gg 0$, so that $|\eta| \sim O(|\xi|) = O(R)$. Then we have

$$\begin{aligned}\frac{\partial}{\partial t}(|z_1 - z| + R^{-1}|\eta_1 - \eta|) \leqslant C\langle tR\rangle^{-\mu} R(|z_1 - z| + R^{-1}|\eta_1 - \eta|) \\ + C\langle tR\rangle^{2-\mu} R^{-1}.\end{aligned}$$

Then by the Duhamel formula, we learn

$$\begin{aligned}|z_1 - z| + R^{-1}|\eta_1 - \eta| &\leqslant C e^{C\int_0^t \langle sR\rangle^{-\mu} R ds} \times \int_0^t \langle sR\rangle^{2-\mu} R^{-1} ds \\ &\leqslant C' R^{-2} \int_0^{Rt} \langle s\rangle^{2-\mu} ds,\end{aligned}$$

since $z_1 = z$, $\eta_1 = \eta$ at $t = 0$. We note

$$\int_0^\sigma \langle s\rangle^{2-\mu} ds \leqslant \int_0^\sigma (1+s)^{2-\mu} ds = \frac{(1+\sigma)^{3-\mu} - 1}{3-\mu} \leqslant C(\sigma + \sigma^{3-\mu}),$$

and hence

$$|z_1 - z| + R^{-1}|\eta_1 - \eta| \leqslant CR^{-2}(Rt + (Rt)^{3-\mu}) \leqslant C' t R^{1-\mu} \tag{A.15}$$

for $t \in [0, T']$, $R \gg 0$. Thus, in particular, (A.12) holds with $\varepsilon_0 = O(R^{1-\mu})$. By contradiction, we learn that (A.12) holds for $t \in [0, T]$ if $|\xi|$ is sufficiently large. (A.15) also implies (A.8) with $\alpha = \beta = 0$. We substitute (A.15) to (A.14) to learn

$$\begin{aligned}\left|\frac{\partial}{\partial t}(\eta_1 - \eta)\right| &\leqslant C\big(t\langle tR\rangle^{-2-\mu} R^{3-\mu} + t\langle tR\rangle^{-1-\mu} R^{3-\mu} + \langle tR\rangle^{1-\mu}\big) \\ &\leqslant C\big(\langle tR\rangle^{-1-\mu} R^{2-\mu} + \langle tR\rangle^{-\mu} R^{2-\mu} + \langle tR\rangle^{1-\mu}\big)\end{aligned}$$

since $t\langle tR\rangle^{-1} \leqslant R^{-1}$. Integrating this inequality, we have

$$\begin{aligned}|\eta_1 - \eta| &\leqslant C\left(R^{1-\mu} \int_0^\infty \langle tR\rangle^{-\mu} R dt + \int_0^t \langle sR\rangle^{1-\mu} ds\right) \\ &\leqslant C'\left(R^{1-\mu} + \frac{\langle tR\rangle^{2-\mu}}{R}\right) \leqslant C'' R^{1-\mu}\end{aligned}$$

for $t \in [0, T]$, and this implies (A.9) with $\alpha = \beta = 0$.

Step 2: We then prove (A.7) mimicking the proof of (A.1). We note

$$\big|\partial_z^\alpha \partial_\eta^\beta \ell(z, \eta)\big| \leqslant C\big(\langle t\eta\rangle^{-\mu-|\alpha|}\langle\eta\rangle^{2-|\beta|} + \langle t\eta\rangle^{2-\mu-|\alpha|}\langle\eta\rangle^{-|\beta|}\big)$$

for any $\alpha, \beta \in \mathbb{Z}_+^n$. We prove them by induction in $|\alpha| + |\beta| = m$. We suppose (A.7) holds for $|\alpha| + |\beta| < m$ and let $|\alpha| + |\beta| = m$. Analogously to the proof of (A.1), it follows from the induction step that if $R \leqslant |\xi| \leqslant 2R$ with $R \gg 0$,

$$\begin{aligned}&\frac{\partial}{\partial t}\big(\big|\partial_x^\alpha \partial_\xi^\beta z_1\big| + R^{-1}\big|\partial_x^\alpha \partial_\xi^\beta \eta_1\big|\big) \\ &\quad\leqslant C\big(\langle tR\rangle^{-\mu} R + \langle tR\rangle^{2-\mu} R^{-1}\big)\big(\big|\partial_x^\alpha \partial_\xi^\beta z_1\big| + R^{-1}\big|\partial_x^\alpha \partial_\xi^\beta \eta_1\big|\big) \\ &\qquad + C\big(\langle tR\rangle^{-\mu} R^{1-|\beta|} + \langle tR\rangle^{2-\mu} R^{-1-|\beta|}\big).\end{aligned}$$

Hence, by using the Duhamel formula again, we obtain

$$\begin{aligned}
&\left|\partial_x^\alpha\partial_\xi^\beta z_1\right| + R^{-1}\left|\partial_x^\alpha\partial_\xi^\beta \eta_1\right| \\
&\quad \leqslant C\exp\biggl(C\int_0^t \langle sR\rangle^{-\mu} R ds + C\int_0^t \langle sR\rangle^{2-\mu}R^{-1}ds\biggr)\times \\
&\qquad \times\biggl(R^{-|\beta|} + \int_0^t \langle sR\rangle^{-\mu}R^{1-|\beta|}ds + \int_0^t \langle sR\rangle^{2-\mu}R^{-1-|\beta|}ds\biggr) \\
&\quad \leqslant C' R^{-|\beta|}
\end{aligned}$$

for $t\in[0,T]$, and (A.7) follows.

Step 3: We now prove (A.10) and (A.11). We again prove it by induction in $m=|\alpha|+|\beta|$. We suppose

$$\text{(A.16)}\quad \left|\partial_x^\alpha\partial_\xi^\beta (z_1-z)\right| \leqslant C|t|\langle\xi\rangle^{1-\mu-|\beta|},\quad \left|\partial_x^\alpha\partial_\xi^\beta (\eta_1-\eta)\right| \leqslant C|t|\langle\xi\rangle^{2-\mu-|\beta|}$$

hold if $|\alpha|+|\beta|<m$. We differentiate (A.13) and (A.14) to learn

$$\begin{aligned}
&\frac{\partial}{\partial t}(\partial_x^\alpha\partial_\xi^\beta(z_1-z)) \\
&\quad = \partial_x^\alpha\partial_\xi^\beta(z_1-z)\cdot\int_0^1 \frac{\partial^2\ell_0}{\partial z\partial\eta}(sz_1+(1-s)z, s\eta_1+(1-s)\eta)ds \\
&\qquad + \partial_x^\alpha\partial_\xi^\beta(\eta_1-\eta)\cdot\int_0^1 \frac{\partial^2\ell_0}{\partial\eta\partial\eta}(sz_1+(1-s)z, s\eta_1+(1-s)\eta)ds \\
&\qquad + r_1,
\end{aligned}$$

where

$$\begin{aligned}
|r_1| &\leqslant C\bigl(|t|\langle tR\rangle^{-1-\mu}R^{2-\mu-|\beta|} + |t|\langle tR\rangle^{-\mu}R^{2-\mu-|\beta|} + \langle tR\rangle^{2-\mu}R^{-1-|\beta|}\bigr) \\
&\leqslant C'\bigl(\langle tR\rangle^{1-\mu}R^{1-\mu-|\beta|} + \langle tR\rangle^{2-\mu}R^{-1-|\beta|}\bigr),
\end{aligned}$$

and

$$\begin{aligned}
\text{(A.17)}\quad &\frac{\partial}{\partial t}(\partial_x^\alpha\partial_\xi^\beta(\eta_1-\eta)) \\
&\quad = -\partial_x^\alpha\partial_\xi^\beta(z_1-z)\cdot\int_0^1 \frac{\partial^2\ell_0}{\partial z\partial z}(sz_1+(1-s)z, s\eta_1+(1-s)\eta)ds \\
&\qquad + \partial_x^\alpha\partial_\xi^\beta(\eta_1-\eta)\cdot\int_0^1 \frac{\partial^2\ell_0}{\partial\eta\partial z}(sz_1+(1-s)z, s\eta_1+(1-s)\eta)ds \\
&\qquad + r_2,
\end{aligned}$$

where

$$\begin{aligned}
|r_2| &\leqslant C\bigl(|t|\langle tR\rangle^{-2-\mu}R^{3-\mu-|\beta|} + |t|\langle tR\rangle^{-1-\mu}R^{3-\mu-|\beta|} + \langle tR\rangle^{1-\mu}R^{-|\beta|}\bigr) \\
&\leqslant C'\bigl(\langle tR\rangle^{-\mu}R^{2-\mu-|\beta|} + \langle tR\rangle^{1-\mu}R^{-|\beta|}\bigr),
\end{aligned}$$

by virtue of (A.16). These imply

$$\begin{aligned}&\frac{\partial}{\partial t}\big(\big|\partial_x^\alpha\partial_\xi^\beta(z_1-z)\big|+R^{-1}\big|\partial_x^\alpha\partial_\xi^\beta(\eta_1-\eta)\big|\big)\\&\quad\leqslant C\big(\big|\partial_x^\alpha\partial_\xi^\beta(z_1-z)\big|+R^{-1}\big|\partial_x^\alpha\partial_\xi^\beta(\eta_1-\eta)\big|\big)\langle tR\rangle^{-\mu}R\\&\qquad+C\big(\langle tR\rangle^{1-\mu}R^{1-\mu-|\beta|}+\langle tR\rangle^{2-\mu}R^{-1-|\beta|}\big),\end{aligned}$$

and by the Duhamel formula again with the vanishing initial conditions, we obtain

$$\big|\partial_x^\alpha\partial_\xi^\beta(z_1-z)\big|+R^{-1}\big|\partial_x^\alpha\partial_\xi^\beta(\eta_1-\eta)\big|\leqslant C|t|R^{1-\mu-|\beta|}$$

for $t\in[0,T]$. This proves (A.16) for $|\alpha|+|\beta|=m$, and we learn (A.16) holds for all α,β. (A.8) follows immediately from (A.16). We substitute (A.16) to (A.17), and we have (A.9) analogously to Step 1. □

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Manuscrit reçu le 4 janvier 2010,
révisé le 3 novembre 2010,
accepté le 4 mai 2011.

Kenichi ITO
University of Tsukuba
Graduate School of Pure and Applied Sciences
1-1-1 Tennodai, Tsukuba Ibaraki, 305-8571 (Japan)
ito_ken@math.tsukuba.ac.jp

Shu NAKAMURA
University of Tokyo
Graduate School of Mathematical Sciences
3-8-1 Komaba, Meguro Tokyo, 153-8914 (Japan)
shu@ms.u-tokyo.ac.jp