ANNALES

DE

L'INSTITUT FOURIER

Martin OLSSON

Integral models for moduli spaces of G-torsors

Tome 62, nº 4 (2012), p. 1483-1549.

<http://aif.cedram.org/item?id=AIF_2012__62_4_1483_0>

Ann. Inst. Fourier, Grenoble
62, 4 (2012) 1483-1549

INTEGRAL MODELS FOR MODULI SPACES OF G-TORSORS

by Martin OLSSON

Abstract. — Given a finite tame group scheme G, we construct compactifications of moduli spaces of G-torsors on algebraic varieties, based on a higher-dimensional version of the theory of twisted stable maps to classifying stacks.

Résumé. — Étant donné un schéma en groupes fini modéré, nous construisons des espaces de modules de G-torseurs sur des variétés algébriques, en utilisant une version en grande dimension de la théorie des morphismes stables tordus dans les champs classifiants.

1. Introduction

The work in this paper is a generalization to higher dimensions of a particular application of the Abramovich-Vistoli theory of twisted stable maps [3] (and its 'tame version' of Abramovich-Olsson-Vistoli [2]).

Let us begin by reviewing how the Abramovich-Vistoli theory gives compactifications of moduli spaces for curves with (possibly non-abelian) level structure. In what follows g denotes an integer $\geqslant 2$.

1.1. Let C/S be a smooth proper curve of genus g over a scheme S, and let P be a finite set of prime numbers which includes all residue characteristics of S. For any section $s : S \to C$ we then obtain, as in [11, 5.5], a pro-object $\pi_1(C/S, s)$ in the category of locally constant sheaves of finite groups on S whose fiber over a geometric $\bar{t} \to S$ is equal to the maximal prime to P quotient of $\pi_1(C_{\bar{t}}, s_{\bar{t}})$.

Now let G be a finite group of order not divisible by any of the primes in P. Let $\mathscr{H}om^{\mathrm{ext}}(\pi_1(C/S, s), G)$ denote the sheaf of homomorphisms

Keywords: Compacitification, moduli spaces, torsors.
Math. classification: 14J15, 14D06, 14D20.

$\pi_1(C/S,s) \to G$ modulo the action of $\pi_1(C/S,s)$ given by conjugation. Then the sheaf $\mathscr{H}om^{\mathrm{ext}}(\pi_1(C/S,s),G)$ is a locally constant sheaf on S which is canonically independent of the section s. It follows that for any smooth proper curve C/S of genus g there is a canonically defined sheaf $\mathscr{H}om^{\mathrm{ext}}(\pi_1(C/S),G)$ even when C/S does not admit a section. Following [11, 5.6], we define a *Teichmüller structure of level* G on C/S to be a section of $\mathscr{H}om^{\mathrm{ext}}(\pi_1(C/S),G)$, which étale locally on S can be represented by a surjective homomorphism $\pi_1(C/S,s) \to G$ for a suitable section s. As in [11, 5.8] we define ${}_G\mathscr{M}_g$ to be the stack over $\mathbb{Z}[1/|G|]$ which to any $\mathbb{Z}[1/|G|]$-scheme S associates the groupoid of pairs $(C/S,\sigma)$, where C/S is a smooth proper genus g curve and σ is a Teichmûller structure of level G on C/S.

1.2. The space ${}_G\mathscr{M}_g$ is connected with the Abramovich-Vistoli theory as follows. Let ${}_G\mathscr{K}_g^\circ$ denote the stack over $\mathbb{Z}[1/|G|]$ which to any $\mathbb{Z}[1/|G|]$-scheme S associates the groupoid of pairs $(C/S,P \to C)$, where C is a smooth proper curve of genus g over S and $P \to C$ is a principal G-bundle, such that for every geometric point $\bar{t} \to S$ the fiber $P_{\bar{t}} \to C_{\bar{t}}$ is connected.

There is a morphism of stacks

$$ {}_G\mathscr{K}_g^\circ \to \mathscr{M}_g $$

sending $(C/S,P \to C)$ to the curve C/S (here $\mathscr{M}_g$ denotes the moduli stack of genus g curves). The fibers of this morphism can be described as follows. Let C/S be a curve defining a morphism

$$ S \to \mathscr{M}_g, $$

and let $s: S \to C$ be a section (since C/S is smooth étale locally on S there exists such a section). The fiber product

$$ \mathscr{S} := S \times_{[C],\mathscr{M}_g} {}_G\mathscr{K}_g^\circ $$

is the stack over S which to any S-scheme S' associates the groupoid of G-torsors $P \to C_{S'}$ such that for every geometric point $\bar{t} \to S'$ the fiber $P_{\bar{t}}$ is connected.

The choice of the section s enables us to describe the stack $\mathscr{S}$ as follows. For any object $P \to C_{S'}$ of $\mathscr{S}(S')$, the pullback s^*P is a G-torsor with action of $\pi_1(C_{S'}/S',s)$ on S'. Étale locally on S' we can choose a trivialization $\tilde{s}: S' \to s^*P$ of the G-torsor s^*P, and such a trivialization defines a homomorphism

$$ \pi_1(C/S,s) \to G. \tag{1.1} $$

Here we use the fact that $\pi_1(C_{S'}/S',s)$ is the pullback to S' of $\pi_1(C/S,s)$.

The assumption on the connectedness of the geometric fibers $P_{\bar{t}}$ implies that the map (1.1) is surjective. It follows that the conjugacy class of the homomorphism $\pi_1(C_S/S, s) \to G$ is independent of the choice of $\tilde{s}$ and also independent of the section s. We therefore get a well-defined section of

$$\mathscr{H}om^{\mathrm{ext}}(\pi_1(C/S), G),$$

even when C/S does not admit a section. In fact we obtain a morphism of stacks over $\mathscr{M}_g$

$$ {}_G\mathscr{K}_g^\circ \to {}_G\mathscr{M}_g, \tag{1.2}$$

and a straightforward verification, which we leave to the reader, shows that this map identifies ${}_G\mathscr{M}_g$ with the relative coarse moduli space, in the sense of [2, §3], of ${}_G\mathscr{K}_g^\circ \to \mathscr{M}_g$.

On the other hand, the category of G-torsors over a smooth genus g-curve C/S is equivalent to the category of morphisms $C \to BG$. Therefore the theory of twisted stable maps developed in [3] gives a natural compactification ${}_G\mathscr{K}_g$ of ${}_G\mathscr{K}_g^\circ$ over $\mathbb{Z}[1/|G|]$. Forgetting the G-torsor defines a morphism ${}_G\mathscr{K}_g \to \overline{\mathscr{M}}_g$ extending (1.2), and therefore by passing to the associated relative coarse moduli space over $\overline{\mathscr{M}}_g$ we obtain a compactification of ${}_G\mathscr{M}_g$.

Example 1.1. — For a finite group G and a proper flat family of curves $C \to S$ with possibly nodal fibers, the stack $\mathscr{X}/S$ which to any $S' \to S$ associates the groupoid of G-torsors on the base change $C_{S'}$ is not proper in general. For a simple example consider the case when S is the spectrum of a discrete valuation ring V with separably closed residue field, and $C \to S$ is a semistable curve whose generic fiber is smooth of genus 1 and whose closed fiber is the nodal curve obtained from $\mathbb{P}^1$ by gluing together 0 and ∞. Let G be the group $\mathbb{Z}/(\ell)$ for some prime ℓ invertible in V. Let $\eta \in S$ (resp. $s \in S$) be the generic (resp. closed) point of S. In this case G-torsors on C are classified by $H^1(C, \mathbb{Z}/(\ell))$, and G-torsors on the geometric generic fiber are classified by $H^1(C_{\bar{\eta}}, \mathbb{Z}/(\ell))$. Now a standard calculation shows that $H^1(C, \mathbb{Z}/(\ell))$ is isomorphic to $\mathbb{Z}/(\ell)$, whereas $H^1(C_{\bar{\eta}}, \mathbb{Z}/(\ell))$ has rank 2 over $\mathbb{Z}/(\ell)$. From this we conclude that there exist $\mathbb{Z}/(\ell)$-torsors over $C_{\bar{\eta}}$ not induced by a torsor over C. Since the groups $H^1(C, \mathbb{Z}/(\ell))$ are invariant under finite base change $V \to V'$ of discrete valuation rings (since $H^1(C, \mathbb{Z}/(\ell)) \simeq H^1(C_s, \mathbb{Z}/(\ell)))$, we conclude that after making a base change $V \to V'$, there exists a G-torsor over C_η which does not extend to C, even after making further base change on S.

1.3. It is natural to ask for an extension of the stable map spaces ${}_G\mathscr{K}_g$ to schemes where $|G|$ is not invertible. This appears intractable in general, but as explained in [2] if G is a tame group scheme, as defined in [1], then one can indeed develop a theory of twisted stable maps and the resulting moduli spaces ${}_G\mathscr{K}_g$ are proper.

An interesting case to consider here is the case of $G = \mu_n$ for some integer n. This group scheme is tame, and so we obtain a proper moduli space ${}_{\mu_n}\mathscr{K}_g$ over $\mathbb{Z}$ classifying μ_n-torsors over twisted curves. The substack ${}_{\mu_n}\mathscr{K}_g^\circ \subset {}_{\mu_n}\mathscr{K}_g$ classifies smooth curves C/S with a μ_n-torsor $P \to C$. Giving such a torsor P is equivalent to giving a pair (L, ι), where L is a line bundle on C and $\iota : L^{\otimes n} \to \mathscr{O}_C$ is an isomorphism. As before there is a projection ${}_{\mu_n}\mathscr{K}_g^\circ \to \mathscr{M}_g$ and since the automorphism group of any pair (L, ι) as above is canonically isomorphic to μ_n acting on L, it follows that the relative coarse moduli space of ${}_{\mu_n}\mathscr{K}_g^\circ \to \mathscr{M}_g$ is equal to the n-torsion subgroup scheme of the Jacobian of the universal curve over $\mathscr{M}_g$. Therefore the spaces ${}_{\mu_n}\mathscr{K}_g$ enable one to obtain proper models over $\mathbb{Z}$ of the n-torsion subgroup of the universal Jacobian. In the case $g = 1$, a discussion of the models obtained in this way and their relationship with the Katz-Mazur regular models for moduli spaces of elliptic curves with level structure is given in [2, §6].

1.4. The construction of the compactification ${}_G\mathscr{K}_g$ is naturally viewed in two parts. First one has the Deligne-Mumford compactification $\mathscr{M}_g \hookrightarrow \overline{\mathscr{M}}_g$ of $\mathscr{M}_g$. Second, over $\overline{\mathscr{M}}_g$ there is the universal stable curve $\overline{\mathscr{C}} \to \overline{\mathscr{M}}_g$ restricting to the universal curve $\mathscr{C} \to \mathscr{M}_g$, and if we view $\overline{\mathscr{M}}_g$ as the base then ${}_G\mathscr{K}_g$ is a compactification over $\overline{\mathscr{M}}_g$ of the stack classifying G-torsors on the *fixed* family of curves $\mathscr{C} \to \mathscr{M}_g$.

Our aim in this paper is to generalize the second part of this construction to higher dimensional varieties. The setup is the following. We consider a flat proper morphism of log schemes $f : (X, M_X) \to (S, M_S)$ which is semi-stable in a suitable sense (see 7.1 for the precise assumptions, and 2.1 for further discussion). For a tame finite flat group scheme G/S we then construct a proper S-stack ${}_G\mathscr{K}_{X/S}$ whose restriction to the open subset $S^\circ \subset S$ where $X \to S$ is smooth is simply the moduli stack of G-torsors on $X^\circ := X \times_S S^\circ$.

In the case when $S = \overline{\mathscr{M}}_g$ and $X = \overline{\mathscr{C}}$ is the universal curve over $\overline{\mathscr{M}}_g$, the log structures $M_{\overline{\mathscr{C}}}$ and $M_{\overline{\mathscr{M}}_g}$ on $\overline{\mathscr{C}}$ and $\overline{\mathscr{M}}_g$ defined by the divisors at infinity define a morphism of log stacks

$$(\overline{\mathscr{C}}, M_{\overline{\mathscr{C}}}) \to (\overline{\mathscr{M}}_g, M_{\overline{\mathscr{M}}_g})$$

and our theory can be applied. In this case we obtain the spaces of stable maps ${}_G\mathscr{K}_g$.

In general, the object of ${}_G\mathscr{K}_{X/S}$ at the boundary can be described as follows. The key observation is that to any simple extension of log structures $j : M_S \hookrightarrow N_S$ on S (see 2.1 for the precise definition) there is an associated tame stack $\mathscr{X}_j \to X$. The stack ${}_G\mathscr{K}_{X/S}$ is the stack which to any S-scheme $g : T \to S$ associates the groupoid of pairs

$$(j : g^* M_S \hookrightarrow N_T, \quad P \to \mathscr{X}_j),$$

where j is a simple extension with associated tame stack $\mathscr{X}_j \to X \times_S T$, and $P \to \mathscr{X}_j$ is a G-torsor (we call such a pair (j, P) a *twisted G-torsor*). Furthermore this data is required to satisfy a suitable stability condition (see section 7). Let us summarize the main properties here:

THEOREM 1.2. — *(i) The stack ${}_G\mathscr{K}_{X/S}$ is a proper algebraic stack over S with finite diagonal.*

(ii) The stack ${}_G\mathscr{K}_{X/S}$ is tame.

Remark 1.3. — The assumptions on $X \to S$ imply that the morphism $X \to S$ is cohomologically flat in dimension 0 and therefore the relative Picard functor $\underline{\mathrm{Pic}}_{X/S}$ is an algebraic space [5, 7.3].

If the connected component of the identity in $\underline{\mathrm{Pic}}_{X/S}$ of X/S is smooth over S, then ${}_G\mathscr{K}_{X/S}$ is flat over S with local complete intersection fibers. This follows from the same argument used in [2, §5].

Remark 1.4. — In the case when $X \to S$ is smooth and proper (so there are no log structures), the stack ${}_G\mathscr{K}_{X/S}$ is simply the stack which to any scheme $S' \to S$ associates the groupoid of G-torsors on $X \times_S S'$. In the case when G is further assumed an étale group scheme over S, the properness of ${}_G\mathscr{K}_{X/S}$ in this case follows from the deformation invariance of the prime-to-p étale fundamental group.

1.5. The paper is organized as follows.

In section 2 we discuss the stacks $\mathscr{X}_j$ associated to a simple morphism of log structures $M_S \hookrightarrow N_S$ as mentioned above. The construction of this stack is a generalized 'root stack' construction, and is an application of the ideas discussed in [19].

In sections 3, 4, and 5 we discuss various extension results for G-torsors, where G is a tame group scheme. The key situation is the following. Let V be a discrete valuation ring, and let G/V be a finite flat tame group scheme. Let P_η be a G-torsor over the field of fractions of V. In section 4 we explain that after making a ramified base change of V, the G-torsor

P_η extends to V. Moreover, we have a good understanding of the required base change.

In section 6, we discuss the basic problem of when a torsor on a stack descends to the coarse moduli space (or in a relative setting to the relative coarse moduli space). These results will be used subsequently to associate to an arbitrary twisted G-torsor a stable twisted G-torsor.

In section 7 we introduce the notion of twisted G-torsors and stability for such objects, and define the stack ${}_G\mathscr{K}_{X/S}$. In section 8 we give a different characterization of the stability condition, which will be useful for the proofs.

Then in section 9 prove that ${}_G\mathscr{K}_{X/S}$ is an Artin stack of finite type over the base S with finite diagonal. Finally in section 10 and 11 we establish the properness of ${}_G\mathscr{K}_{X/S}$, and in section 12 we show that ${}_G\mathscr{K}_{X/S}$ is tame.

1.6. **Acknowledgements:** The author is grateful to Dan Abramovich, David Rydh, and Angelo Vistoli for helpful conversations and comments. We thank the referee for providing a very large number of helpful comments and corrections which greatly improved the paper. The author was partially supported by NSF grant DMS-0714086, NSF CAREER grant DMS-0748718, and an Alfred P. Sloan Research Fellowship.

1.7. **Conventions:** We assume the reader is familiar with the basic of logarithmic geometry in the sense of Fontaine and Illusie (see for example [16] or [18]). In this paper we only consider fine log structures, and therefore usually omit the adjective “fine”.

By a *tame group scheme* G over a scheme S we mean a finite flat linearly reductive group scheme $G \to S$. See [1] for further discussion.

If $\mathscr{S}$ is a Deligne-Mumford stack, then a *geometric point* of $\mathscr{S}$ is a morphism $\mathrm{Spec}(k) \to \mathscr{S}$ with k a separably closed field (this follows the conventions of [10, Arcata, II 3.1]).

2. Stacks associated to simple extensions.

2.1. In general if Z is a scheme and $h : M \to N$ is a morphism of fine log structures on Z we say that h is *logarithmically semistable* if for every geometric point $\bar z \to Z$ there exists isomorphisms $\overline{M}_{\bar z} \simeq \mathbb{N}^r$ and $\overline{N}_{\bar z} \simeq \mathbb{N}^{r+s}$ such that the map $\overline{M}_{\bar z} \to \overline{N}_{\bar z}$ is given by

$$e_i \mapsto \begin{cases} e_i & \text{if } i \neq r \\ e_r + e_{r+1} + \cdots + e_{r+s} & \text{if } i = r, \end{cases} \tag{2.1}$$

where e_i denotes the i-th standard generator of $\mathbb{N}^r$.

We say that h is a *simple morphism* if for every geometric point $\bar{z} \to Z$ there exists isomorphisms $\overline{M}_{\bar{z}} \simeq \mathbb{N}^r$ and $\overline{N}_{\bar{z}} \simeq \mathbb{N}^r$ for some integer r such that the morphism $\overline{M}_{\bar{z}} \to \overline{N}_{\bar{z}}$ is given by

$$\mathbb{N}^r \to \mathbb{N}^r, \quad e_i \mapsto a_i \cdot e_i \tag{2.2}$$

for some collection of natural numbers $a_1, \dots, a_r \in \mathbb{N}$.

2.2. Let $f : (X, M_X) \to (S, M_S)$ be a log smooth morphism with underlying morphism $X \to S$ proper. Assume further that the map of log structures $f^*M_S \to M_X$ is logarithmically semistable. This implies that for every geometric point $\bar{x} \to X$, with image in the singular locus of the morphism $X \to S$, there exists a unique irreducible element in $\overline{M}_{S,f(\bar{x})}$ whose image in $\overline{M}_{X,\bar{x}}$ is not irreducible. In this way we obtain for every geometric point $\bar{s} \to S$ a set map

$$\{\text{singular points of } X_{\bar{s}}\} \to \{\text{irreducible elements in } \overline{M}_{S,\bar{s}}\}.$$

The morphism $f : (X, M_X) \to (S, M_S)$ is *special and essentially semistable* if for every geometric point $\bar{s} \to S$ this map induces a bijection between the connected components of the singular locus of $X_{\bar{s}}$ and the set of irreducible elements in $\overline{M}_{S,\bar{s}}$.

Remark 2.1. — Let $f : X \to S$ be a proper flat morphism of schemes of finite type over an excellent Dedekind ring such that for every geometric point $\bar{s} \to S$ and point of the fiber $x \in X_{\bar{s}}$, there exists an étale neighborhood U of x in $X_{\bar{s}}$ and an étale morphism

$$U \to \mathrm{Spec}(k(\bar{s})[X_1, \dots, X_n]/(X_1 \cdots X_l)),$$

for some positive integers n and l, sending the point x to the point defined by $X_1 = \dots = X_n = 0$. In this case it follows from [20, 2.7] that if there exists data (M_S, M_X, f^b) consisting of a log structure M_S on S, a log structure M_X on X, and a morphism of log structures $f^b : f^*M_S \to M_X$ such that the induced morphism of log schemes

$$(f, f^b) : (X, M_X) \to (S, M_S)$$

is special and essentially semistable, then the triple (M_S, M_X, f^b) is unique up to unique isomorphism. Moreover, if there exists data (N_S, N_X, g^b) consisting of a log structure N_S on S, a log structure N_X on X, and a morphism $g^b : f^*N_S \to N_X$ such that

$$(f, g^b) : (X, N_X) \to (S, N_S)$$

is log smooth, integral, and vertical, then there exists a triple (M_S, M_X, f^b) extending f to a special and essentially semistable morphism of log schemes (again by [20, 2.7]).

This can for example be applied as follows. Suppose k is an algebraically closed field, and S/k is a smooth k-scheme. Let $f : X \to S$ be a morphism of schemes as above. Suppose there exists a divisor with normal crossings $D \subset S$ such that the following hold: if $\bar{s} \to D$ is a geometric point and $t_1, \dots, t_r \in \mathscr{O}_{S,\bar{s}}$ are coordinates for the branches of D at $\bar{s}$, then for every geometric point $\bar{x} \to X_{\bar{s}}$ there exists an étale neighborhood U of $\bar{x}$ in $X \times_S \operatorname{Spec}(\mathscr{O}_{S,\bar{s}})$ and an index $i \in [1, r]$ such that U admits an étale morphism to

$$\operatorname{Spec}(\mathscr{O}_{S,\bar{s}}[X_1, \dots, X_n]/(X_1 \cdots X_l - t_i)),$$

for some integers n and l, mapping $\bar{x}$ to the point defined by $X_1 = \cdots = X_n = 0$. Let N_S (resp. N_X) be the log structure on S (resp. X) defined by the divisor D (resp. $f^{-1}(D)$). Then the resulting morphism of log schemes

$$(X, N_X) \to (S, N_S)$$

is log smooth, integral, and vertical. By [20, 2.7], we can therefore also find a triple (M_S, M_X, f^b) such that

$$(X, M_X) \to (S, M_S)$$

is special and essentially semistable, and the results of this paper apply.

2.3. Let $f : (X, M_X) \to (S, M_S)$ be a log smooth proper special and essentially semistable morphism. For a simple morphism $j : M_S \hookrightarrow N_S$ of fine log structures on S we define an algebraic stack $\mathscr{X}_j$ over X as follows. For any X-scheme $g : Y \to X$ define $\mathscr{X}_j(Y)$ to be the groupoid of commutative diagrams of fine log structures on Y

$$\begin{tikzcd} g^*M_X \arrow[r, "a"] & N_Y \\ g^*f^*M_S \arrow[u] \arrow[r, "j"] & g^*f^*N_S, \arrow[u, "b"'] \end{tikzcd}$$

where a is a simple morphism and b is logarithmically semistable.

2.4. In local coordinates the stack $\mathscr{X}_j$ can be described as follows. First of all, as discussed in [20, 2.2] and [17, 3.4] we can fppf locally on S and X

find a diagram of charts

$$\begin{tikzcd}
 & M_X & & \\
\mathbb{N}^{r+s} \arrow[ur, "\beta"] & & & \\
 & M_S \arrow[uu, "f^b"'] \arrow[rr, "j"] & & N_S \\
\mathbb{N}^r \arrow[uu, "h"] \arrow[ur, "\gamma"] \arrow[rr, "\underline{a}"] & & \mathbb{N}^r, \arrow[ur, "\gamma'"] &
\end{tikzcd}$$

where h is given by the formula (2.1) and $\underline{a}$ denotes the map (2.2) for $\underline{a} = (a_1, \dots, a_r)$ (note that in [17, 3.4] it is assumed that the integers a_i are invertible in S, but the same proof applies in our context if we allow fppf localization and not just étale localization). Let

$$\chi_{\underline{a}} : \mathbb{N}^{r+s} \to \mathbb{N}^{r+s}$$

denote the map given by

$$e_i \mapsto a_i e_i, \quad i < r$$

and

$$e_i \mapsto a_r e_i, \quad i \geqslant r.$$

We then have a commutative diagram

$$\begin{tikzcd}
\mathbb{N}^{r+s} \arrow[r, "\chi_{\underline{a}}"] & \mathbb{N}^{r+s} \\
\mathbb{N}^r \arrow[u, "h"] \arrow[r, "\underline{a}"] & \mathbb{N}^r. \arrow[u, "h"']
\end{tikzcd} \tag{2.3}$$

Let Δ denote the kernel of the map of group schemes

$$\prod_{i=0}^{s} \mu_{a_r} \to \mu_{a_r} \quad (\zeta_0, \dots, \zeta_s) \mapsto \zeta_0 \cdots \zeta_s.$$

The commutative square (2.3) defines a morphism

$$\delta : \mathbb{Z}[\mathbb{N}^r] \otimes_{\underline{a}, \mathbb{Z}[\mathbb{N}^r], h} \mathbb{Z}[\mathbb{N}^{r+s}] \to \mathbb{Z}[\mathbb{N}^{r+s}].$$

The group scheme Δ acts on $\mathrm{Spec}(\mathbb{Z}[\mathbb{N}^{r+s}])$ over

$$\mathrm{Spec}(\mathbb{Z}[\mathbb{N}^r] \otimes_{\underline{a}, \mathbb{Z}[\mathbb{N}^r], h} \mathbb{Z}[\mathbb{N}^{r+s}]).$$

If we write $\mathbb{Z}[z_1, \dots, z_{r+s}]$ for $\mathbb{Z}[\mathbb{N}^{r+s}]$ then $(\zeta_0, \dots, \zeta_s) \in \Delta$ acts by

$$z_i \mapsto z_i \quad (i < r), \quad z_i \mapsto \zeta_{i-r} z_i \quad (i \geqslant r).$$

Proposition 2.2. — *The stack $\mathscr{X}_j$ is isomorphic to the fiber product of the diagram*

$$\begin{tikzcd} & {[\mathrm{Spec}(\mathbb{Z}[\mathbb{N}^{r+s}])/\Delta]} \arrow[d] \\ X \arrow[r] & \mathrm{Spec}(\mathbb{Z}[\mathbb{N}^r] \otimes_{\underline{a},\mathbb{Z}[\mathbb{N}^r],h} \mathbb{Z}[\mathbb{N}^{r+s}]). \end{tikzcd} \tag{2.4}$$

Proof. — Let $\mathcal{S}$ denote the fiber product of the diagram (2.4). The stack $\mathcal{S}$ can be described using the theory of toric stacks in [19] as follows. Let Q denote the pushout in the category of integral monoids of the diagram

$$\begin{tikzcd} \mathbb{N}^r \arrow[r, "h"] \arrow[d, "\underline{a}"] & \mathbb{N}^{r+s} \\ \mathbb{N}^r, & \end{tikzcd}$$

and let P denote $\mathbb{N}^{r+s}$. Let

$$l : Q \to P$$

be the morphism defined by the commutative square (2.3). Let $\mathrm{Spec}(\mathbb{Z}[P])$ (resp. $\mathrm{Spec}(\mathbb{Z}[Q])$) denote the monoid scheme over S defined by P (resp. Q), and let $\mathscr{S}_P$ (resp. $\mathscr{S}_Q$) denote the stack quotient of $\mathrm{Spec}(\mathbb{Z}[P])$ (resp. $\mathrm{Spec}(\mathbb{Z}[Q])$) by the action of the group scheme $\mathrm{Spec}(\mathbb{Z}[P^{\mathrm{gp}}])$ (resp. $\mathrm{Spec}(\mathbb{Z}[Q^{\mathrm{gp}}])$) (see [19, p. 777] for the notation). The map l induces a morphism of stacks

$$\mathscr{S}(l) : \mathscr{S}_P \to \mathscr{S}_Q.$$

The group scheme Δ is the kernel of the morphism of group schemes

$$\mathrm{Spec}(\mathbb{Z}[P^{\mathrm{gp}}]) \to \mathrm{Spec}(\mathbb{Z}[Q^{\mathrm{gp}}])$$

induced by l, so from the projection maps

$$\mathrm{Spec}(\mathbb{Z}[\mathbb{N}^r] \otimes_{\underline{a},\mathbb{Z}[\mathbb{N}^r],h} \mathbb{Z}[\mathbb{N}^{r+s}]) \to \mathscr{S}_Q, \quad \mathrm{Spec}(\mathbb{Z}[\mathbb{N}^{r+s}]) = \mathrm{Spec}(\mathbb{Z}[P]) \to \mathscr{S}_P$$

we obtain a commutative diagram

$$\begin{tikzcd} {[\mathrm{Spec}(\mathbb{Z}[\mathbb{N}^{r+s}])/\Delta]} \arrow[r] \arrow[d] & \mathscr{S}_P \arrow[d] \\ \mathrm{Spec}(\mathbb{Z}[\mathbb{N}^r] \otimes_{\underline{a},\mathbb{Z}[\mathbb{N}^r],h} \mathbb{Z}[\mathbb{N}^{r+s}]) \arrow[r] & \mathscr{S}_Q, \end{tikzcd}$$

which is cartesian by the following lemma.

Lemma 2.3. — *Let S be a scheme, and let*

$$1 \longrightarrow K \xrightarrow{i} G \xrightarrow{\pi} H \longrightarrow 1$$

be an exact sequence of flat finitely presented S-group schemes. Let X be a locally finitely presented S-scheme with G-action, let Y be a locally finitely presented S-scheme with H-action, and let $f : X \to Y$ be an S-morphism such that the diagram

$$\begin{tikzcd} G \times X \arrow[r, "\text{action}"] \arrow[d, "\pi\times f"'] & X \arrow[d, "f"] \\ H \times Y \arrow[r, "\text{action}"] & Y \end{tikzcd}$$

commutes. Then the induced commutative square of stacks

$$\begin{tikzcd} {[X/K]} \arrow[r] \arrow[d] & {[X/G]} \arrow[d] \\ Y \arrow[r] & {[Y/H]} \end{tikzcd}$$

is cartesian.

Proof. — Let $\mathscr{R}$ denote the fiber product

$$\mathscr{R} := Y \times_{[Y/H]} [X/G],$$

and let $\gamma : [X/K] \to \mathscr{R}$ be the induced map. The stack $\mathscr{R}$ associates to any Y-scheme T the groupoid of triples (P, ϵ, δ), where $P \to T$ is G-torsor, $\delta : P \to X$ is a G-equivariant Y-morphism, and $\epsilon : T \to \pi_* P$ is a trivialization of the H-torsor $\pi_* P$ obtained from P by pushout. On the other hand, $[X/K]$ is the stack which to a Y-scheme T associates the groupoid of pairs (Z, λ), where $Z \to T$ is a K-torsor and $\lambda : Z \to X$ is a K-equivariant Y-morphism. The map

$$\gamma_T : [X/K](T) \to \mathscr{R}(T)$$

sends such a pair (Z, λ) to the triple $(i_* Z, \epsilon_{\text{can}}, i_* \lambda)$, where $i_* Z$ is the pushout to a G-torsor of Z, $i_* \lambda : i_* Z \to X$ is the G-equivariant map induced by λ, and ϵ_{can} is the canonical trivialization of $\pi_* i_* Z$. On the other hand, we also have a map

$$I_T : \mathscr{R}(T) \to [X/K](T)$$

sending (P, ϵ, δ) to the K-torsor Z of trivialization of P lifting the trivialization δ together with the map to X obtained by restricting g. It follows immediately from the construction that the maps γ_T and I_T are inverse equivalences which implies the lemma. □

From this and [19, 5.20] we get that the stack $\mathcal{S}$ is isomorphic to the stack which to any scheme $g : Y \to X$ associates the groupoid of commutative diagrams of fine log structures

$$\begin{tikzcd} g^*M_X \arrow[r, "a"] & N_Y \\ g^*f^*M_S \arrow[u] \arrow[r, "j"] & g^*f^*N_S, \arrow[u, "b"'] \end{tikzcd} \tag{2.5}$$

together with a morphism $\tau : \mathbb{N}^{r+s} \to \overline{N}_Y$ which locally on Y lifts to a chart and such that the diagram

$$\begin{tikzcd} \mathbb{N}^{r+s} \arrow[rr, "\chi_{\underline{a}}"] \arrow[dr, "\beta"] & & \mathbb{N}^{r+s} \arrow[dr, "\tau"] & \\ & g^{-1}\overline{M}_X \arrow[rr, "a"] & & \overline{N}_Y \\ \mathbb{N}^r \arrow[uu, "h"] \arrow[rr, "\underline{a}"] \arrow[dr, "\gamma"] & & \mathbb{N}^r \arrow[uu, "h"] \arrow[dr, "\gamma'"] & \\ & (fg)^{-1}\overline{M}_S \arrow[uu, "f^b"] \arrow[rr, "j"] & & (fg)^{-1}N_S \arrow[uu, "b"'] \end{tikzcd} \tag{2.6}$$

commutes. Note that the commutativity of (2.6) implies that the square (2.5) defines an object of $\mathscr{X}_j(Y)$. We therefore obtain a morphism of stacks

$$\mathcal{S} \to \mathscr{X}_j$$

over X. Furthermore, to prove that this is an equivalence it suffices to show that for any square (2.5) over $g : Y \to X$ defining an object of $\mathscr{X}_j(Y)$ there is a unique morphism $\tau : \mathbb{N}^{r+s} \to \overline{N}_Y$ which locally lifts to a chart and such that the diagram (2.6) commutes. By the definition of $\mathscr{X}_j$ for any geometric point $\bar{y} \to Y$ there exists such a map $\tau : \mathbb{N}^{r+s} \to \overline{N}_{Y,\bar{y}}$, and by spreading out we obtain the map τ in some étale neighborhood of $\bar{y}$. On the other hand, the map τ is clearly unique as the cokernel of $\chi_{\underline{a}} : \mathbb{N}^{r+s} \to \mathbb{N}^{r+s}$ is torsion and $\overline{N}_Y^{gp}$ is torsion free. Therefore these locally defined maps τ glue to a unique global map. This completes the proof of Proposition 2.2. □

Corollary 2.4. — *The stack $\mathscr{X}_j$ is tame.*

Proof. — Indeed this can be verified fppf-locally on X, where it follows from 2.2 which shows that $\mathscr{X}_j$ is the quotient of a scheme by a finite flat tame group scheme. □

Example 2.5. — Let $S = \mathrm{Spec}(k)$ be the spectrum of an algebraically closed field, and let X/k be a proper k-scheme. Assume that X has two irreducible components Y_1 and Y_2, both of which are smooth over k, and that the intersection $Z := Y_1 \cap Y_2$ is a smooth connected divisor in both Y_1 and Y_2. Assume further that in an étale neighborhood of any point of the singular locus of X there exists a smooth morphism

$$X \to \mathrm{Spec}(k[x_1, x_2]/(x_1 x_2)).$$

Fix an integer n.

Let $i : I_s \subset \mathscr{O}_{Y_s}$ $(s = 1, 2)$ be the ideal defining Z, so I_s is an invertible sheaf. We can then consider the n-th root stack of Y_s with respect to I_s (see for example [8, §2]). This is the stack $\mathscr{Y}_s \to Y_s$ which to any morphism $t : T \to Y_s$ associates the groupoid of triples (L, ρ, σ), where L is an invertible sheaf on T, $\rho : L \to \mathscr{O}_T$ is a morphism of $\mathscr{O}_T$-modules, and $\sigma : L^{\otimes n} \to t^* I_s$ is an isomorphism of invertible sheaves such that the diagram

$$\begin{tikzcd} L^{\otimes n} \arrow[rr, "\sigma"] \arrow[dr, "\rho^{\otimes n}"'] & & t^* I_s \arrow[dl, "t^* i"] \\ & \mathscr{O}_T & \end{tikzcd}$$

commutes. We refer to the data (L, ρ, σ) as an *n-th root of* $(I_s \to \mathscr{O}_{Y_s})$. Note that this definition makes sense for any line bundle I with a map $I \to \mathscr{O}_{Y_s}$ (for example the zero map). In local coordinates, if $f \in \mathscr{O}_{Y_s}$ is an element defining I_s, then we have

$$\mathscr{Y}_s = [\mathrm{Spec}(\mathscr{O}_{Y_s}[w]/(w^n - f))/\mu_n],$$

where $\zeta \in \mu_n$ acts by $w \mapsto \zeta w$. In particular, the restriction

$$\mathscr{G}_s := (\mathscr{Y}_s \times_{Y_j} Z)_{\mathrm{red}} \subset \mathscr{Y}_s$$

is a smooth divisor and the projection

$$\mathscr{G}_s \to Z$$

is a μ_n-gerbe over Z. The corresponding cohomology class (here cohomology is taken with respect to the fppf topology)

$$\mathrm{cl}(\mathscr{G}_s) \in H^2(Z, \mu_n)$$

is the first Chern class of the line bundle $I_s|_Z$.

Combining [13, 2.3] and [15, 11.7 (2)], we get that the structure morphism $X \to S$ extends to a log smooth special and essentially semistable morphism

$$(X, M_X) \to (\mathrm{Spec}(k), M_k)$$

if and only if

$$I_1|_Z \otimes I_2|_Z \simeq \mathscr{O}_Z. \tag{2.7}$$

Assume this is the case for the rest of this example.

Note that M_k is non-canonically isomorphic to $k^* \oplus \mathbb{N}$ (since Z is connected). Fix such an isomorphism and consider the map

$$k^* \oplus \mathbb{N} \to k^* \oplus \mathbb{N}, \quad (u, m) \mapsto (u, nm).$$

This map defines a simple extension $j : M_k \hookrightarrow N_k$ (though N_k is isomorphic to M_k we differentiate the two in the notation). We can then consider the resulting stack

$$\mathscr{X}_j \to X.$$

This stack $\mathscr{X}_j$ glues together the two stacks $\mathscr{Y}_s \to Y_s$ along an isomorphism of stacks (not of μ_n-gerbes)

$$\mathscr{G}_1 \to \mathscr{G}_2.$$

This isomorphism is linear with respect to the map $\iota : \mu_n \to \mu_n$ sending u to u^{-1}, or equivalently is induced from a morphism of μ_n-gerbes

$$\mathscr{G}_1 \times^{\iota, \mu_n} \mu_n \to \mathscr{G}_2.$$

Note that this is consistent with the assumption that we have the isomorphism (2.7) which implies that

$$\mathrm{cl}(\mathscr{G}_1) = -\mathrm{cl}(\mathscr{G}_2)$$

in $H^2(Z, \mu_n)$.

3. A description of G-torsors.

It follows from [1, 2.17] that if G/S is a finite flat tame group scheme, then étale locally on S the group G can be written as an extension

$$1 \to \Delta \to G \to H \to 1,$$

where Δ is a diagonalizable group scheme and H is an étale group scheme of order invertible in S (a *tame étale* group scheme). In what follows, when studying G-torsors it will often be convenient to reduce problems to the diagonalizable and tame étale cases separately. To this end, we discuss in this section some generalities about G-torsors for G an extension as above.

3.1. Let S be a scheme, and let G/S be a finite flat group scheme which sits in an extension of group schemes

$$1 \to \Delta \to G \to H \to 1,$$

where Δ is diagonalizable and H is constant. For a G-torsor $P \to \mathscr{X}$ over an algebraic S-stack $\mathscr{X}/S$, let $\overline{P}$ denote the quotient P/Δ, which is an H-torsor over $\mathscr{X}$.

Since Δ is abelian the conjugation action of G on Δ descends to an action

$$\rho : H \to \mathrm{Aut}(\Delta).$$

For $h \in H$ the map $\rho_h : \Delta \to \Delta$ sends a local section $\delta \in \Delta$ to $\tilde{h}\delta\tilde{h}^{-1}$, where $\tilde{h} \in G$ is a (local) lifting of h.

3.2. For $h \in H$, let Q_h denote the sheaf of liftings of h to G. Then Q_h is a (Δ, Δ)-bitorsor through the left and right translation actions.

For a Δ-torsor $P \to \mathscr{X}$ over an algebraic stack $\mathscr{X}/S$, let $Q_h \wedge P$ denote

$$Q_h \wedge P := Q_h \times_S P/\sim,$$

where $\sim$ is the equivalence relation

$$(q, p) \sim (q\delta^{-1}, \delta p), \quad q \in Q_h, \quad p \in P, \quad \delta \in \Delta$$

The space $Q_h \wedge P$ is a Δ-torsor over $\mathscr{X}$ with $\delta \in \Delta$ acting by

$$\delta * (q, p) = (\delta q, p).$$

Note that there is a canonical isomorphism of torsors

$$Q_{h'} \wedge Q_h \simeq Q_{h'h},$$

for $h, h' \in H$.

3.3. Fix an algebraic stack $\mathscr{X}/S$. Define $\mathscr{C}$ to be the category whose objects are collections of data

$$(\overline{P}, P \to \overline{P}, \{\chi_h\}_{h \in H}),$$

as follows:

(1) $\overline{P}$ is an H-torsor over $\mathscr{X}$.
(2) P is a Δ-torsor over $\overline{P}$.
(3) For each $h \in H$

$$\chi_h : Q_h \wedge P \to h^* P$$

is a morphism of Δ-torsors over $\overline{P}$.

(4) For any two elements $h, h' \in H$ the diagram

$$\begin{tikzcd} Q_{h'} \wedge (Q_h \wedge P) \arrow[r, "\chi_h"] \arrow[d, "\simeq"] & Q_{h'} \wedge h^*P \arrow[d, "\simeq"] \\ Q_{h'} \wedge Q_h \wedge P \arrow[d, "\simeq"] & h^*(Q_{h'} \wedge P) \arrow[d, "\chi_{h'}"] \\ Q_{h'h} \wedge P \arrow[r, "\chi_{h'h}"] & (h'h)^*P \end{tikzcd} \tag{3.1}$$

commutes.

3.4. There is a functor

$$F : (G\text{-torsors over } \mathscr{X}) \to \mathscr{C}$$

defined as follows.

Given a G-torsor $P \to \mathscr{X}$ let $F(P)$ be the object of $\mathscr{C}$ given by the Δ-torsor $P \to \overline{P}$ over $\overline{P}$, and the maps

$$\chi_h : Q_h \wedge P \to h^*P$$

given by the map

$$Q_h \times P \to P$$

provided by the action of G on P. Then one verifies immediately that $(\overline{P}, P \to \overline{P}, \{\chi_h\})$ is an object of $\mathscr{C}$.

Proposition 3.1. — *The functor F is an equivalence of categories.*

Proof. — Given an object $(\overline{P}, P \to \overline{P}, \{\chi_h\})$ the Δ-torsor $P \to \mathscr{X}$ inherits an action of G by noting that

$$G = \coprod_{h \in H} Q_h$$

and therefore the maps

$$\chi_h : Q_h \times P \to P$$

define a map

$$G \times P \to P.$$

We leave to the reader the verification that this action makes P a G-torsor, and that we obtain a quasi-inverse to F. $\square$

PROPOSITION 3.2. — *Let T be a regular scheme, and let $T^\circ \subset T$ be an open subset with* $\operatorname{codim}(T\backslash T^\circ, T) \geqslant 2$. *Let G/T be a finite flat tame group scheme. Then the restriction functor*

$$(G\text{-}\textit{torsors over } T) \to (G\text{-}\textit{torsors over } T^\circ) \tag{3.2}$$

is an equivalence of categories.

Proof. — In the case when G is an étale group scheme this follows from [14, X.3.3].

In the case when G is diagonalizable the result can be seen as follows. The scheme G is isomorphic to a finite product of group schemes of the form μ_n so it suffices to consider the case when $G = \mu_n$. In this case the category of G-torsors is equivalent to the category of pairs (L, ι), where L is a line bundle and ι is a trivialization of L^n. The result in this case therefore follows from the fact that the restriction functor

$$(\text{line bundles on } T) \to (\text{line bundles on } T^\circ)$$

is an equivalence of categories.

For the general case, we may work étale locally on T, and may therefore assume that there exists a short exact sequence

$$1 \to \Delta \to G \to H \to 1,$$

where Δ is diagonalizable and H is étale.

Let $\mathscr{C}$ (resp. $\mathscr{C}^\circ$) be the category defined in paragraph 3.3 with $S = T$ (resp. $S = T^\circ$). Then it suffices to show that the restriction functor

$$R : \mathscr{C} \to \mathscr{C}^\circ$$

is an equivalence. For the essential surjectivity, let

$$(\overline{P}^\circ, P^\circ \to \overline{P}^\circ, \{\chi_h\}) \in \mathscr{C}^\circ$$

be an object. By the case of an étale group scheme, $\overline{P}^\circ$ extends uniquely to an H-torsor $\overline{P} \to T$. Now by the case of a diagonalizable group scheme applied to $\overline{P}^\circ \subset \overline{P}$ the Δ-torsor $P^\circ \to \overline{P}^\circ$ extends uniquely to a Δ-torsor $P \to \overline{P}$, and furthermore the maps χ_h also extend uniquely. The resulting collection of data

$$(\overline{P}, P \to \overline{P}, \{\chi_h\})$$

defines an object of $\mathscr{C}$ as the commutativity of (3.1) can be verified after restricting to T°. This proves the essential surjectivity of R.

The full faithfulness is shown similarly. □

For later use, let us also record the following mild generalization of 3.2:

Corollary 3.3. — *Let $\mathscr{X}$ be a regular algebraic stack over a base scheme S, and let $\mathscr{X}^\circ \subset \mathscr{X}$ be an open substack with complement of codimension $\geqslant 2$. Then for any tame group scheme G over S the restriction functor*

$$(G\text{-torsors over } \mathscr{X}) \to (G\text{-torsors over } \mathscr{X}^\circ)$$

is an equivalence of categories.

Proof. — Let $T \to \mathscr{X}$ be a smooth surjection, and let $T^\circ \subset T$ be the inverse image of $\mathscr{X}^\circ$. By descent theory the category of G-torsors over $\mathscr{X}$ (resp. $\mathscr{X}^\circ$) is equivalent to the category of pairs (P, ι), where P is a G-torsor over T (resp. T°) and

$$\iota : \mathrm{pr}_1^* P \to \mathrm{pr}_2^* P$$

is an isomorphism of G-torsors over $T \times_{\mathscr{X}} T$ (resp. $T^\circ \times_{\mathscr{X}^\circ} T^\circ$) such that the usual cocycle condition on $T \times_{\mathscr{X}} T \times_{\mathscr{X}} T$ (resp. $T^\circ \times_{\mathscr{X}^\circ} T^\circ \times_{\mathscr{X}^\circ} T^\circ$) holds. Therefore it suffices to show that for any $i \geqslant 1$ the restriction functor

$$(G\text{-torsors over } T^{(i)}) \to (G\text{-torsors over } T^{(i)\circ})$$

is an equivalence of categories, where $T^{(i)}$ (resp. $T^{(i)\circ}$) denotes the i-fold fiber product of T (resp. T°) with itself over $\mathscr{X}$ (resp. $\mathscr{X}^\circ$). This reduces the proof to the case when $\mathscr{X}$ is an algebraic space. In this case, repeating the above argument with an étale cover of $\mathscr{X}$ by a scheme we are then further reduced to the case of a scheme, which is Proposition 3.2. □

4. Extending torsors over discrete valuation rings.

4.1. Let A be a strictly henselian discrete valuation ring over k with residue field L and fraction field K. Fix a uniformizer $\pi \in A$.

Let G/A be a tame finite flat group scheme. Let Δ be the connected component of the identity so we have an exact sequence

$$1 \to \Delta \to G \to H \to 1,$$

where H is tame and étale. Let $P_\eta \to \mathrm{Spec}(K)$ be a G-torsor.

For a positive integer r, let A_r denote the finite A-algebra

$$A[t]/(t^r - \pi).$$

Let K_r denote the field of fractions of A_r. Note that A_r is a strictly henselian discrete valuation ring with uniformizer t, and that $K_r \simeq A_r \otimes_A K$.

Proposition 4.1. — *There exists a unique integer $e = e(P_\eta)$ such that $P_\eta|_{K_e}$ extends to A_e, and such that if f is any other positive integer for which $P_\eta|_{K_f}$ extends to A_f then $e|f$.*

Moreover, the integer e is independent of the choice of the uniformizer π, and for any integer f divisible by e an extension of $P_\eta|_{K_f}$ to $\mathrm{Spec}(A_f)$ is unique up to unique isomorphism.

Proof. — Let us first make some elementary observations about μ_n-torsors over a scheme S. Consider the short exact sequence of fppf-sheaves

$$0 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \xrightarrow{u \mapsto u^n} \mathbb{G}_m \longrightarrow 0$$

which gives rise to a long exact sequence

$$0 \longrightarrow \mu_n(S) \longrightarrow \Gamma(S, \mathscr{O}_S^*) \xrightarrow{u \mapsto u^n} \Gamma(S, \mathscr{O}_S^*) \longrightarrow H^1(S, \mu_n) \longrightarrow \mathrm{Pic}(S).$$

If $\mathrm{Pic}(S) = 0$, which we assume for the rest of the proof (since we are interested in the case $S = \mathrm{Spec}(A_f)$), then this sequence identifies the isomorphism classes of μ_n-torsors on S with

$$\Gamma(S, \mathscr{O}_S^*)/\Gamma(S, \mathscr{O}_S^*)^n.$$

If $P \to S$ is a μ_n-torsor, then the corresponding class

$$[P] \in \Gamma(S, \mathscr{O}_S^*)/\Gamma(S, \mathscr{O}_S^*)^n$$

can be described as follows. The μ_n-torsor P corresponds to a pair (L, ι), where L is a line bundle on S and $\iota : L^{\otimes n} \to \mathscr{O}_S$ is an isomorphism. Now since $\mathrm{Pic}(S)$ is trivial, the line bundle L is trivial. Let $f \in L(S)$ be a basis. Then the image of $f^{\otimes n}$ under ι is an element $u \in \Gamma(S, \mathscr{O}_S^*)$ and we define $[P]$ to be the class of u. Note that a different choice of basis f changes u by an element of $\Gamma(S, \mathscr{O}_S^*)^n$ so the class $[P]$ is independent of this choice.

We apply this observation with S either the spectrum of A or K, or one of the extensions A_f or K_f. Let $\nu : K^* \to \mathbb{Z}$ be the valuation, normalized so that $\nu(\pi) = 1$. From the snake lemma applied to the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mu_n(A) \arrow[r, "\simeq"] \arrow[d] & \mu_n(K) \arrow[d] & & \\
0 \arrow[r] & A^* \arrow[r] \arrow[d, "\cdot n"] & K^* \arrow[r, "\nu"] \arrow[d, "\cdot n"] & \mathbb{Z} \arrow[r] \arrow[d, "\cdot n"] & 0 \\
0 \arrow[r] & A^* \arrow[r] & K^* \arrow[r, "\nu"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

we obtain an exact sequence

$$0 \longrightarrow A^*/(A^*)^n \longrightarrow K^*/(K^*)^n \xrightarrow{\bar{\nu}} \mathbb{Z}/(n) \longrightarrow 0.$$

If $P_\eta \to \mathrm{Spec}(K)$ is a μ_n-torsor over K, then it follows that P_η extends to a μ_n-torsor over A if and only if $\bar{\nu}([P_\eta]) = 0$.

These remarks enable us to prove the proposition in the case when $G = \Delta$ is a diagonalizable group scheme as follows. If the Cartier dual of Δ is equal to $\mathbb{Z}/(n_1) \times \cdots \times \mathbb{Z}/(n_r)$, then the torsor P_η corresponds to a collection of pairs $\{(L_i, \iota_i)\}_{i=1}^r$ where L_i is a line bundle on $\mathrm{Spec}(K)$ and $\iota_i : L_i^{\otimes n_i} \to K$ is an isomorphism. Fix trivializations $s_i \in L_i$ and let $u_i \in K^*$ be the image of $s_i^{\otimes n_i}$ under ι_i. Then it follows from the preceding discussion that $P_\eta|_{K_f}$ extends to A_f if and only if

$$\nu(u_i) \cdot f \equiv 0 \pmod{n_i}$$

for all i. In this case we therefore take e to be the least common multiple of the integers $n_i/\gcd(n_i, \nu(u_i))$.

Note also that it is immediate that the functor

$$(\Delta\text{-torsors over } \mathrm{Spec}(A)) \to (\Delta\text{-torsors over } \mathrm{Spec}(K))$$

is fully faithful.

Next consider the case when $G = H$ is a tame étale group scheme. Fix an algebraic closure $K \subset \overline{K}$ and let $\pi_1(K)'$ be the prime-to-p fundamental group of K with respect to this base point, where p is the characteristic of the residue field L. Then since L is separably closed we have a canonical isomorphism $\pi_1(K)' \simeq \widehat{\mathbb{Z}}(1)'$, where $\widehat{\mathbb{Z}}(1)'$ denotes

$$\widehat{\mathbb{Z}}(1)' := \varprojlim_{n, p \nmid n} \mu_n(\overline{K}).$$

The torsor P_η is then defined by an H-conjugacy class of homomorphisms

$$\rho_{P_\eta} : \widehat{\mathbb{Z}}(1)' \to H,$$

and P_η extends to $\mathrm{Spec}(A)$ if and only if ρ_{P_η} is trivial.

For any integer f, the restriction of P_η to K_f corresponds to the conjugacy class of homomorphisms defined by

$$\pi_1(K_f)' \simeq \widehat{\mathbb{Z}}(1)' \xrightarrow{\cdot f} \widehat{\mathbb{Z}}(1)' \xrightarrow{\rho_{P_\eta}} H.$$

It follows that in the case when G is tame and étale we can take e in the proposition to be the order of the image of ρ_{P_η} (which depends only on the conjugacy class of ρ_{P_η}).

For the general case we combine the previous two cases as follows. Let $\overline{P}_\eta$ be the H-torsor P_η/Δ over $\mathrm{Spec}(K)$. Let $\bar{e}$ denote $e(\overline{P}_\eta)$, so we have an extension $\overline{P} \to \mathrm{Spec}(A_{\bar{e}})$. Since the residue field of $A_{\bar{e}}$ is separably closed the torsor $\overline{P}$ is trivial, and therefore the underlying scheme of $\overline{P}$ is non-canonically isomorphic to

$$\coprod_{h \in H} \mathrm{Spec}(A_{\bar{e}}).$$

We can therefore apply the diagonalizable case to the Δ-torsor $P_\eta \to \overline{P}_\eta$ for each connected component of $\overline{P}$. Let e' be the least common multiple of these integers and let $e = e' \cdot \bar{e}$ (note that e does not depend on the choice of π). Then by construction we obtain a Δ-torsor $P \to \overline{P}|_{A_e}$ restricting to $P_\eta|_{K_e}$. Moreover, by the uniqueness of the extension in the diagonalizable case we obtain maps

$$\chi_h : Q_h \wedge P \to h^*P$$

of Δ-torsors over $\overline{P}_{A_e}$ as in paragraph 3.3, inducing the given maps over K_e. These maps give P the structure of a G-torsor extending $P_\eta|_{K_e}$.

By a similar reasoning one obtains the uniqueness of the extension of P_η and the remaining statements in the proposition. □

Remark 4.2. — Note that it follows from the proof that the integer e in Proposition 4.1 divides the order of G.

Corollary 4.3. — *Let V be a discrete valuation ring, and let Y/V be a quasi-compact smooth V-scheme. Let G/V be a tame finite flat group scheme, and assume $P_\eta \to Y_\eta$ is a G-torsor over the generic fiber Y_η of Y. Then after making a finite flat base change $V \to V'$ there exists a unique extension $P \to Y$ of P_η to a G-torsor over Y.*

Proof. — Let $\pi \in V$ be a uniformizer. For each geometric point $\bar{\eta} \to Y$ mapping to a generic point of the closed fiber, the ring $\mathscr{O}_{Y,\bar{\eta}}$ is a discrete valuation ring with uniformizer the image of π. Therefore by 4.1 after making a finite ramified base change $V \to V'$ we may assume that P_η extends to $\mathrm{Spec}(\mathscr{O}_{Y,\bar{\eta}})$ for each geometric generic point of the closed fiber. By a standard limit argument, it follows that P_η extends to an étale neighborhood of each geometric generic point of the closed fiber. By the uniqueness statement in Proposition 4.1 these extensions are unique up to unique isomorphism, and therefore we get by descent an open subset $U \subset Y$ containing the generic fiber Y_η and each generic point of the closed fiber, such that P_η extends to U. By Proposition 3.2 it follows that P_η extends to all of Y. □

4.2. With notation as in 4.1, let f be an integer divisible by e. Let $\mathscr{V}_f$ denote the stack-theoretic quotient of $\mathrm{Spec}(A_f)$ by the action of μ_f for which $\zeta \in \mu_f$ sends t to ζt. Then

$$\mathscr{V}_f \times_{\mathrm{Spec}(A)} \mathrm{Spec}(K) \simeq [\mathrm{Spec}(K_f)/\mu_f] \simeq \mathrm{Spec}(K),$$

and the resulting inclusion

$$\mathrm{Spec}(K) \hookrightarrow \mathscr{V}_f$$

is a dense open immersion.

The stack $\mathscr{V}_f$ has the following modular description: For any A-scheme $g : T \to \mathrm{Spec}(A)$ the category $\mathscr{V}_f(T)$ is equivalent to the category of triples (M, γ, ι), where M is an invertible sheaf on T, $\gamma : M \to \mathscr{O}_T$ is a morphism of line bundles, and $\iota : M^{\otimes f} \to \mathscr{O}_T$ is an isomorphism of line bundles on T such that the diagram

$$\begin{tikzcd} M^{\otimes f} \arrow[rr, "\iota"] \arrow[dr, "\gamma^{\otimes f}"'] & & \mathscr{O}_T \arrow[dl, "\pi"] \\ & \mathscr{O}_T & \end{tikzcd}$$

commutes.

In terms of this description, the inclusion $\mathrm{Spec}(K) \hookrightarrow \mathscr{V}_f$ corresponds to the trivial line bundle $L = K \cdot b$ (where b denotes a basis element) with the map γ sending b to $1 \in K$ and the isomorphism ι being given by multiplication by π^{-1}.

Proposition 4.4. — *The torsor $P_\eta \to \mathrm{Spec}(K)$ extends uniquely to a G-torsor $\mathscr{P}_f \to \mathscr{V}_f$.*

Proof. — Let $P \to \mathrm{Spec}(A_f)$ denote the extension of $P_\eta|_{K_f}$ provided by Proposition 4.1. Let

$$\rho : \mathrm{Spec}(A_f) \times \mu_f \to \mathrm{Spec}(A_f)$$

be the map giving the action. To give an extension of P to $\mathscr{V}_f$ is equivalent to giving an isomorphism

$$\gamma : \mathrm{pr}_1^* P \to \rho^* P$$

over $\mathrm{Spec}(A_f) \times \mu_f$ satisfying the following cocycle condition (thereby giving descent data for P to $\mathscr{V}_f$). Let

$$m : \mu_f \times \mu_f \to \mu_f$$

be the map giving the group multiplication. Then the cocycle condition on γ is that the diagram over $\mathrm{Spec}(A_f) \times \mu_f \times \mu_f$

$$\begin{tikzcd} \mathrm{pr}_1^* P \arrow[r, "\simeq"] \arrow[d, "\simeq"] & (1 \times m)^* \mathrm{pr}_1^* P \arrow[r, "(1\times m)^*\gamma"] & (1 \times m)^* \rho^* P \arrow[d, "\simeq"] \\ (\mathrm{pr}_1 \times 1)^* \mathrm{pr}_1^* P \arrow[r, "\gamma"] & (\rho \times 1)^* \mathrm{pr}_1^* P \arrow[r, "(\rho\times 1)^*\gamma"] & (\rho \times 1)^* \rho^* P \end{tikzcd} \tag{4.1}$$

commutes. From this it follows that if an extension exists then it is unique up to unique isomorphism.

To construct the extension $\mathscr{P}_f$, consider first the case when $G = \mu_n$ for some integer n. In this case, the torsor P_η corresponds to the trivial line bundle $L_\eta = K \cdot b$ with an isomorphism

$$\sigma_\eta : L_\eta^{\otimes n} = K \cdot b^{\otimes n} \to K$$

sending $b^{\otimes n}$ to some element $g \in K^*$. After possibly changing our choice of basis b for L_η we may assume that $g \in A$. Write

$$g = u\pi^r,$$

where $u \in A^*$ and $r \geqslant 0$. By definition of the integer e, the product fr is divisible by n. Let $L_f = A_f \cdot b'$ denote the trivial line bundle on $\mathrm{Spec}(A_f)$, and let

$$\sigma : L_f^{\otimes n} \to A_f$$

be the isomorphism sending $b'^{\otimes n}$ to u. This data defines a μ_n-torsor over $\mathrm{Spec}(A_f)$. Moreover, the isomorphism

$$L_f \otimes_{A_f} K_f \to L_\eta \otimes_K K_f, \quad b' \mapsto \pi_f^{-fr/n} \cdot b$$

is compatible with the maps to K_f and therefore defines an isomorphism of μ_n-torsors over K_f. The μ_f-action on $\mathrm{Spec}(A_f)$ lifts to a μ_f-action on (L_f, σ) as follows. Let R be an A-algebra and let $\zeta \in \mu_f(R)$ be an R-valued point. Then the action of ζ on $A_f \otimes_A R$ is given by the map

$$\zeta_* : A_f \otimes_A R \to A_f \otimes_A R, \quad \pi_f \mapsto \zeta\pi_f.$$

Let

$$\tilde{\zeta} : L_f \otimes_{A_f} (A_f \otimes_A R) \to L_f \otimes_{A_f} (A_f \otimes_A R)$$

be the ζ_*-linear map given by

$$b' \mapsto \zeta^{-fr/n} \cdot b'.$$

Then this defines a μ_f-action on the pair (L_f, σ), so the μ_n-torsor corresponding to this pair descends to a μ_n-torsor $\mathscr{P}_f$ over $\mathscr{V}_f$. Furthermore, note that the isomorphism between $(L_f, \sigma)|_{K_f}$ and $(L_\eta, \sigma_\eta)|_{K_f}$ is compatible with the μ_f-actions. Therefore $\mathscr{P}_f$ gives the desired extension of $\mathscr{P}_\eta$ to $\mathscr{V}_f$. This completes the case when $G = \mu_n$.

From this we also obtain the extension in the case when G is diagonalizable as in the proof of Proposition 4.1.

In the case when $G = H$ is a tame étale group scheme, let $\mathscr{P} \to \mathscr{V}_f$ be the normalization of $\mathscr{V}_f$ in P_η. By the construction of the integer e, the stack $\mathscr{P}$ is an H-torsor over $\mathscr{V}_f$ as this can be verified after pulling back to $\mathrm{Spec}(A_f)$. In fact the pullback of $\mathscr{P}$ to $\mathrm{Spec}(A_f)$ is trivial (as A_f is strictly henselian local) and therefore as a stack $\mathscr{P}$ is isomorphic to a finite disjoint union of the form

$$\mathscr{P} \simeq \coprod_j [\mathrm{Spec}(A_f)/\mu_{f_j}],$$

for some integers $f_j | f$, where the action of μ_{f_j} is through the natural inclusion $\mu_{f_j} \subset \mu_f$.

For the general case one proceeds as in the proof of Proposition 4.1 using the diagonalizable and tame étale cases by writing G as an extension

$$1 \to \Delta \to G \to H \to 1,$$

where Δ is diagonalizable and H is tame and étale, and considering the quotient H-torsor P_η/Δ. We leave the details to the reader. □

4.3. There is a more global version of Proposition 4.4 which we now describe.

Let X be a smooth scheme over a separably closed field, and let $D \subset X$ be a connected divisor with local normal crossings. Let X° denote the complement $X \backslash D$, and let $P^\circ \to X^\circ$ be a G-torsor.

The divisor D defines a log structure M_D on X. For a geometric point $\bar{x} \to X$ the stalk $\overline{M}_{D,\bar{x}}$ is equal to the free monoid on generators the branches of D at $\bar{x}$. Let $\mathscr{X}_{D,r}$ be the stack over X which to any X-scheme $f : T \to X$ associates the groupoid $\mathscr{X}_{D,r}(T)$ of morphisms of log structures $f^* M_D \to M_T$ on T such that for every geometric point $\bar{t} \to T$ there exists a commutative diagram

$$\begin{tikzcd} \overline{M}_{D,\bar{t}} \arrow[r] \arrow[d, "\simeq"] & \overline{M}_{T,\bar{t}} \arrow[d, "\simeq"] \\ \mathbb{N}^q \arrow[r, "\times r"] & \mathbb{N}^q \end{tikzcd}$$

for some integer q.

Locally the stack $\mathscr{X}_{D,r}$ can be described as follows. Suppose given elements $f_1, \dots, f_q \in \mathscr{O}_X$ such that D is defined by $f_1 \cdots f_q$ and the zero locus of each f_i is a smooth divisor in X. Then $\mathscr{X}_{D,r}$ is the quotient of

$$\mathrm{Spec}_X(\mathscr{O}_X[t_1, \dots, t_q]/(t_1^r - f_1, \dots, t_q^r - f_q))$$

by the action of μ_r^q given by

$$(\zeta_1, \dots, \zeta_q) * t_i = \zeta_i \cdot t_i.$$

Note that this local description implies that $\mathscr{X}_{D,r}$ is flat over X.

If $\bar{\eta} \to D$ is a geometric generic point and $A := \mathscr{O}_{X,\bar{\eta}}$ (which is a discrete valuation ring with separably closed residue field) then we have

$$\mathscr{V}_r \simeq \mathscr{X}_{D,r} \times_X \mathrm{Spec}(A),$$

where $\mathscr{V}_r$ is defined as in paragraph 4.2. Let $e_{\bar{\eta}}$ denote the least integer such that the G-torsor

$$\mathrm{Spec}(A) \times_X P^\circ \to \mathrm{Spec}(A) \times_X X^\circ$$

extends to $\mathscr{V}_{e_{\bar{\eta}}}$.

Proposition 4.5. — *The G-torsor $P^\circ \to X^\circ$ extends to $\mathscr{X}_{D,r}$ if and only if $e_{\bar{\eta}}$ divides r for all geometric generic points $\bar{\eta} \to D$. In this case the extension is unique up to unique isomorphism.*

Proof. — Let $\bar{\eta} \to D$ be a geometric generic point, and set $A := \mathscr{O}_{X,\bar{\eta}}$.

If P° extends to $\mathscr{X}_{D,r}$ then $P^\circ \times_X \mathrm{Spec}(A)$ also extends to

$$\mathscr{X}_{D,r} \times_X \mathrm{Spec}(A) \simeq \mathscr{V}_r,$$

and therefore by Proposition 4.4 we must have $e_{\bar{\eta}}|r$.

Conversely suppose $e_{\bar{\eta}}|r$ for all $\bar{\eta}$. Then P° extends to $\mathscr{X}_{D,r} \times_X \mathrm{Spec}(A)$, and by a standard limit argument there exists an étale neighborhood $U \to X$ of the generic points of D such that P° extends to $\mathscr{X}_{D,r} \times_X U$. By the uniqueness if $V \subset X$ denotes the union of X° and the image of U then P° extends to $V \times_X \mathscr{X}_{D,r}$. By Corollary 3.3 it follows that P° extends uniquely to all of $\mathscr{X}_{D,r}$. □

5. More extension results.

In this section we gather together some more local results about extending G-torsors. These results will be used in section 11 to verify the valuative criterion for properness for the stacks ${}_G\mathscr{K}_{X/S}$.

Throughout this section V denotes a discrete valuation ring with residue characteristic p (possibly equal to 0).

Proposition 5.1. — *Let $n \geqslant 2$ be an integer, and let X denote the spectrum of the strict henselization of*

$$V[x_1, \dots, x_n]/(x_1 \cdots x_s), \quad (s \leqslant n)$$

at the point $\{x_1 = \cdots x_n = 0\}$. Let $X_{\mathrm{sing}} \subset X$ denote the singular locus of X, and let $X^\circ \subset X$ denote the complement of the intersection of X_{sing} with the closed fiber. Let G/V be a tame finite flat group scheme. Then the restriction functor

$$(G\text{-torsors on } X) \to (G\text{-torsors on } X^\circ) \tag{5.1}$$

is an equivalence of categories.

Remark 5.2. — Note that since X is strictly henselian local, if G is étale over V then any G-torsor on X is trivial. Since both X and X° are connected, proposition 5.1 is therefore equivalent in this case to the statement that any G-torsor on X° is trivial.

The proof of Proposition 5.1 will be in several steps (5.1)–(5.6).

5.1. Let $\mathcal{R}$ denote the coordinate ring of X, and let $\mathfrak{m} \subset \mathcal{R}$ denote the maximal ideal. We begin the proof with some observations about the ring $\mathcal{R}$. For a subset $I \subset \{1, \dots, s\}$, let $\mathcal{R}_I$ denote the quotient of $\mathcal{R}$ by the ideal generated by x_i $(i \in I)$. More generally, for an $\mathcal{R}$-algebra A, let A_I denote $A \otimes_{\mathcal{R}} \mathcal{R}_I$. Let $C^\cdot_A$ denote the complex

$$0 \to A \to \prod_{I, |I|=1} A_I \to \prod_{I, |I|=2} A_I \to \cdots,$$

where for $I \subset \{1, \dots, s\}$ we write $|I|$ for the cardinality of I. The differentials in $C^\cdot_A$ are obtained by taking alternating sums of the restriction maps. If $A = \mathcal{R}$ we write simply $C^\cdot$ for $C^\cdot_{\mathcal{R}}$, and if $A = \mathcal{R}/\mathfrak{m}^{t+1}$ we write $C^\cdot_t$ for $C^\cdot_{\mathcal{R}/\mathfrak{m}^{t+1}}$. Finally we write $\mathfrak{m}^t C^\cdot_A$ for the image of the natural map of complexes

$$\mathfrak{m}^t \otimes_{\mathcal{R}} C^\cdot_A \to C^\cdot_A.$$

Note that $\mathfrak{m}^0 C^\cdot_A = C^\cdot_A$. As above we write simply $\mathfrak{m}^t C^\cdot$ for $\mathfrak{m}^t C^\cdot_{\mathcal{R}}$.

Lemma 5.3. — *For every $t \geqslant 0$, the complex $\mathfrak{m}^t C^\cdot$ is acyclic.*

Proof. — Let $S = V[x_1, \dots, x_n]/(x_1 \cdots x_s)$, and for $I \subset \{1, \dots, s\}$ let S_I denote the quotient of S by the ideal generated by x_i $(i \in I)$. Let $K^\cdot$ denote the complex

$$0 \to S \to \prod_{I, |I|=1} S_I \to \prod_{I, |I|=2} S_I \to \cdots,$$

defined analogously to $C^\cdot$. Also let $J \subset S$ be the ideal generated by $(x_1, \dots, x_n)$, and let $J^tK^\cdot$ denote the image of the natural map of complexes

$$J^t \otimes K^\cdot \to K^\cdot.$$

Then $J^tK^\cdot$ defines a complex of coherent sheaves on $\mathrm{Spec}(S)$, whose stalk (in the étale topology) at the point defined by $x_1 = x_2 = \cdots = x_n = 0$ is the complex $\mathfrak{m}^tC^\cdot$. It therefore suffices to show that the complex $J^tK^\cdot$ is acyclic.

For this note that there is a natural grading on $J^tK^\cdot$ obtained by looking at monomials

$$x_1^{a_1} \cdots x_n^{a_n}, \quad \sum a_i \geqslant t.$$

This grading breaks up $J^tK^\cdot$ into the direct sum of complexes of the following form. Let $(a_1, \dots, a_n) \in \mathbb{N}^n$ be such that $\sum a_i \geqslant t$, and let $\Sigma = \{i \in \{1, \dots, s\} | a_i \neq 0\}$. Associated to this data is the complex

$$0 \to V \to \prod_{|I|=1, I\cap\Sigma=\emptyset} V \to \prod_{|I|=2, I\cap\Sigma=\emptyset} V \to \cdots,$$

where again the transition maps are given by restriction. Since this complex is acyclic by remark 5.4 below, and $J^tK^\cdot$ is isomorphic to a direct sum of such complexes with $\Sigma \neq \{1, \dots, s\}$, it follows that $J^tK^\cdot$ is also acyclic. □

Remark 5.4. — Note that if J is a nonempty finite set and H an abelian group, then the complex

$$0 \to H \to \prod_{I\subset J, |I|=1} H \to \prod_{I\subset J, |I|=2} H \to \cdots$$

is acyclic. Indeed, if we delete the first H, the resulting complex computes the cellular homology of the $|J|$-simplex with coefficients in H, and in particular is quasi-isomorphic to H.

For later use we also record the following immediate corollary of 5.3.

Corollary 5.5. — *For every $t \geqslant 0$ the complex*

$$\kappa_t := \mathfrak{m}^tC^\cdot/\mathfrak{m}^{t+1}C^\cdot$$

is acyclic.

5.2. We will also need a multiplicative version of the above results. For an $\mathcal{R}$-algebra A we can also consider the following complex, which will be denoted $D^\cdot_A$,

$$A^* \to \prod_{I, |I|=1} A_I^* \to \prod_{I, |I|=2} A_I^* \to \cdots,$$

where as before the transition maps are obtained from the natural restriction maps.

Fix now a *flat* $\mathcal{R}$-algebra A which is equal to the strict henselization of a finite type V-algebra at some point, and let A_t denote $A/\mathfrak{m}^{t+1}A$.

Lemma 5.6. — *For any $t \geqslant 0$ the complex $D^{\cdot}_{A_t}$ is acyclic.*

Proof. — Let $J \subset \{1, \dots, s\}$ denote the subset of indices i for which x_i maps to the maximal ideal of A. Then the complex $D^{\cdot}_A$ is given by

$$A^* \to \prod_{I \subset J, |I|=1} A^*_I \to \prod_{I \subset J, |I|=2} A^*_I \to \cdots,$$

and similarly for $D^{\cdot}_{A_t}$.

We prove the lemma by induction on t. For $t = 0$, it suffices to apply remark 5.4 with $H = A^*_0$.

For general t, note that there is a natural exact sequence of complexes (using the flatness of A over $\mathcal{R}$)

$$0 \to \kappa_t \otimes_{\mathcal{R}} A \to D^{\cdot}_{A_t} \to D^{\cdot}_{A_{t-1}} \to 0,$$

and by corollary 5.5 the complex κ_t is acyclic. □

Lemma 5.7. — *Let $\{M^{\cdot}_t\}_{t \geqslant 0}$ be a projective system of complexes of abelian groups such that for every t and i the map $M^i_t \to M^i_{t-1}$ is surjective. Let $M^{\cdot}$ denote $\varprojlim M^{\cdot}_t$. If each $M^{\cdot}_t$ is acyclic, then $M^{\cdot}$ is also acyclic.*

Proof. — This is a well-known fact; see for example [12, 0_{III} 13.2.3]. □

Corollary 5.8. — *Let $\widehat{A}$ denote the $\mathfrak{m}$-adic completion of A. Then the complex $D^{\cdot}_{\widehat{A}}$ is acyclic.*

Proof. — This follows from Lemmas 5.6 and 5.7, combined with the fact that $D^{\cdot}_{\widehat{A}} = \varprojlim_t D^{\cdot}_{A_t}$. □

Lemma 5.9. — *The complex $D^{\cdot}_A$ is acyclic.*

Proof. — Let $x \in D^i_A$ be a closed element. Consider the functor

$$F_x : (A\text{-algebras}) \to \text{Set},$$

which to any A-algebra B associates the set

$$\{y \in D^{i-1}_B | dy = x\},$$

where we abusively write also x for its image in D^i_B. This functor is clearly locally of finite presentation in the sense of [4, 1.5], and $F_x(\widehat{A})$ is non-empty by Corollary 5.8. By the Artin approximation theorem [4, 1.12] it follows that $F_x(A)$ is also nonempty. □

5.3. For $I \subset \{1, \ldots, s\}$, let $X_I \subset X$ denote $\mathrm{Spec}(\mathcal{R}_I) \subset \mathrm{Spec}(\mathcal{R})$. We write simply $\mathscr{O}_{X_I}$ and $\mathscr{O}^*_{X_I}$ for the pushforwards of these sheaves to X. The above results can be sheafified as follows. Again using the natural restriction maps we get a complex

$$1 \to \mathscr{O}^*_X \to \prod_{I, |I|=1} \mathscr{O}^*_{X_I} \to \prod_{I, |I|=2} \mathscr{O}^*_{X_I} \to \cdots . \tag{5.2}$$

Lemma 5.10. — *The complex (5.2) is acyclic.*

Proof. — Note that if $\bar{x} \to X$ is a geometric point, then the stalk of (5.2) at $\bar{x}$ is the complex $D^{\cdot}_{\mathscr{O}_{X,\bar{x}}}$. □

5.4. Turning now to the proof of Proposition 5.1, consider first the case when G is a tame étale group scheme. In this case we prove the proposition by showing that any finite étale covering of X° is trivial as in remark 5.2.

For this, let X_i $(i = 1, \ldots, s)$ denote the irreducible component defined by $x_i = 0$, and for indices $i_1, \ldots, i_t$ define

$$X_{i_1 \cdots i_t} := X_{i_1} \cap X_{i_2} \cap \cdots \cap X_{i_t}. \tag{5.3}$$

By proper descent theory for étale morphisms, the category $\mathrm{Fet}(X)$ of finite étale X-schemes is equivalent to the category of collections $\{(P_i \to X_i, \sigma_{ij})\}$, where $P_i \to X_i$ is a finite étale morphism and for every i, j

$$\sigma_{ij} : P_i|_{X_{ij}} \to P_j|_{X_{ij}} \tag{5.4}$$

is an isomorphism such that the diagrams over the triple intersections

$$\begin{tikzcd} P_i|_{X_{ijk}} \arrow[r, "\sigma_{ij}"] \arrow[rr, bend left, "\sigma_{ik}"] & P_j|_{X_{ijk}} \arrow[r, "\sigma_{jk}"] & P_k|_{X_{ijk}} \end{tikzcd} \tag{5.5}$$

commute.

Similarly if we set $X_i^o := X_i \cap X^o$ etc., then giving a finite étale morphism $P^o \to X^o$ is equivalent to giving a collection of data $\{(P_i^o \to X_i^o), \sigma_{ij}^o\}$, where $P_i^o \to X_i^o$ is a finite étale morphism and

$$\sigma^o_{ij} : P^o_i|_{X^o_{ij}} \to P^o_j|_{X^o_{ij}} \tag{5.6}$$

is an isomorphism such that the analogue of (5.5) commutes.

Now given such a collection of data $\{(P_i^o \to X_i^o), \sigma_{ij}^o\}$, let $P_i \to X_i$ be the normalization of X_i in P_i^o. Since X_i is regular and of dimension $\geqslant 2$ (by our assumption on n), the morphism $P_i \to X_i$ is étale by purity. Furthermore, since X_{ij} is normal the restriction functor

$$\mathrm{Fet}(X_{ij}) \to \mathrm{Fet}(X^o_{ij}) \tag{5.7}$$

is fully faithful. It follows that the σ^o_{ij} extend uniquely to isomorphisms

$$\sigma_{ij} : P_i|_{X_{ij}} \to P_j|_{X_{ij}}. \tag{5.8}$$

Furthermore, since the intersections X_{ijk} are also normal the diagram (5.5) commutes as its restriction to X^o_{ijk} commutes.

We therefore obtain a collection of data $\{(P_i \to X_i), \sigma_{ij}\}$ inducing $\{(P^o_i \to X^o_i), \sigma^o_{ij}\}$.

5.5. For the case of general G, note first that by making a finite étale extension of V (which is permitted by descent theory), we may assume that G fits into an exact sequence

$$1 \to \Delta \to G \to H \to 1,$$

where Δ is diagonalizable and H is étale and tame.

Let us first show that the restriction functor (5.1) is fully faithful. Let P_1 and P_2 be two G-torsors over X, and set

$$\overline{P}_i := P_i/\Delta,$$

which is an H-torsor over X. Assume given an isomorphism of G-torsors

$$\rho^\circ : P^\circ_1 \to P^\circ_2$$

over X°. By the étale case, the induced isomorphism

$$\bar{\rho}^\circ : \overline{P}^\circ_1 \to \overline{P}^\circ_2$$

extends uniquely to an isomorphism of H-torsors over X. Since the $\overline{P}_i$ are trivial H-torsors (being étale torsors over the spectrum of a strictly henselian local ring), the underlying scheme of $\overline{P}_i$ is isomorphic to a disjoint union of copies of X. Now to find a dotted arrow

$$\begin{tikzcd} P_1 \arrow[r, dashed, "\rho"] \arrow[d] & P_2 \arrow[d] \\ \overline{P}_1 \arrow[r, "\bar{\rho}"] & \overline{P}_2 \end{tikzcd}$$

filling in the diagram and restricting to ρ°, it suffices to find an extension of ρ° which is compatible with the Δ-action (since the compatibility with the G-action can be verified after restriction to X°). This reduces the proof of full faithfulness to the case when $G = \Delta$. Since Δ is equal to a finite product of copies of μ_n for various n, one further reduces to the case of $\Delta = \mu_n$ for some integer n. In this case the category of μ_n-torsors (over either X or X°) is equivalent to the category of pairs (L, ι), where L is a line bundle and ι is a trivialization of L^n.

Let P_i correspond to (L_i, ι_i) $(i = 1, 2)$. The isomorphism ρ° corresponds to an isomorphism $\lambda : \mathscr{O}_{X^\circ} \to L_2 \otimes L_1^{-1}$ such that the induced isomorphism

$$\lambda^{\otimes n} : \mathscr{O}_{X^\circ} \to L_2^{\otimes n} \otimes L_1^{\otimes -n}$$

is equal to $\iota_2 \otimes \iota_1^{-1}$. The full faithfulness therefore follows from part (i) of the following lemma.

Lemma 5.11. — *(i) The restriction map*

$$\Gamma(X, \mathscr{O}_X^*) \to \Gamma(X^\circ, \mathscr{O}_{X^\circ}^*)$$

is an isomorphism.

(ii) The restriction map

$$H^1(X, \mu_n) \to H^1(X^\circ, \mu_n)$$

is an isomorphism (here cohomology is taken with respect to the fppf-topology).

Proof. — Statement (i) follows immediately from the exactness of (5.2) (by Lemma 5.10), and the fact that the X_I $(I \neq \emptyset)$ are normal, and of dimension $\geqslant 2$ for $|I| = 1$.

For (ii), consider the short exact sequence of sheaves on X

$$0 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \xrightarrow{u \mapsto u^n} \mathbb{G}_m \longrightarrow 0,$$

which upon taking cohomology gives a commutative diagram with exact rows

$$\begin{array}{ccccccc}
\Gamma(X, \mathscr{O}_X^*) & \xrightarrow{\times n} & \Gamma(X, \mathscr{O}_X^*) & \longrightarrow & H^1(X, \mu_n) & \longrightarrow & \mathrm{Pic}(X) \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow & & \downarrow \\
\Gamma(X^\circ, \mathscr{O}_{X^\circ}^*) & \xrightarrow{\times n} & \Gamma(X^\circ, \mathscr{O}_{X^\circ}^*) & \longrightarrow & H^1(X^\circ, \mu_n) & \longrightarrow & \mathrm{Pic}(X^\circ),
\end{array}$$

where the vertical maps are the restriction maps. The two left-most vertical arrows are isomorphisms since X is normal and $X - X^\circ$ has codimension $\geqslant 2$, and also $\mathrm{Pic}(X) = 0$ since X is strictly henselian local. Therefore to prove (ii) it suffices to show that the torsion subgroup

$$\mathrm{Pic}(X^\circ)_{\mathrm{tors}} \subset \mathrm{Pic}(X^\circ)$$

is zero.

For this let X_I° denote $X_I \cap X^\circ$, and consider the resolution (5.2) of $\mathscr{O}_X^*$ restricted to X°. The complex

$$\mathscr{K}^\cdot : \prod_{I, |I|=1} \mathscr{O}_{X_I^\circ}^* \to \prod_{I, |I|=2} \mathscr{O}_{X_I^\circ}^* \to \cdots$$

is filtered by the subcomplexes $\sigma_{\geqslant p}\mathscr{K}^\cdot$ with

$$(\sigma_{\geqslant p}\mathscr{K}^\cdot)^n = \begin{cases} 0 & \text{if } n < p \\ \mathscr{K}^n & \text{if } n \geqslant p. \end{cases}$$

The hypercohomology spectral sequence of a filtered complex (see for example [9, 1.4.5]) applied to $\mathscr{K}^\cdot$ with this filtration can then be written as

$$E_1^{pq} = \prod_{I, |I|=p+1} H^q(X_I^\circ, \mathscr{O}_{X_I^\circ}^*) \implies H^{p+q}(X^\circ, \mathscr{O}_{X^\circ}^*).$$

In particular we have an exact sequence

$$0 \to H^1(E_1^{\cdot 0}) \to \mathrm{Pic}(X^\circ) \to \mathrm{Ker}(\prod_{I,|I|=1} \mathrm{Pic}(X_I^\circ) \to \prod_{I,|I|=2} \mathrm{Pic}(X_I^\circ)).$$

Now for $|I| = 1$ the complement of $X_I^\circ \subset X_I$ has codimension $\geqslant 2$ so the restriction map

$$0 = \mathrm{Pic}(X_I) \to \mathrm{Pic}(X_I^\circ)$$

is an isomorphism. Therefore

$$H^1(E_1^{\cdot 0}) \to \mathrm{Pic}(X^\circ)$$

is an isomorphism.

The complex $E_1^{\cdot 0}$ is the complex of global sections

$$\prod_{I,|I|=1} \Gamma(X^\circ, \mathscr{O}_{X_I^\circ}^*) \to \prod_{I,|I|=2} \Gamma(X^\circ, \mathscr{O}_{X_I^\circ}^*) \to \prod_{I,|I|=3} \Gamma(X^\circ, \mathscr{O}_{X_I^\circ}^*) \to \cdots.$$

We have a commutative diagram

(5.9)
$$\begin{array}{ccccccc} \prod_{I,|I|=1} \Gamma(X, \mathscr{O}_{X_I}^*) & \longrightarrow & \prod_{I,|I|=2} \Gamma(X, \mathscr{O}_{X_I}^*) & \longrightarrow & \prod_{I,|I|=3} \Gamma(X, \mathscr{O}_{X_I}^*) & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ \prod_{I,|I|=1} \Gamma(X^\circ, \mathscr{O}_{X_I^\circ}^*) & \to & \prod_{I,|I|=2} \Gamma(X^\circ, \mathscr{O}_{X_I^\circ}^*) & \to & \prod_{I,|I|=3} \Gamma(X^\circ, \mathscr{O}_{X_I^\circ}^*) & \to & \cdots, \end{array}$$

where the vertical maps are restriction maps, and the top complex has no cohomology in degrees > 0 by Lemma 5.9. If $s > 2$ then for $|I| \leqslant 2$ the map

$$\Gamma(X_I, \mathscr{O}_{X_I}^*) \to \Gamma(X_I^\circ, \mathscr{O}_{X_I^\circ}^*) \tag{5.10}$$

is an isomorphism (since $X_I - X_I^\circ$ has codimension $\geqslant 2$) and for $|I| = 3$ the map (5.10) is injective. Therefore when $s > 2$ the map of complexes (5.9) induces an isomorphism on H^1 whence $H^1(E_1^{\cdot 0}) = 0$.

If $s = 2$, then the terms in (5.9) with $|I| = 3$ are zero, the first vertical map is still an isomorphism, and the second vertical map is identified with the map

$$R^* \to (R[1/\pi])^*, \tag{5.11}$$

where R denotes the strict henselization of $V[x_3, \dots, x_n]$ at the origin of the closed fiber and $\pi \in V$ is a uniformizer. If

$$\eta_0 \in \mathrm{Spec}(R)$$

denotes the generic point of the closed fiber, then the local ring at η_0 is a discrete valuation ring with uniformizer π, and the associated discrete valuation induces an isomorphism between $\mathbb{Z}$ and the cokernel of (5.11). We therefore find that if $s = 2$ then $H^1(E_1^{\cdot 0}) \simeq \mathbb{Z}$, and in particular this group is torsion free. □

5.6. Returning finally to the proof of essential surjectivity in Proposition 5.1, note that by a similar reduction as in the start of the proof it suffices to consider the case when $G = \mu_n$. In this case the result follows from Lemma 5.11 (ii). □

6. Coverings of tame stacks and their coarse spaces.

6.1. Let S be a scheme and let $\mathscr{X}/S$ be a tame stack. Let $\pi : \mathscr{X} \to X$ be the coarse moduli space. For every geometric point $\bar{x} : \mathrm{Spec}(k) \to \mathscr{X}$ let $G_{\bar{x}}$ denote the automorphism group scheme of $\bar{x}$ (a finite tame group scheme over k). If L is a line bundle on $\mathscr{X}$, then the pullback $\bar{x}^*L$ is a rank 1 representation of $G_{\bar{x}}$.

Similarly, if $P \to \mathscr{X}$ is a finite étale morphism of stacks, then the fiber product $P_{\bar{x}} := \bar{x} \times_{\mathscr{X}} P$ is a finite disjoint union of copies of $\mathrm{Spec}(k)$ and there is a natural action of $G_{\bar{x}}$ on $P_{\bar{x}}$.

Proposition 6.1. — *The pullback functor*

$$\pi^* : (\textit{line bundles on } X) \to (\textit{line bundles on } \mathscr{X})$$

induces an equivalence of categories between the category of line bundles on X and the category of line bundles L on $\mathscr{X}$ such that for every geometric point $\bar{x} : \mathrm{Spec}(k) \to \mathscr{X}$ the action of $G_{\bar{x}}$ on $\bar{x}^*L$ is trivial.

Proof. — It is clear that if M is a line bundle on X and $L := \pi^*M$, then for any geometric point $\bar{x}$ of $\mathscr{X}$ the action of $G_{\bar{x}}$ on $\bar{x}^*L$ is trivial. Therefore π^* induces a functor between the indicated categories.

To see that π^* is fully faithful, assume M and M' are line bundles on X. Giving a map $M \to M'$ is equivalent to giving a global section of $M^{-1} \otimes M'$, and similarly giving a map $\pi^*M \to \pi^*M'$ is equivalent to giving a global section of

$$\pi^*M^{-1} \otimes_{\mathscr{O}_{\mathscr{X}}} \pi^*M' \simeq \pi^*(M^{-1} \otimes_{\mathscr{O}_X} M').$$

To prove the full faithfulness it therefore suffices to show that the natural map

$$M^{-1} \otimes M' \to \pi_*\pi^*(M^{-1} \otimes M')$$

is an isomorphism. This can be verified locally on X, so it further suffices to consider the case when M and M' are trivial. In this case it follows from the fact that $\mathscr{O}_X \to \pi_*\mathscr{O}_{\mathscr{X}}$ is an isomorphism (see for example [3, 2.2.1 (5)]).

For the essential surjectivity, let L be a line bundle on $\mathscr{X}$ such that for every geometric point $\bar{x} : \mathrm{Spec}(k) \to \mathscr{X}$ the action of $G_{\bar{x}}$ on $\bar{x}^*L$ is trivial. We claim that in this case the sheaf π_*L is a line bundle on X and that the adjunction map

$$\pi^*\pi_*L \to L$$

is an isomorphism.

This can be verified locally in the fppf topology on X. From this, a standard limit argument, and [1, 3.2 (c)] we may assume that $X = \mathrm{Spec}(R)$ for some strictly henselian local ring R and

$$\mathscr{X} = [\mathrm{Spec}(A)/G]$$

where A is a finite R-algebra and G is a tame group scheme acting on A. In fact we can even base change to the completion of R along its maximal ideal, so may further assume that R is complete local with maximal ideal $\mathfrak{m}_R$. Let N be the free A-module of rank 1 with action of G corresponding to L, and let N_n denote $N/\mathfrak{m}_R^n N$. Since A is finite over R, N is a finitely generated R module so

$$N = \varprojlim N_n,$$

and since G is linearly reductive we also have

$$N^G = \varprojlim N_n^G.$$

It therefore suffices to show that for every n the $R/\mathfrak{m}_R^n$-module N_n^G is free of rank 1, and that the map

$$N_n^G \otimes_{R/\mathfrak{m}_R^n} (A/\mathfrak{m}_R^n A) \to N_n$$

is an isomorphism. This reduces the proof to the case when R is artinian local with separably closed residue field k. By [1, 3.7], the inclusion morphism

$$\mathrm{Spec}(k) \hookrightarrow \mathrm{Spec}(R)$$

lifts to a morphism

$$\xi : \mathrm{Spec}(k) \to \mathscr{X}.$$

Let G_ξ denote the stabilizer group scheme (which is linearly reductive since $\mathscr{X}$ is tame), so we have a closed immersion

$$BG_\xi \hookrightarrow \mathscr{X}.$$

As in [1, Proof of 3.6, Step 1], the trivial torsor

$$\mathrm{Spec}(k) \to BG_\xi$$

lifts to a G_ξ-torsor

$$\mathrm{Spec}(A_\xi) \to \mathscr{X}$$

which gives a presentation

$$\mathscr{X} \simeq [\mathrm{Spec}(A_\xi)/G_\xi].$$

Replacing A by A_ξ and G by G_ξ we may assume that A is artinian local with residue field k, and that G acts trivially on the residue field. In this case by assumption the action of G on $N/\mathfrak{m}_A N$ is trivial. Since G is linearly reductive we can find an invariant section $e \in N^G$ which maps to a basis for $N/\mathfrak{m}_A N$. Since N is a free module over A of rank 1, we conclude that the map

$$A \to N$$

defined by e is an isomorphism. Then $\pi_* L$ corresponds to the R-module $A^G \cdot e = R \cdot e$ which is of rank 1 and the map

$$A \otimes_R N^G \to N$$

is an isomorphism. □

Proposition 6.2. — *Let* $\mathrm{Fet}(X)$ *(resp.* $\mathrm{Fet}(\mathscr{X})$*) denote the category of finite étale morphisms* $Y \to X$ *(resp.* $\mathscr{Y} \to \mathscr{X}$*).*

The pullback functor

$$\pi^* : \mathrm{Fet}(X) \to \mathrm{Fet}(\mathscr{X})$$

induces an equivalence of categories between $\mathrm{Fet}(X)$ *and the category of finite étale morphisms of stacks* $P \to \mathscr{X}$ *such that for every geometric point* $\bar{x} : \mathrm{Spec}(k) \to \mathscr{X}$ *the action of* $G_{\bar{x}}$ *on* $P_{\bar{x}}$ *is trivial.*

Remark 6.3. — A priori, $\mathrm{Fet}(\mathscr{X})$ is a 2-category. However, it is in fact equivalent to a 1-category since a finite étale morphism is in particular representable.

Proof. — For the full faithfulness let P_1 and P_2 be two finite étale X-schemes. Let $\mathscr{M}or(P_1, P_2)$ be the sheaf of sets on the étale site of X given by

$$(U \to X) \mapsto \{U\text{-morphisms } P_{1,U} \to P_{2,U}\}.$$

Similarly define $\mathscr{M}or(\pi^* P_1, \pi^* P_2)$ to be the sheaf on the étale site of X (not $\mathscr{X}$!) given by

$$(U \to X) \mapsto \{\mathscr{X}_U\text{-morphisms } \pi^* P_1|_{\mathscr{X}_U} \to \pi^* P_2|_{\mathscr{X}_U}\}.$$

There is a natural map of sheaves

$$\mathscr{M}or(P_1, P_2) \to \mathscr{M}or(\pi^* P_1, \pi^* P_2)$$

which we need to show is an isomorphism. For this we may work locally on X, and hence may assume that both P_1 and P_2 are trivial in which case the result is immediate.

For the essential surjectivity let $P \to \mathscr{X}$ be a finite étale morphism such that for every geometric point $\bar{x} : \mathrm{Spec}(k) \to \mathscr{X}$ the action of $G_{\bar{x}}$ on $P_{\bar{x}}$ is trivial.

By the full faithfulness already shown, to prove that P is in the essential image of π^* we may work fppf locally on X. By [1, 3.2 (c)] we may therefore assume that

$$\mathscr{X} = [\mathrm{Spec}(A)/G]$$

and

$$X = \mathrm{Spec}(A^G),$$

as in the proof of Proposition 6.1, and by a standard limit argument we may even assume that $R = A^G$ is strictly henselian local. In this case A is also strictly henselian local and so the restriction of P to $\mathrm{Spec}(A)$ is trivial. It follows that $P|_{\mathrm{Spec}(A)}$ is isomorphic to

$$\pi_0(P|_{\mathrm{Spec}(A)}) \times \mathrm{Spec}(A)$$

and so to show that P is obtained by pullback from $\mathrm{Spec}(R)$ it suffices to show that the action of G on $\pi_0(P|_{\mathrm{Spec}(A)})$ is trivial. This can be verified after base change to the closed point of $\mathrm{Spec}(A)$ in which case it holds by assumption. $\square$

6.2. Let D/S be a tame finite flat group scheme. For a geometric point $\bar{x} : \mathrm{Spec}(k) \to \mathscr{X}$ we write $D_{\bar{x}}$ for the pullback of D along the composite morphism

$$\mathrm{Spec}(k) \xrightarrow{\bar{x}} \mathscr{X} \longrightarrow S.$$

If $P \to \mathscr{X}$ is a D-torsor over $\mathscr{X}$ then the pullback $\bar{x}^*P$ is a $D_{\bar{x}}$-torsor over $\mathrm{Spec}(k)$ which comes equipped with an action of $G_{\bar{x}}$, compatible with the action of $D_{\bar{x}}$.

Proposition 6.4. — *The pullback functor*

$$\pi^* : (D\text{-}torsors\ on\ X) \to (D\text{-}torsors\ on\ \mathscr{X})$$

induces an equivalence of categories between the category of D-torsors on X and the category of D-torsors $P \to \mathscr{X}$ on $\mathscr{X}$ such that for every geometric point $\bar{x} : \mathrm{Spec}(k) \to \mathscr{X}$ the action of $G_{\bar{x}}$ on $P_{\bar{x}}$ is trivial.

Proof. — In the case when D is an étale group scheme the result follows from Proposition 6.2.

If D is diagonalizable the proposition can be seen as follows. Let T be the Cartier dual of D (a finite abelian group) and write

$$T = \mathbb{Z}/(a_1) \times \cdots \times \mathbb{Z}/(a_r)$$

for some positive integers a_i. Then the category of D-torsors on X is equivalent to the category of collections $\{(L_i, \iota_i)\}_{i=1}^r$, where L_i is a line bundle on X and

$$\iota_i : L_i^{a_i} \to \mathscr{O}_X$$

is an isomorphism of line bundles. The category of D-torsors on $\mathscr{X}$ is described similarly. In the case when D is diagonalizable Proposition 6.4 therefore follows from Proposition 6.1.

For the general case, we may work fpqc locally on S and therefore by [1, 2.16] may assume that D fits into an exact sequence

$$1 \to \Delta \to D \to H \to 1,$$

where H is tame and constant and Δ is diagonalizable.

For the full faithfulness we proceed as in the proof of Proposition 6.2. Let P_1 and P_2 be two D-torsors over X and define $\mathscr{M}or(P_1, P_2)$ (resp. $\mathscr{M}or(\pi^*P_1, \pi^*P_2)$) to be the sheaf on the fppf site of X which to any $U \to X$ associates the set of morphisms $P_{1,U} \to P_{2,U}$ of D-torsors over U (resp. the set of morphisms $\pi^*P_1|_{\mathscr{X}_U} \to \pi^*P_2|_{\mathscr{X}_U}$ of D-torsors over $\mathscr{X}_U$). We then need to show that the map of fppf sheaves

$$\mathscr{M}or(P_1, P_2) \to \mathscr{M}or(\pi^*P_1, \pi^*P_2) \tag{6.1}$$

is an isomorphism. This is an fppf-local assertion so it suffices to consider the case when P_1 and P_2 are trivial. Fixing trivializations, the map on global sections defined by the map (6.1) is identified with the natural map

$$\mathrm{Hom}_S(X, D) \to \mathrm{Hom}_S(\mathscr{X}, D)$$

which is a bijection since X is the coarse moduli space of $\mathscr{X}$.

For the essential surjectivity, let $P \to \mathscr{X}$ be a D-torsor such that for every geometric point $\bar{x} : \mathrm{Spec}(k) \to \mathscr{X}$ the action of $G_{\bar{x}}$ on $P_{\bar{x}}$ is trivial. We descend P to a D-torsor on X as follows. Let $\overline{P}$ denote the quotient of P by the Δ-action, so that $\overline{P}$ is an H-torsor over $\mathscr{X}$. Then by the case when D is an étale group scheme there exists an H-torsor $\overline{Q} \to X$ such that $\overline{Q} \times_X \mathscr{X} \simeq \overline{P}$. Observe that $\overline{Q}$ is the coarse moduli space of $\overline{P}$ by [3, 2.2.2 (1)].

Now P is a Δ-torsor over $\overline{P}$ so by the case of a diagonalizable group scheme applied to the stack $\overline{P}$ we obtain a Δ-torsor $Q \to \overline{Q}$ inducing P. Furthermore, by the full faithfulness already shown, the maps of Δ-torsors

$$\chi_h : Q_h \wedge P \to h^* P$$

over $\overline{P}$ defined as in paragraph 3.3 , descend to maps of Δ-torsors

$$Q_h \wedge Q \to h^* Q$$

giving Q the structure of a D-torsor, inducing the D-torsor structure on P. □

6.3. There is also a relative version of Proposition 6.4.

Let $f : \mathscr{X} \to \mathscr{Y}$ be a morphism of tame stacks, and let

$$\mathscr{X} \xrightarrow{\pi} X_{\mathscr{Y}} \longrightarrow \mathscr{Y}$$

be the relative moduli space (see for example [2, §3]).

For any geometric point $\bar{x} \to \mathscr{X}$ we have a map of linearly reductive finite flat group schemes over $\bar{x}$

$$\underline{\mathrm{Aut}}_{\mathscr{X}}(\bar{x}) \to \underline{\mathrm{Aut}}_{\mathscr{Y}}(f \circ \bar{x}).$$

Let $K_{\bar{x}}$ be the kernel of this homomorphism, so $K_{\bar{x}}$ is also a linearly reductive group scheme over $\bar{x}$ [1, 2.5].

Now let D be a linearly reductive finite flat group scheme over S and let $P \to \mathscr{X}$ be a D-torsor. Then as in paragraph 6.2 there is an induced action of $K_{\bar{x}}$ on $P_{\bar{x}}$ compatible with the $D_{\bar{x}}$-action.

Proposition 6.5. — *The functor*

$$\pi^* : (D\text{-}torsors\ on\ X_{\mathscr{Y}}) \to (D\text{-}torsors\ on\ \mathscr{X})$$

induces an equivalence between the category of D-torsors on $X_{\mathscr{Y}}$ and the category of D-torsors P on $\mathscr{X}$ such that for every geometric point $\bar{x} \to \mathscr{X}$ the action of $K_{\bar{x}}$ on $P_{\bar{x}}$ is trivial.

Proof. — All the categories considered in the proposition are stacks over $\mathscr{Y}$ with the fppf topology. It therefore suffices to consider the case when $\mathscr{Y}$ is a scheme, in which case the result is Proposition 6.4. □

Proposition 6.6. — *With notation as in paragraph 6.3, let $P \to \mathscr{X}$ be a D-torsor. Then there exists an open substack $\mathscr{U} \subset \mathscr{X}$ such that a geometric point $\bar{x} \to \mathscr{X}$ factors through $\mathscr{U}$ if and only if the action of $K_{\bar{x}}$ on $P_{\bar{x}}$ is trivial.*

Proof. — Let $Y \to \mathscr{Y}$ be a smooth surjection with Y a scheme. Then it clearly suffices to prove the proposition for $\mathscr{X} \times_{\mathscr{Y}} Y \to Y$. We may therefore assume that $\mathscr{Y} = Y$ is a scheme. Let $\mathscr{X} \to X$ be the coarse moduli space. We may then further assume that $Y = X$.

Let $\mathscr{I} \to \mathscr{X}$ be the inertia stack of $\mathscr{X}$, and let $\mathscr{G}$ denote the automorphism group scheme over $\mathscr{X}$ of P. Since P is a D-torsor the group scheme $\mathscr{G}$ over $\mathscr{X}$ is a twisted form of D. There is a canonical homomorphism of relative group schemes over $\mathscr{X}$

$$\rho : \mathscr{I} \to \mathscr{G}.$$

We need to show that there exists an open substack $\mathscr{U} \subset \mathscr{X}$ such that for any geometric point $\bar{x} \to \mathscr{X}$ the homomorphism

$$\rho_{\bar{x}} : \bar{x}^*\mathscr{I} \to \bar{x}^*\mathscr{G}$$

is trivial if and only if $\bar{x}$ has image in $\mathscr{U}$.

This follows from the following lemma. □

Lemma 6.7. — *Let S be a noetherian scheme, let I/S be a finite (but not necessarily flat) group scheme such that for every point $s \in S$ the fiber $I_s \to \mathrm{Spec}(k(s))$ is linearly reductive, and let G/S be a linearly reductive finite flat group scheme. Fix a homomorphism*

$$\rho : I \to G.$$

Then the set of points $s \in S$ for which the homomorphism of group schemes over $k(s)$

$$\rho_s : I_s \to G_s$$

is trivial is an open subset of S.

Proof. — We show that the set of points s for which ρ_s is trivial is constructible and stable under generization.

For the constructibility, note that since I is finite over S there exists a stratification of S such that the restriction of I to each stratum is finite and flat. It therefore suffices to consider the case when I is finite and flat over S. In this case we claim that the set of points $s \in S$ for which ρ_s is trivial is an open and closed subset of S.

For this note that we can without loss of generality replace S by a fpqc covering, so we may by [1, 2.16] assume that both I and G are well-split in the sense of [1, 2.6]. Write

$$I = \Delta \rtimes H, \quad G = \Sigma \rtimes \Gamma,$$

where Δ and Σ are diagonalizable of order a power of a prime p, and H and Γ are tame étale group schemes of order prime to p.

The homomorphism ρ is then induced by homomorphisms

$$\rho_1 : \Delta \to \Sigma, \quad \rho_2 : H \to \Gamma.$$

The condition that ρ_2 is trivial is clearly an open and closed condition, and the condition that ρ_1 is trivial is equivalent to the condition that the map on Cartier duals

$$\mathrm{Hom}(\Sigma, \mathbb{G}_m) \to \mathrm{Hom}(\Delta, \mathbb{G}_m)$$

is trivial, which is also clearly an open and closed condition. This completes the proof of the constructibility of the set of points $s \in S$ for which ρ_s is trivial.

For the stability under generization, we may without loss of generality assume that $S = \mathrm{Spec}(V)$ is the spectrum of a discrete valuation ring. Let $\eta \in S$ (resp. $s \in S$) be the generic (resp. closed) point of S. We need to show that if ρ_s is trivial, then ρ_η is also trivial.

Let $I' \subset I$ denote the scheme-theoretic closure of I_η in I. Then I' is a finite flat group scheme over S and I' is linearly reductive as the closed fiber of I' is a closed subgroup scheme of a linearly reductive group scheme. Moreover the map $I'_s \to G_s$ is trivial since ρ_s is trivial, and hence by the same reasoning used above it follows that ρ_η is also trivial. □

6.4. We conclude this section with a consequence of Proposition 6.5 which we will use in what follows. With notation as in paragraph 4.1, let A be a strictly henselian discrete valuation ring over k with residue field L and fraction field K. Fix a uniformizer $\pi \in A$, and let G/A be a tame finite flat group scheme.

Let f be an integer and let $\mathscr{V}_f$ be as in paragraph 4.2, and let $P \to \mathscr{V}_f$ be a G-torsor, and let $P_\eta \to \operatorname{Spec}(K)$ be the restriction of P to the generic point of $\mathscr{V}_f$. From this torsor and Proposition 4.1, we then get an integer e dividing f.

On the other hand, there is a natural inclusion

$$B\mu_f = [\operatorname{Spec}(L)/\mu_f] \hookrightarrow \mathscr{V}_f,$$

induced by the closed point $\operatorname{Spec}(L) \hookrightarrow \operatorname{Spec}(A_f)$. Pulling P back along this closed immersion we get a G-torsor over $B\mu_f$, or equivalently a conjugacy class of homomorphisms

$$\mu_f \to G. \tag{6.2}$$

Let δ denote the order of the kernel of this homomorphism (note that it depends only on the conjugacy class of the map).

Lemma 6.8. — *We have $\delta = f/e$.*

Proof. — We have a commutative diagram

$$\begin{tikzcd} B\mu_f \arrow[r, hook] \arrow[d] & \mathscr{V}_f \arrow[d] \\ B\mu_e \arrow[r, hook] & \mathscr{V}_e, \end{tikzcd}$$

and by Proposition 4.4 the torsor P is pulled back from $\mathscr{V}_e$. Therefore the kernel of the homomorphism

$$\mu_f \to \mu_e, \quad u \mapsto u^{f/e}$$

is in the kernel of the homomorphism (6.2), and f/e divides δ.

This implies that the number f/δ divides e and we get a diagram of stacks

$$\mathscr{V}_f \to \mathscr{V}_e \to \mathscr{V}_{f/\delta}.$$

To prove the lemma it suffices to show that P descends to $\mathscr{V}_{f/\delta}$, for then by the minimality of e we must have $e = f/\delta$. This follows from Proposition 6.5, since the relative moduli space of

$$\mathscr{V}_f \to \mathscr{V}_{f/\delta}$$

is $\mathscr{V}_{f/\delta}$ itself, and by construction for every geometric point $\bar{x} \to \mathscr{V}_f$ the kernel of the induced map of inertia groups acts trivially on $P_{\bar{x}}$ (there are only two points two check, and this condition on the generic point is trivial). $\square$

7. Twisted G-torsors and stability.

In this section we introduce the notion of a stable twisted G-torsor, and define the stack ${}_G\mathscr{K}_{X/S}$. The setup will be the following:

Setup 7.1. — *Let $(X, M_X) \to (S, M_S)$ be a log smooth special and essentially semistable morphism (in the sense of paragraph 2.2), with $X \to S$ proper.*

7.1. Let G/S be a tame finite flat group scheme. We define a *twisted G-torsor on X* to be a pair $(j : M_S \hookrightarrow N_S, P)$, where j is a simple extension and $P \to \mathscr{X}_j$ is a G-torsor on $\mathscr{X}_j$, where $\mathscr{X}_j$ is defined as in paragraph 2.3.

7.2. A key notion in what follows will be the notion of a stable twisted G-torsor which we now define. This notion will be defined pointwise, so let us first discuss the definition over a separably closed field.

Assume that $S = \mathrm{Spec}(k)$ is the spectrum of a separably closed field and that $(j : M_S \hookrightarrow N_S, P)$ is a twisted G-torsor. Let $Z \subset X$ be the singular locus of X with the reduced structure. Étale locally X is isomorphic to

$$\mathrm{Spec}(k[x_1, \ldots, x_n]/(x_1 \cdots x_r))$$

for some $r \leqslant n$, in which case Z is the union of the closed subschemes defined by the ideals (x_i, x_j) $(1 \leqslant i < j \leqslant r)$.

Let $Z_1, \ldots, Z_t$ be the connected components of Z. Recall (see the discussion in paragraph 2.2) that there is a canonical bijection between the connected components Z_i of Z and the irreducible elements in $\overline{M}_S \simeq \mathbb{N}^r$. This bijection is characterized by the condition that for any geometric point $\bar{z} \to Z_i$ the irreducible element $e_i \in \overline{M}_S$ is the unique irreducible element whose image in $\overline{M}_{X,\bar{z}}$ is not irreducible.

Notice also that the map $\mathscr{X}_j \to X$ is an isomorphism over the complement of Z. This follows for example from the local description in Proposition 2.2.

For each irreducible element $e_i \in \overline{M}_S$ there exists a unique integer a_i such that the image of e_i in $\overline{N}_S$ is equal to a_i times an irreducible element of $\overline{N}_S$. In this way we associate to each connected component Z_i an integer a_i.

On the other hand, consider a geometric generic point $\bar{\eta} \to Z_i$ for some connected component Z_i of Z. Then $\mathrm{Spec}(\mathscr{O}_{X,\bar{\eta}})$ has exactly two irreducible components, say W_1 and W_2. Moreover, the rings $\mathscr{O}_{W_p,\bar{\eta}}$ are strictly henselian discrete valuation rings. Let $\pi_p \in \mathscr{O}_{W_p,\bar{\eta}}$ $(p = 1, 2)$ be a uniformizer, and let $W_{p,\bar{\eta}}$ denote $\mathrm{Spec}(\mathscr{O}_{W_p,\bar{\eta}})$. Let $W^\circ_{p,\bar{\eta}} \subset W_{p,\bar{\eta}}$ be the generic point, and let

$$P^\circ_p \to W^\circ_{p,\bar{\eta}}$$

be the restriction of P (since $\mathscr{X}_j \to X$ is an isomorphism over $W^\circ_{p,\bar\eta}$ this makes sense). By Proposition 4.4 there exists a minimal integer $e_{\bar\eta,p}$ such that P°_p extends to

$$\mathscr{W}_{p,e_{\bar\eta,p}} := [\mathrm{Spec}(\mathscr{O}_{W_p,\bar\eta}[t]/(t^{e_{\bar\eta,p}} - \pi_p))/\mu_{e_{\bar\eta,p}}].$$

Lemma 7.2. — *We have $e_{\bar\eta,1} = e_{\bar\eta,2}$.*

Proof. — Note that we have a natural isomorphism

$$\mathscr{W}_{p,a_i} \simeq (W_p \times_X \mathscr{X}_j)_{\mathrm{red}}. \tag{7.1}$$

Indeed it follows from the local description of the stack $\mathscr{X}_j$ in Proposition 2.2, that we can find two elements $x_1, x_2 \in \mathscr{O}_{X,\bar\eta}$ defining the irreducible components W_1 and W_2 respectively, such that the stack

$$\mathrm{Spec}(\mathscr{O}_{X,\bar\eta}) \times_X \mathscr{X}_j$$

is isomorphic to the stack

$$[\mathrm{Spec}(\mathscr{O}_{X,\bar\eta}[t_1,t_2]/(t_1^{a_i} = x_1, t_2^{a_i} = x_2, t_1t_2))/\mu_{a_i}],$$

where $\zeta \in \mu_{a_i}$ acts by $t_1 \mapsto \zeta \cdot t_1$ and $t_2 \mapsto \zeta^{-1} \cdot t_2$. From this we find that

$$(W_1 \times_X \mathscr{X}_j)_{\mathrm{red}} \simeq [\mathrm{Spec}(\mathscr{O}_{W_1}[t_2]/t_2^{a_i} = x_2)/\mu_{a_i}]$$

and

$$(W_2 \times_X \mathscr{X}_j)_{\mathrm{red}} \simeq [\mathrm{Spec}(\mathscr{O}_{W_2}[t_1]/t_1^{a_i} = x_1)/\mu_{a_i}].$$

This gives the isomorphism (7.1) using the choice of coordinates. To make it canonical, note that if

$$\lambda_p : (W_p \times_X \mathscr{X}_j)_{\mathrm{red}} \to W_p$$

is the projection, then

$$(\lambda_p^{-1}(W_1 \cap W_2))_{\mathrm{red}} \subset (W_p \times_X \mathscr{X}_j)_{\mathrm{red}}$$

is an effective Cartier divisor whose a_i-th power is $\lambda_p^{-1}(W_1 \cap W_2)$. From this and the modular description of $\mathscr{W}_{p,a_i}$ in paragraph 4.2, we get a morphism

$$(W_p \times_X \mathscr{X}_j)_{\mathrm{red}} \to \mathscr{W}_{p,a_i}$$

which the calculation in coordinates shows is an isomorphism.

In particular the torsor $P \to \mathscr{X}_j$ restricts to a torsor $P_p \to \mathscr{W}_{p,a_i}$.

Therefore we must have $e_{\bar\eta,p} | a_i$. Let f_p denote the ratio $a_i/e_{\bar\eta,p}$. We can then describe the integer f_p as follows. The pullback of $\mathscr{W}_{p,a_i}$ to the closed point of W_p is a nilpotent thickening of the classifying stack $B\mu_{a_i}$, so we have an inclusion

$$B\mu_{a_i} \hookrightarrow \mathscr{W}_{p,a_i}.$$

Pulling back P_p along this inclusion we obtain a G-torsor over $B\mu_{a_i}$ which corresponds to a conjugacy class of homomorphisms $\rho_p : \mu_{a_i} \to G$. The order of the kernel of this homomorphism depends only on the conjugacy class and by 6.8 is equal to f_p.

Now observe that the two composite inclusions for $p = 1, 2$

$$B\mu_{a_i} \hookrightarrow \mathscr{W}_{p,a_i} \hookrightarrow \mathscr{X}_j \times_X \operatorname{Spec}(\mathscr{O}_{X,\bar{\eta}})$$

differ by the automorphism of $B\mu_{a_i}$ given by $\zeta \mapsto \zeta^{-1}$. It follows that $f_1 = f_2$, and therefore also $e_{\bar{\eta},1} = e_{\bar{\eta},2}$. □

Remark 7.3. — In what follows we write simply $e_{\bar{\eta}}$ for $e_{\bar{\eta},1} = e_{\bar{\eta},2}$.

Definition 7.4. — *We say that* $(j : M_S \hookrightarrow N_S, P)$ *is* stable *if for each connected component* Z_i *of* Z *the integer* a_i *is equal to the least common multiple of the integers* $e_{\bar{\eta}}$ *as* η *varies over the generic points of* Z_i.

Remark 7.5. — Note that for each geometric point $\bar{\eta} \to Z$ mapping to a generic point of some connected component Z_i, the integer $e_{\bar{\eta}}$ divides $|G|$ by Remark 4.2. It follows that if $(j : M_S \hookrightarrow N_S, P)$ is stable then the integer a_i is also a divisor of $|G|$.

Example 7.6. — Suppose X/k is a nodal curve. In this case the subscheme $Z \subset X$ is the set of nodes with the reduced structure, and $\mathscr{X}_j \to X$ is a twisted curve in the sense of [2, 2.1]. A torsor $P \to \mathscr{X}_j$ is stable if and only if the map

$$\mathscr{X}_j \to BG$$

corresponding to P is representable.

7.3. For a general base scheme S, a twisted G-torsor $(j : M_S \hookrightarrow N_S, P)$ over S is *stable* if for every geometric point $\bar{s}$ the pullback of this twisted G-torsor to $\bar{s}$ is stable.

7.4. Let ${}_G\mathscr{K}_{X/S}$ denote the stack over S which to any S-scheme T associates the groupoid of stable twisted G-torsors $(j : M_S|_T \hookrightarrow N_T, P \to \mathscr{X}_j)$ over T. The main result, whose proof occupies the following sections 8 through 12, is then theorem 1.2.

Example 7.7. — The stacks ${}_G\mathscr{K}_{X/S}$ are connected with the Abramovich-Vistoli theory of twisted stable maps to BG as follows.

Fix a genus $g \geqslant 2$ and a finite flat tame group scheme G/S. Let $\overline{\mathscr{M}}_g$ denote the moduli stack of genus g stable curves over S. The universal stable curve $\mathscr{Y} \to \overline{\mathscr{M}}_g$ extends naturally to a morphism of log stacks

$$(\mathscr{Y}, M_{\mathscr{Y}}) \to (\overline{\mathscr{M}}_g, M_{\overline{\mathscr{M}}_g}),$$

which is special and essentially semistable [22, §3]. Let ${}_G\mathscr{K} \to \overline{\mathscr{M}}_g$ denote the stack which to any morphism $S \to \overline{\mathscr{M}}_g$, corresponding to a stable curve C/S, associates the groupoid ${}_G\mathscr{K}_{C/S}(S)$, where the extension of $C \to S$ to a morphism of log schemes

$$(C, M_C) \to (S, M_S)$$

is obtained by pulling back the log structures $M_{\mathscr{Y}}$ and $M_{\overline{\mathscr{M}}_g}$. So an object of ${}_G\mathscr{K}(S)$ consists of a stable curve C/S, a simple extension $j : M_S \hookrightarrow N_S$, and a G-torsor $P \to \mathscr{C}_j$ such that (using example 7.6) the corresponding map

$$\mathscr{C}_j \to BG$$

is representable. By [2, A.5], the forgetful functor associating to

$$(C/S, j, P \to \mathscr{C}_j)$$

the twisted stable map $\mathscr{C}_j \to BG$ defines an equivalence of stacks between ${}_G\mathscr{K}$ and the stack $\mathscr{K}_g(BG)$ of twisted stable maps to BG (notation as in [2, 4.2]).

Example 7.8. — Continuing with the example 2.5, let $G = \mu_n$. In this special case we can describe the stack ${}_{\mu_n}\mathscr{K}_{X/k}$ as follows. First of all for any k-scheme S and simple extension $j : M_k|_S \hookrightarrow N_S$, the sheaf $\overline{N}_S$ is canonically isomorphic to $\mathbb{N}$ (because $\mathbb{N}$ has no automorphisms), and the map j defines an inclusion

$$\mathbb{N}_S \hookrightarrow \overline{N}_S$$

of constant sheaves on S. The quotient is therefore a locally constant sheaf of finite order on S, which defines a locally constant function d_S on S. It follows that we have a decomposition

$${}_{\mu_n}\mathscr{K}_{X/k} \simeq \coprod_{d|n} {}_{\mu_n}\mathscr{K}^{(d)}_{X/k},$$

where ${}_{\mu_n}\mathscr{K}^{(d)}_{X/k} \subset {}_{\mu_n}\mathscr{K}_{X/k}$ is the substack classifying (j, P) for which the cokernel of $\overline{M}_k|_S \hookrightarrow \overline{N}_S$ is isomorphic to $\mathbb{Z}/(d)$.

To describe the stack ${}_{\mu_n}\mathscr{K}^{(d)}_{X/k}$ we will use the relationship between line bundles and log structures (see for example [16, complement 1] and in a very general setting [7] for more details). Recall that if T is a scheme then there is an equivalence of categories between the groupoid of pairs $(M_T, \beta : \mathbb{N}^r \to \overline{M}_T)$, where M_T is a fine log structure on T and β is a map which étale locally on T lifts to a chart for M_T, and the groupoid of collections of data

$$\{(\beta_i : L_i \to \mathscr{O}_T)\}_{i=1}^r, \tag{7.2}$$

consisting of r line bundles L_i on T and maps $\beta_i : L_i \to \mathscr{O}_T$ of invertible sheaves. This equivalence is obtained by associating to $(M_T, \beta : \mathbb{N}^r \to \overline{M}_T)$ the following:

(1) L_i is the line bundle corresponding to the $\mathscr{O}_T^*$-torsor of liftings of β applied to the i-th standard generator of $\mathbb{N}^r$.

(2) The map $\beta_i : L_i \to \mathscr{O}_T$ is induced by the map $\alpha : M_T \to \mathscr{O}_T$.

The groupoid of collections (7.2) is also equivalent to the groupoid of morphisms

$$T \to [\mathbb{A}^r/\mathbb{G}_m^r],$$

where the action of $\mathbb{G}_m^r$ on $\mathbb{A}^r$ is the standard action.

Fix an isomorphism $k^* \oplus \mathbb{N} \simeq M_k$. The stack classifying simple extensions $j : M_k \hookrightarrow N$ for which the cokernel $\overline{N}/\overline{M}_k$ is isomorphic to $\mathbb{Z}/(d)$ is, using the above discussion, isomorphic to the stack

$$\mathscr{S}_d := [\mathrm{Spec}(k[T]/T^d)/\mu_d],$$

where μ_d acts by multiplication by on T. This stack can also be viewed as the fiber product of the diagram

$$\begin{tikzcd} & {[\mathbb{A}^1/\mathbb{G}_m]} \arrow[d, "m_d"] \\ \mathrm{Spec}(k) \arrow[r] & {[\mathbb{A}^1/\mathbb{G}_m]}, \end{tikzcd} \tag{7.3}$$

where the map m_d is induced by the maps

$$\mathbb{A}^1 \to \mathbb{A}^1, \quad f \mapsto f^d,$$
$$\mathbb{G}_m \to \mathbb{G}_m, \quad u \mapsto u^d,$$

and the map $\mathrm{Spec}(k) \to [\mathbb{A}^1/\mathbb{G}_m]$ is induced by the zero section $\mathrm{Spec}(k) \hookrightarrow \mathbb{A}^1$.

For any geometric point $\bar{x} \to X$ mapping to Z, the stalk $\overline{M}_{X,\bar{x}}$ is isomorphic to $\mathbb{N}^2$ and the two generators are canonically in bijection with the branches of X at $\bar{x}$. Namely if $f \in M_{X,\bar{x}}$ is a lifting of one of the standard generators of $\mathbb{N}^2$, then the image of f defines an ideal whose closed subscheme is one of

$$Y_s \times_X \mathrm{Spec}(\mathscr{O}_{X,\bar{x}}) \quad (s = 1, 2).$$

Since our components are ordered we get a global map $\mathbb{N}^2 \to \overline{M}_X|_Z$ over Z such that the first (resp. second) standard generator of $\mathbb{N}^2$ corresponds to Y_1 (resp. Y_2). This map extends to a map $b : \mathbb{N}^2 \to \overline{M}_X$ over X. At a geometric point $\bar{x} \to Y_s \backslash Z$, where the stalk $\overline{M}_{X,\bar{x}} \simeq \mathbb{N}$ the map sends e_s to 1 and the other generator to 0. Let $\mathscr{L}_s$ be the line bundle corresponding to the $\mathscr{O}_X^*$-torsor of liftings of $b(e_s)$ to M_X. The morphism $M_X \to \mathscr{O}_X$ defines a morphism of line bundles $\beta_s : L_s \to \mathscr{O}_X$. Taking the d-th roots of these pairs $(\beta_s : L_s \to \mathscr{O}_X)$ we get a stack

$$\mathscr{X} \to X.$$

This stack is the fiber product of the diagram

$$\begin{tikzcd} & {[\mathbb{A}^2/\mathbb{G}_m^2]} \arrow[d, "m_d"] \\ X \arrow[r, "(\beta_s : L_s \to \mathscr{O}_X)"] & {[\mathbb{A}^2/\mathbb{G}_m^2],} \end{tikzcd} \tag{7.4}$$

where again m_d is obtained by raising to the d-th power. It can also be viewed as the stack classifying simple extensions $M_X \hookrightarrow N_X$ such that for any geometric point $\bar{x} \to X$ there exists an isomorphism $\overline{N}_{X,\bar{x}} \simeq \overline{M}_{X,\bar{x}}$ which identifies the map

$$\overline{M}_{X,\bar{x}} \to \overline{N}_{X,\bar{x}}$$

with the map

$$\cdot d : \overline{M}_{X,\bar{x}} \to \overline{M}_{X,\bar{x}}.$$

Notice that the map $M_k|_X \to M_X$ induces a trivialization

$$\epsilon : L_1 \otimes L_2 \simeq \mathscr{O}_X$$

and that

$$\beta_1 \otimes \beta_2 : L_1 \otimes L_2 \to \mathscr{O}_X$$

is the zero map. Given d-th roots $(M_s, \rho_s, \sigma_s)_{s=1,2}$ of $(L_s \to \mathscr{O}_X)_{s=1,2}$ over an X-scheme T, defining a morphism $T \to \mathscr{X}$, the tensor product $M_1 \otimes M_2$ with the maps

$$\rho_1 \otimes \rho_2 : M_1 \otimes M_2 \to \mathscr{O}_T, \quad \sigma_1 \otimes \sigma_2 : (M_1 \otimes M_2)^d \simeq L_1|_T \otimes L_2|_T \simeq \mathscr{O}_T$$

defines a d-th root of $0 : \mathscr{O}_T \to \mathscr{O}_T$ and therefore a map

$$T \to \mathscr{S}_d.$$

In this way we get a morphism of stacks

$$\mathscr{X} \to \mathscr{S}_d.$$

This morphism fits into a commutative diagram

$$\begin{tikzcd}
\mathscr{X} \arrow[rr] \arrow[dr] \arrow[dd] & & {[\mathbb{A}^2/\mathbb{G}_m^2]} \arrow[dr, "m_d"] \arrow[dd, "\delta" near start] & \\
& X \arrow[rr] \arrow[dd] & & {[\mathbb{A}^2/\mathbb{G}_m^2]} \arrow[dd, "\delta"] \\
\mathscr{S}_d \arrow[rr] \arrow[dr] & & {[\mathbb{A}^1/\mathbb{G}_m]} \arrow[dr, "m_d"] & \\
& \mathrm{Spec}(k) \arrow[rr] & & {[\mathbb{A}^1/\mathbb{G}_m].}
\end{tikzcd}$$

Here the bottom (resp. top) face of the cube is as in (7.3) (resp. (7.4)) and the map δ is the map induced by the multiplication maps

$$\mathbb{A}^2 \to \mathbb{A}^1, \quad \mathbb{G}_m^2 \to \mathbb{G}_m.$$

For any morphism $T \to \mathscr{S}_d$ corresponding to a simple extension $j : M_k|_T \hookrightarrow N_T$ the fiber product of the diagram

$$\begin{tikzcd}
& \mathscr{X} \arrow[d] \\
T \arrow[r] & \mathscr{S}_d
\end{tikzcd}$$

is the stack $\mathscr{X}_j$ over X_T associated to the simple extension j as in paragraph 2.3.

Let $\mathscr{P}ic_{\mathscr{X}/\mathscr{S}_d}$ denote the stack over $\mathscr{S}_d$ which to any $T \to \mathscr{S}_d$ is the groupoid of line bundles on $\mathscr{X} \times_{\mathscr{S}_d} T$. Raising to the n-th power defines a morphism of stacks

$$\cdot n : \mathscr{P}ic_{\mathscr{X}/\mathscr{S}_d} \to \mathscr{P}ic_{\mathscr{X}/\mathscr{S}_d}.$$

Let $\mathscr{P}ic_{\mathscr{X}/\mathscr{S}_d}[n]$ denote the fiber product of the diagram of stacks

$$\begin{tikzcd}
& \mathscr{S}_d \arrow[d, "{[\mathscr{O}_{\mathscr{X}}]}"] \\
\mathscr{P}ic_{\mathscr{X}/\mathscr{S}_d} \arrow[r, "\cdot n"] & \mathscr{P}ic_{\mathscr{X}/\mathscr{S}_d}.
\end{tikzcd}$$

As in [2, 2.7], an application of Artin's criteria for verifying that a stack is algebraic shows that the stacks $\mathscr{P}ic_{\mathscr{X}/\mathscr{S}_d}$ and $\mathscr{P}ic_{\mathscr{X}/\mathscr{S}_d}[n]$ are algebraic stacks. The stack $\mathscr{P}ic_{\mathscr{X}/\mathscr{S}_d}$ is the stack which to any k-scheme T associates the groupoid of triples $(j, \mathscr{M}, \iota)$, where $j : M_k|_T \hookrightarrow N_T$ is a simple extension defining a morphism $T \to \mathscr{S}_d$, $\mathscr{M}$ is a line bundle on $\mathscr{X}_j$, and ι is a trivialization of $\mathscr{M}^{\otimes n}$. Such a pair $(\mathscr{M}, \iota)$ is in turn equivalent to a μ_n-torsor, so we can identify $\mathscr{P}ic_{\mathscr{X}/\mathscr{S}_d}[n]$ with the stack whose fiber over a k-scheme T is the groupoid of pairs (j, P), where $j : M_k|_T \hookrightarrow N_T$ is a simple extension defining a morphism $T \to \mathscr{S}_d$, and $P \to \mathscr{X}_j$ is a μ_n-torsor.

From this we see that ${}_{\mu_n}\mathscr{K}^{(d)}_{X/k}$ is a substack of $\mathscr{P}ic_{\mathscr{X}/\mathscr{S}_d}[n]$. Let

$$\underline{\mathrm{Hom}}(\mu_d, \mu_n)$$

denote the constant sheaf on the big étale site of $\mathrm{Spec}(k)$ of homomorphisms $\mu_d \to \mu_n$, and let

$$\underline{\mathrm{Hom}}'(\mu_d, \mu_n) \subset \underline{\mathrm{Hom}}(\mu_d, \mu_n)$$

denote the constant subsheaf of injective homomorphisms $\mu_d \to \mu_n$.

There is a morphism of stacks

$$\mathscr{P}ic_{\mathscr{X}/\mathscr{S}_d}[n] \to \underline{\mathrm{Hom}}(\mu_d, \mu_n) \tag{7.5}$$

defined as follows. Let $\mathscr{V}_1 \subset \mathscr{X}$ denote the substack

$$(\mathscr{X} \times_X Y_1)_{\mathrm{red}}.$$

In local coordinates, if we have a smooth morphism

$$X \to \mathrm{Spec}(k[x_1, x_2]/(x_1 x_2))$$

with Y_s the zero locus of x_s, then $\mathscr{X}$ is the stack

$$[\mathrm{Spec}_X(\mathscr{O}_X[w_1, w_2]/(w_1^d = x_1, w_2^d = x_2))/\mu_d^2],$$

where $(\zeta_1, \zeta_2) \in \mu_d^2$ acts by

$$w_s \mapsto \zeta_s w_s.$$

The stack $\mathscr{V}_1$ is then the stack

$$[\mathrm{Spec}_{Y_1}(\mathscr{O}_{Y_1}[w_2]/w_2^d = x_2)/\mu_d^2].$$

This implies that the map

$$\mathscr{V}_1 \to \mathscr{S}_d$$

factors through

$$B\mu_d \xhookrightarrow{T=0} \mathscr{S}_d$$

and also that the square

$$\begin{tikzcd} \mathscr{V}_1 \arrow[r] \arrow[d] & \mathscr{Y}_1 \arrow[d] \\ B\mu_d \arrow[r] & \mathrm{Spec}(k) \end{tikzcd}$$

is cartesian. Here $\mathscr{Y}_1$ denotes the stack over Y_1 of d-th roots of the divisor Z, and the map $\mathscr{V}_1 \to \mathscr{Y}_1$ is induced by restriction to $\mathscr{V}_1$ the universal d-th root of L_2 over $\mathscr{X}$.

In particular, for any morphism $T \to \mathscr{S}_d$ the pullback of $B\mu_d \hookrightarrow \mathscr{S}_d$ is a closed subscheme $T' \hookrightarrow T$ defined by a nilpotent ideal, and over T' we have a closed immersion

$$\mathscr{Y}_{1,T'} \hookrightarrow \mathscr{X} \times_{\mathscr{S}_d} T'.$$

Given a μ_n-torsor P on $\mathscr{X} \times_{\mathscr{S}_d} T$, we can restrict P to the μ_d-gerbe

$$(\mathscr{Y}_1 \times_{Y_1} Z)_{\mathrm{red}} \times T' \hookrightarrow Z_{T'}.$$

Since μ_n is abelian this restricted torsor defines a homomorphism $\mu_d \to \mu_n$ over $Z_{T'}$ which since Z is connected gives a homomorphism $\mu_d \to \mu_n$ over T'. Furthermore, since $T' \hookrightarrow T$ is defined by a nilpotent ideal, this homomorphism lifts uniquely to a homomorphism $\mu_d \to \mu_n$ over T. In this way we obtain the morphism of stacks (7.5). By the definition of stability, the stack ${}_{\mu_n}\mathscr{K}^{(d)}_{X/k}$ is equal to the preimage of $\underline{\mathrm{Hom}}'(\mu_d, \mu_n)$ under (7.5).

8. Reformulation of the stability condition.

Let $(X, M_X)/(S, M_S)$ be as in paragraph 7.1, and let $(j : M_S \hookrightarrow N_S, P)$ be a twisted G-torsor.

8.1. Assume first that $S = \mathrm{Spec}(k)$ is the spectrum of a separably closed field k. Let $\widetilde{X} \to X$ be the normalization of X, and let $\widetilde{Z} \subset \widetilde{X}$ be the preimage (with the reduced structure) of the singular locus $Z \subset X$. Note that $\widetilde{X}$ is smooth and that $\widetilde{Z}$ is a divisor with local normal crossings on $\widetilde{X}$.

Let $\{Z_1, \dots, Z_r\}$ be the connected components of Z, and let $\widetilde{Z}_i \subset \widetilde{Z}$ be the preimage of Z_i so that

$$\widetilde{Z} = \widetilde{Z}_1 \coprod \widetilde{Z}_2 \cdots \coprod \widetilde{Z}_r.$$

Also let $\mathbb{N}^r \simeq \overline{M}_S$ be the isomorphism obtained by ordering the components Z_i, and let $\{a_i\}_{i=1}^r$ be the integers such that the map $\overline{M}_S \to \overline{N}_S$ is isomorphic to the map $\cdot\underline{a} : \mathbb{N}^r \to \mathbb{N}^r$.

Let $\widetilde{X}^\circ \subset \widetilde{X}$ denote the complement of $\widetilde{Z}$, and let $\widetilde{P}^\circ \to \widetilde{X}^\circ$ denote the pullback of P.

By 4.5 there exists a unique minimal set of integers $b_1, \dots, b_r$ such that $\widetilde{P}^\circ$ extends to (notation as in Proposition 4.3)

$$\mathscr{X}_{\widetilde{Z},\underline{b}} := \mathscr{X}_{\widetilde{Z}_1,b_1} \times_{\widetilde{X}} \mathscr{X}_{\widetilde{Z}_2,b_2} \times_{\widetilde{X}} \cdots \times_{\widetilde{X}} \mathscr{X}_{\widetilde{Z}_r,b_r}.$$

Lemma 8.1. — *The twisted G-torsor (j, P) is stable if and only if $a_i = b_i$ for all i.*

Proof. — For each geometric point $\bar{x} \to \widetilde{X}$ mapping to a generic point of $\widetilde{Z}_i$, the ring $\mathscr{O}_{\widetilde{X},\bar{x}}$ is a discrete valuation ring, and P restricts to a G-torsor over the generic point of $\mathrm{Spec}(\mathscr{O}_{\widetilde{X},\bar{x}})$. By Proposition 4.1 we therefore have an integer $e_{\bar{x}}$ associated to $\bar{x}$. By Proposition 4.5 the integer b_i is equal to the least common multiple of the integers $e_{\bar{x}}$ as $\bar{x}$ ranges over geometric generic points of $\widetilde{Z}_i$.

Now the integer $e_{\bar{x}}$ is also equal to the integer associated to the composite $\bar{x} \to \widetilde{X} \to X$ as in paragraph 7.2. This implies the lemma. □

8.2. Next consider the case when $S = \mathrm{Spec}(V)$ is the spectrum of a discrete valuation ring and M_S is induced by a chart $\mathbb{N}^r \to V$ sending all nonzero elements to 0. Let η (resp. s) denote the generic (resp. closed) point of S.

Lemma 8.2. — *The restriction to the closed fiber (j_s, P_s) is stable if and only if the restriction to the generic fiber (j_η, P_η) is stable.*

Proof. — As above, let $\widetilde{X}$ be the normalization of X, and let $Z_1, \dots, Z_r$ be the connected components of the singular locus $Z \subset X$, ordered according to the isomorphism $\mathbb{N}^r \simeq \overline{M}_S$ coming from our chosen chart. Let $\widetilde{Z}_i \subset \widetilde{X}$ be the preimage of Z_i with the reduced structure. For a sequence of integers $\underline{d} = (d_1, \dots, d_r)$ set

$$\mathscr{X}_{\widetilde{Z},\underline{d}} := \mathscr{X}_{\widetilde{Z}_1,d_1} \times_{\widetilde{X}} \mathscr{X}_{\widetilde{Z}_2,d_2} \times_{\widetilde{X}} \cdots \times_{\widetilde{X}} \mathscr{X}_{\widetilde{Z}_r,d_r}.$$

Let $\underline{b}$ (resp. $\underline{b}'$) be the minimal sequence of integers for which $\widetilde{P}^\circ_\eta \to \widetilde{X}^\circ_\eta$ (resp. $\widetilde{P}^\circ_s \to \widetilde{X}^\circ_s$) extends to $\mathscr{X}_{\widetilde{Z},\underline{b},\eta}$ (resp. $\mathscr{X}_{\widetilde{Z},\underline{b}',s}$). It then suffices to show that $\underline{b} = \underline{b}'$.

We claim that the torsor $\widetilde{P}^\circ \to \mathscr{X}^\circ_{\widetilde{Z},\underline{b}}$ extends to $\mathscr{X}_{\widetilde{Z},\underline{b}}$. Indeed let $\mathscr{U} \subset \mathscr{X}_{\widetilde{Z},\underline{b}}$ denote $\mathscr{X}_{\widetilde{Z},\underline{b},\eta} \cup \mathscr{X}^\circ_{\widetilde{Z},\underline{b}}$. Then by construction the torsor $\widetilde{P}^\circ$ extends to $\mathscr{U}$. Since $\mathscr{X}_{\widetilde{Z},\underline{b}}$ is regular and the complement of $\mathscr{U}$ has codimension $\geqslant 2$, it then follows from Corollary 3.3 that $\widetilde{P}^\circ$ extends uniquely to $\mathscr{X}_{\widetilde{Z},\underline{b}}$. In particular, we have $b'_i | b_i$ for all i.

On the other hand, let f_i denote b_i/b'_i, and let

$$\mathscr{Z}_i \subset \mathscr{X}_{\widetilde{Z},\underline{b}}$$

be the smooth locus of the preimage of $\widetilde{Z}_i$ (with the reduced structure). The inertia stack $I_{\mathscr{Z}_i}$ is locally isomorphic to μ_{b_i}. The f_i define a well-defined closed subgroup $\mathscr{K}_i \subset I_{\mathscr{Z}_i}$ of the inertia stack of $\mathscr{Z}_i$, and the restriction of

P defines a conjugacy class of homomorphisms $\mathscr{K}_i \to G$. The restriction to the closed fiber of this conjugacy class of homomorphisms is the zero class, and since $\mathscr{K}_i$ and G are tame group schemes it follows that it is zero over all of $\mathscr{Z}_i$ (see Lemma 6.7). From this we conclude that $f_i = 1$ for all i. $\square$

8.3. A similar argument also shows the following. Again assume that $S = \mathrm{Spec}(V)$ is the spectrum of a discrete valuation ring, but we do not assume that there exists a chart $\mathbb{N}^r \to V$ sending all nonzero elements to zero (so in particular the closed fiber could have more singular components than the generic fiber).

Lemma 8.3. — *If the restriction to the closed fiber (j_s, P_s) is stable then the generic fiber (j_η, P_η) is also stable.*

Proof. — After possibly replacing V by a finite extension, we may assume that there exists a chart $\mathbb{N}^r \to V$ for M_S sending all nonzero elements to the maximal ideal. Let $Z_i \subset X$ be the connected component of the singular locus corresponding to the i-th standard generator of $\mathbb{N}^r$.

After possibly reordering the generators of $\mathbb{N}^r$ we may assume that $Z_1, \dots, Z_s$ have nonempty generic fibers (for some $s \leqslant r$), and that $Z_{s+1}, \dots, Z_r$ are contained in the closed fiber.

For $1 \leqslant i \leqslant s$, let b_i (resp. b'_i) denote the least common multiple of the integers $e_{\bar{x}}$ as $\bar{x}$ varies over geometric generic points of the generic (resp. closed) fiber of Z_i. Then the proof of Lemma 8.2 shows that $b_i = b'_i$. If the closed fiber (j_s, P_s) is stable then $a_i = b'_i$ for all $i \leqslant s$ and therefore $b_i = a_i$ also. This implies that the generic fiber of (j, P) is stable. $\square$

Corollary 8.4. — *For arbitrary S there exists an open subset $U \subset S$ such that a geometric point $\bar{s} \to S$ has image in U if and only if the fiber of (j, P) over $\bar{s}$ is stable.*

Proof. — Combining Lemmas 8.2 and 8.3 one sees that the set of points $s \in S$ for which the fiber of (j, P) is stable is a constructible set stable under generization, and hence open. $\square$

9. Algebraicity of ${}_G\mathscr{K}_{X/S}$

Let $(X, M_X)/(S, M_S)$ be as in paragraph 7.1.

9.1. Let $\mathscr{S} \to S$ be the stack classifying simple extensions $j : M_S \hookrightarrow N$ as in [2, A.6]. As in loc. cit. the stack $\mathscr{S}$ is an algebraic stack. The universal simple extension over $\mathscr{S}$ then defines an algebraic stack $\mathscr{X}_{\mathscr{S}} \to X_{\mathscr{S}}$ over $\mathscr{S}$. Now consider the relative hom-stack

$$\mathscr{H} := \underline{\mathrm{Hom}}_{\mathscr{S}}(\mathscr{X}_{\mathscr{S}}, BG)$$

over $\mathscr{S}$. By [2, C.2] the stack $\mathscr{H}$ is algebraic and of finite type over $\mathscr{S}$ and the diagonal

$$\mathscr{H} \to \mathscr{H} \times_{\mathscr{S}} \mathscr{H}$$

is quasi-compact and separated.

The stack ${}_G\mathscr{K}_{X/S}$ is a substack of $\mathscr{H}$, which is an open substack by Corollary 8.4. In particular ${}_G\mathscr{K}_{X/S}$ is algebraic.

9.2. By the same argument as in the proof of [22, 1.11], for any integer N there is a quasi-compact open substack $\mathscr{S}^{\leqslant N} \subset \mathscr{S}$ classifying simple extensions $M_S \hookrightarrow N_S$ such that for every geometric point $\bar{s} \to S$ the order of

$$\mathrm{Coker}(\overline{M}^{\mathrm{gp}}_{S,\bar{s}} \to \overline{N}^{\mathrm{gp}}_{S,\bar{s}})$$

is less than or equal to N. Let M be an integer such that for any geometric point $\bar{s} \to S$ the number of connected components of the singular locus of $X_{\bar{s}}$ is less than or equal to M. Then it follows from Remark 7.5 that ${}_G\mathscr{K}_{X/S}$ has image in $\mathscr{S}^{\leqslant N}$, where $N = (\#G)^M$.

In particular the stack ${}_G\mathscr{K}_{X/S}$ is quasi-compact and hence of finite type over S.

Lemma 9.1. — *The diagonal of ${}_G\mathscr{K}_{X/S}$ is quasi-finite.*

Proof. — It suffices to show that if k is an algebraically closed field, M_k is a log structure on k, and $(j : M_k \hookrightarrow N_k, P \to \mathscr{X}_j)$ is an object of ${}_G\mathscr{K}_{X/S}$, then the automorphism group of this object is finite. Since the automorphism group of N_k over M_k is finite, it further suffices to show that the automorphism group of the G-torsor P on $\mathscr{X}_j$ is finite. Write G as an extension

$$1 \to \Delta \to G \to H \to 1,$$

where Δ is diagonalizable and H is tame and étale. Let $\overline{P}$ denote the H-torsor P/Δ. The sheaf on $\mathscr{X}_j$ of automorphisms of $\overline{P}$, which commute with the H-action, is a locally constant sheaf $\mathscr{H}$ locally isomorphic to H. Furthermore, since Δ is abelian the automorphism group of P over $\overline{P}$ is isomorphic to Δ. To prove the lemma, it therefore suffices to prove the following two statements:

(a) Let $\mathscr{X}/k$ be a proper algebraic stack, and let $\mathscr{H}$ be a locally constant sheaf of finite groups on $\mathscr{X}$. Then $\Gamma(\mathscr{X}, \mathscr{H})$ is a finite set.

(b) Let $\mathscr{Y}/k$ be a proper reduced algebraic stack. Then the set of maps $\mathscr{Y} \to \Delta$ is finite.

For (a), let $p : Y \to \mathscr{X}$ be a proper surjection with Y a scheme (such a morphism exists by Chow's lemma [21, 1.1]). Then the map

$$\Gamma(\mathscr{X}, \mathscr{H}) \to \Gamma(Y, p^*\mathscr{H})$$

is injective. It therefore suffices to prove (a) in the case when $\mathscr{X}$ is a scheme, in which case it follows from [6, XIV, 1.1].

For (b) note that any morphism $\mathscr{Y} \to \Delta$ factors uniquely through $\mathrm{Spec}(\Gamma(\mathscr{Y}, \mathscr{O}_{\mathscr{Y}}))$. Since $\mathscr{Y}$ is assumed reduced, the scheme $\mathrm{Spec}(\Gamma(\mathscr{Y}, \mathscr{O}_{\mathscr{Y}}))$ is a finite disjoint union of copies of $\mathrm{Spec}(k)$. Statement (b) therefore follows from the fact that $\Delta(k)$ is finite. □

Proposition 9.2. — *The stack ${}_G\mathscr{K}_{X/S}$ is an Artin stack of finite type over S with finite diagonal.*

Proof. — It remains to verify the valuative criterion for properness for the diagonal of ${}_G\mathscr{K}_{X/S}$.

So assume that $S = \mathrm{Spec}(V)$ is the spectrum of a discrete valuation ring V and let $\pi \in V$ be a uniformizer. We may also assume that we have a chart $\mathbb{N}^r \to M_S$ defining an isomorphism $\mathbb{N}^r \simeq \overline{M}_{S,\bar{s}}$ (where $s \in S$ is the closed point). Observe then that for any simple extension $M_S \hookrightarrow N_S$ we obtain an isomorphism $\overline{N}_{S,\bar{s}} \simeq \mathbb{N}^r$ and a sequence of integers $a_1, \dots, a_r$ such that the map $\overline{M}_{S,\bar{s}} \to \overline{N}_{S,\bar{s}}$ is identified with the map

$$\cdot\, \underline{a} : \mathbb{N}^r \to \mathbb{N}^r. \qquad (9.1)$$

Now assume given two simple extensions $j : M_S \hookrightarrow N_S$ and $j' : M_S \hookrightarrow N'_S$ which become isomorphic over the generic fiber, and let $P \to \mathscr{X}_j$ and $P' \to \mathscr{X}_{j'}$ be two stable G-torsors whose restriction to $\mathscr{X}_{j,\eta} \simeq \mathscr{X}_{j',\eta}$ are equal. We claim that, after possible making a base change $V \to V'$, there is an isomorphism of log structures $\tau : N_S \to N'_S$ such that the triangle

$$\begin{tikzcd} & M_S \arrow[dl, "j"'] \arrow[dr, "j'"] & \\ N_S \arrow[rr, "\tau"] & & N'_S \end{tikzcd}$$

commutes. Let $\underline{a}$ (resp. $\underline{a}'$) be the sequence of integers obtained from j (resp. j') as above. To prove that j is isomorphic to j' it suffices to show that $a_i = a'_i$ for all i.

This can be verified on the closed fiber X_s of X, where it follows from the definition of stable (note that the restrictions $P^\circ_{\bar{s}} \to X^\circ_{\bar{s}}$ and $P'^\circ_{\bar{s}} \to X^\circ_{\bar{s}}$

are isomorphic, as the two torsors $P^\circ \to X^\circ$ and $P'^\circ \to X^\circ$ are isomorphic by Corollary 4.3).

Fix now a simple extension $j : M_S \hookrightarrow N_S$ with corresponding sequence of integers $\underline{a}$.

Let $f_1, \dots, f_r \in V$ be the images of the standard generators of $\mathbb{N}^r$ under the composite $\mathbb{N}^r \to M_S \to V$. Then the stack classifying simple extensions $M_S \hookrightarrow N$, such that the induced map $\overline{M}_{S,\bar{s}} \to \overline{N}_{\bar{s}}$ is given by (9.1), is isomorphic to

$$[\mathrm{Spec}(V[x_1, \dots, x_r]/(x_i^{a_i} - f_i)_{i=1}^r/\mu_{a_1} \times \cdots \times \mu_{a_r}].$$

In particular this is a proper stack over V.

It follows that to verify the properness of the diagonal of ${}_G\mathscr{K}_{X/S}$ it suffices to consider the following situation.

Assume given a simple extension $j : M_S \hookrightarrow N_S$ and let $\mathscr{X}_j \to X$ be the resulting stack. Let $P_i \to \mathscr{X}_j$ $(i = 1, 2)$ be two G-torsors, and assume given an isomorphism $\iota_\eta : P_{1,\eta} \to P_{2,\eta}$ over the generic point $\eta \in S$. We then must show that ι_η extends uniquely to an isomorphism $\iota : P_1 \to P_2$.

By the uniqueness we may work étale locally on S, so we may assume that G is equal to an extension

$$1 \to \Delta \to G \to H \to 1,$$

where Δ is diagonalizable and H is étale. Let $\overline{P}_i$ denote P_i/Δ, so $\overline{P}_i$ is an H-torsor and the projection $P_i \to \overline{P}_i$ realizes P_i as a Δ-torsor over $\overline{P}_i$.

By a standard limit argument it suffices to construct the extension of ι_η over the base change of $\mathscr{X}_j$ to the strict henselization of X at a geometric point in the closed fiber. As in Proposition 2.2 we can write such a base change as

$$[\mathrm{Spec}(R)/D]$$

where R is either equal to the strict henselization at the point $z_1 = \cdots = z_r = 0$ of

$$V[z_1, \dots, z_r]/(z_1 \cdots z_s = \pi^a) \tag{9.2}$$

for some $a \neq 0$, or the strict henselization of

$$V[z_1, \dots, z_r]/(z_1 \cdots z_s = 0). \tag{9.3}$$

Now any extension of ι_η over $\mathrm{Spec}(R)$ is automatically compatible with the D-action by flatness, so to construct the extension of ι_η it suffices to construct an extension over $\mathrm{Spec}(R)$.

First observe that the isomorphism $\overline{P}_{1,\eta} \to \overline{P}_{2,\eta}$ defined by ι_η extends uniquely to an isomorphism $\overline{P}_1 \to \overline{P}_2$ over $\mathrm{Spec}(R)$. In the case when R

is of the form (9.2) this is immediate as R is normal, which implies that $\overline{P}_i$ is equal to the normalization of $\mathrm{Spec}(R)$ in $\overline{P}_{i,\eta}$ ($i = 1, 2$). In the case when R is of the form (9.3) note that the category of torsors on $\mathrm{Spec}(R)$ is equivalent to the category of collections (P_i, σ_{ij}), where P_i is a torsor over $\mathrm{Spec}(R/(z_i))$ ($i = 1, \dots, s$), and $\sigma_{ij} : P_i|_{R/(z_i,z_j)} \to P_j|_{R/(z_i,z_j)}$ is an isomorphism of torsors over $\mathrm{Spec}(R/(z_i, z_j))$ satisfying a suitable cocycle condition over the triple overlaps (as in the proof of 5.1). Since the rings $R/(z_i)$ and $R/(z_i, z_j)$ are normal, the isomorphisms over the generic fiber extend uniquely.

Now observe that $\overline{P}_i$ is isomorphic to a finite disjoint union of copies of $\mathrm{Spec}(R)$, so to lift this isomorphism $\overline{P}_1 \to \overline{P}_2$ to the P_i we are reduced to the case when G is diagonalizable. In this case G is isomorphic to a finite product of schemes of the form μ_n (for various n), which further reduces the proof to the case μ_n. In this case P_i corresponds to a pair (L_i, γ_i), where L_i is a line bundle and $\gamma_i : L_i^n \to R$ is an isomorphism. Let M denote $L_1 \otimes L_2^{-1}$. Then M comes equipped with a trivialization $f \in M^{\otimes n}$ as well as a trivialization $g_\eta \in M_\eta$ such that g_η^n is equal to the restriction of f. The claim is then that g_η extends to a trivialization of M.

In the case when R is of the form (9.2) this is immediate as R is normal, and in the case (9.3) this follows from Proposition 5.1. This completes the proof that the diagonal of ${}_G\mathscr{K}_{X/S}$ is finite. □

10. Stabilization

10.1. Let $(X, M_X)/(S, M_S)$ be as in paragraph 7.1 and let $(j : M_S \hookrightarrow N_S, P)$ be a twisted G-torsor. We explain in this section how to construct a subextension

$$\begin{tikzcd} M_S \arrow[r, "j'"] \arrow[rr, bend left, "j"] & N'_S \arrow[r] & N_S \end{tikzcd} \tag{10.1}$$

and a natural morphism $\pi : \mathscr{X}_j \to \mathscr{X}_{j'}$ such that P descends to a G-torsor $P' \to \mathscr{X}_{j'}$ such that (j', P') is stable.

10.2. Let us begin by considering the case when $S = \mathrm{Spec}(k)$ is the spectrum of a separably closed field k.

Fix an ordering $Z_1, \dots, Z_r$ of the connected components of the singular locus of X, and recall that this defines isomorphisms

$$\mathbb{N}^r \to \overline{M}_S, \quad \mathbb{N}^r \to \overline{N}_S$$

such that we have a commutative diagram

$$\begin{tikzcd} \mathbb{N}^r \arrow[r, "\underline{a}"] \arrow[d, "\simeq"] & \mathbb{N}^r \arrow[d, "\simeq"] \\ \overline{M}_S \arrow[r] & \overline{N}_S, \end{tikzcd}$$

for some sequence of positive integers $\underline{a} = (a_1, \dots, a_r)$.

On the other hand, for any geometric generic point $\bar{\eta} \to Z_i$ we obtain by Lemma 7.2 an integer $e_{\bar{\eta}}$, and we let b_i denote the least common multiple of the integers $e_{\bar{\eta}}$ as $\bar{\eta}$ ranges over generic points of Z_i. By the definition of the integers $e_{\bar{\eta}}$ we have $b_i | a_i$. Let f_i denote a_i / b_i.

Define $N'_S \subset N_S$ to be the fiber product of the diagram

$$\begin{tikzcd} & & N_S \arrow[d] \\ \mathbb{N}^r \arrow[r, "\underline{f}"] & \mathbb{N}^r \arrow[r, "\simeq"] & \overline{N}_S, \end{tikzcd}$$

so we have a diagram of simple extensions

$$M_S \xrightarrow{j'} N'_S \longrightarrow N_S.$$

10.3. There is a natural morphism of stacks

$$\pi' : \mathscr{X}_j \to \mathscr{X}_{j'}$$

defined as follows.

Let $g : Y \to X$ be a morphism of schemes, and suppose given an object

$$\begin{tikzcd} g^*M_X \arrow[r, "a"] & N_Y \\ g^*f^*M_S \arrow[u] \arrow[r, "j"] & g^*f^*N_S \arrow[u, "b"'] \end{tikzcd} \tag{10.2}$$

of $\mathscr{X}_j(Y)$. We claim that there exists a unique sub-log structure $N'_Y \subset N_Y$ containing the images of g^*M_X and $g^*f^*N'_S$ such that the resulting diagram

$$\begin{tikzcd} g^*M_X \arrow[r, "a'"] & N'_Y \\ g^*f^*M_S \arrow[u] \arrow[r, "j'"] & g^*f^*N'_S \arrow[u, "b'"'] \end{tikzcd} \tag{10.3}$$

is an object of $\mathscr{X}_{j'}(Y)$. The functor π' is defined by sending (10.2) to (10.3).

For this note first that N'_Y is determined by the subsheaf of monoids $\overline{N}'_Y \subset \overline{N}_Y$, and this subsheaf is in turn determined by its stalks. Now for a geometric point $\bar{y}$ we can identify the diagram of stalks

$$\begin{tikzcd} \overline{M}_{X,g(\bar{y})} \arrow[rr] & & \overline{N}_{Y,\bar{y}} \\ \overline{M}_{S,fg(\bar{y})} \arrow[u] \arrow[r] & \overline{N}'_{S,fg(\bar{y})} \arrow[r] & \overline{N}_{S,fg(\bar{y})} \arrow[u] \end{tikzcd}$$

with the diagram

$$\begin{tikzcd} \mathbb{N}^{r+s} \arrow[rr, "\chi_{\underline{a}}"] & & \mathbb{N}^{r+s} \\ \mathbb{N}^r \arrow[u, "h"] \arrow[r, "\underline{b}"] \arrow[rr, bend right, "\underline{a}"'] & \mathbb{N}^r \arrow[r] & \mathbb{N}^r, \arrow[u, "h"'] \end{tikzcd}$$

where h and $\chi_{\underline{a}}$ are defined as in paragraph 2.4, and $\underline{a}$ and $\underline{b}$ are collections of integers as in paragraph 10.2 with $b_i|a_i$ for all i. Let $F_{\bar{y}} \subset \overline{N}_{Y,\bar{y}} \simeq \mathbb{N}^{r+s}$ be the submonoid generated by the elements f_ie_i $(i < r)$ and f_re_i $(i \geqslant r)$. Then we want a sub-log structure $N'_Y \subset N_Y$ such that $\overline{N}_{Y',\bar{y}} = F_{\bar{y}}$ for all $\bar{y} \to Y$.

From this the uniqueness follows, and also it suffices to construct N'_Y étale locally on X. Furthermore, it suffices to construct N'_Y in the universal case, over $\mathscr{X}_j$.

We can therefore assume that $\mathscr{X}_j$ is given by a fiber product as in Proposition 2.2. Let

$$\tau : \mathbb{N}^r \to \mathbb{N}^r$$

be the map sending e_i to $(a/b_i)e_i$, and let

$$\xi : \mathbb{N}^{r+s} \to \mathbb{N}^{r+s}$$

be the map sending e_i to $(a_i/b_i)e_i$ for $i < r$ and $(a_r/b_r)e_i$ for $i \geqslant r$. Also let Δ' denote the kernel of the map

$$\prod_{i=0}^{s} \mu_{b_r} \to \mu_{b_r}, \quad (\zeta_0, \dots, \zeta_s) \mapsto \zeta_0 \cdots \zeta_s.$$

We then have a commutative diagram

(10.4)

$$\begin{tikzcd} & {[\mathrm{Spec}(\mathbb{Z}[\mathbb{N}^{r+s}])/\Delta]} \arrow[r, "\xi"] \arrow[d] & {[\mathrm{Spec}(\mathbb{Z}[\mathbb{N}^{r+s}])/\Delta']} \arrow[d] \\ X \arrow[r] & \mathrm{Spec}(\mathbb{Z}[\mathbb{N}^r] \otimes_{\underline{a},\mathbb{Z}[\mathbb{N}^r],h} \mathbb{Z}[\mathbb{N}^{r+s}]) \arrow[r, "\tau\otimes 1"] & \mathrm{Spec}(\mathbb{Z}[\mathbb{N}^r] \otimes_{\underline{b},\mathbb{Z}[\mathbb{N}^r],h} \mathbb{Z}[\mathbb{N}^{r+s}]) \end{tikzcd}$$

which defines a morphism

$$\pi' : \mathscr{X}_j \to \mathscr{X}_{j'}$$

by base changing to X. By construction there is a natural map $\pi'^* N_{\mathscr{X}_{j'}} \to N_{\mathscr{X}_j}$ of tautological log structures, which has the desired properties on stalks.

Remark 10.1. — Note that the preceding construction of the map

$$\pi : \mathscr{X}_j \to \mathscr{X}_{j'}$$

works over any base scheme S and not just in the case when S is the spectrum of a separably closed field.

Remark 10.2. — The local description of the map $\pi' : \mathscr{X}_j \to \mathscr{X}_{j'}$ in terms of the diagram (10.4) also shows that the relative coarse moduli space of $\mathscr{X}_j \to \mathscr{X}_{j'}$ is equal to $\mathscr{X}_{j'}$.

10.4. We claim that P descends to a torsor $P' \to \mathscr{X}_{j'}$. By construction this torsor P' will be stable.

For a geometric point $\bar{x} \to \mathscr{X}_j$ let $K_{\bar{x}}$ denote the kernel of the map of group schemes

$$\underline{\mathrm{Aut}}_{\mathscr{X}_j}(\bar{x}) \to \underline{\mathrm{Aut}}_{\mathscr{X}_{j'}}(\pi \circ \bar{x}). \tag{10.5}$$

By Proposition 6.5 it suffices to show that for every geometric point $\bar{x} \to \mathscr{X}_j$ the action of $K_{\bar{x}}$ on $P_{\bar{x}}$ is trivial.

10.5. If $\bar{x}$ has image in the smooth locus of X then this is trivial since the projections $\mathscr{X}_j \to X$ and $\mathscr{X}_{j'} \to X$ are isomorphisms over the smooth locus of X. Therefore assume that $\bar{x}$ has image in Z_i. Write the strict henselization $\mathscr{O}_{X,\bar{x}}$ as

$$\mathscr{O}_{X,\bar{x}} \simeq (k[x_1, \dots, x_n]/(x_1 \cdots x_s))^{\mathrm{sh}},$$

where the right side denotes the strict henselization at the origin. For an integer c let $H_c \subset \mu_c^s$ denote the kernel of the sum map

$$\mu_c^s \to \mu_c, \quad (\zeta_1, \dots, \zeta_s) \mapsto \prod_{i=1}^{s} \zeta_i,$$

and let $\mathscr{Q}_c$ denote the stack quotient

$$\mathscr{Q}_c := [\mathrm{Spec}((k[z_1, \dots, z_n]/(z_1 \cdots z_s))^{\mathrm{sh}})/H_c],$$

where $\underline{\zeta} \in H_c$ acts by

$$z_i \mapsto \zeta_i z_i \quad (i \leqslant s), \quad z_i \mapsto z_i \quad (i > s).$$

Then by 2.2 we have

$$\mathscr{X}_j \times_X \operatorname{Spec}(\mathscr{O}_{X,\bar{x}}) \simeq \mathscr{Q}_{a_i}, \quad \mathscr{X}_{j'} \times_X \operatorname{Spec}(\mathscr{O}_{X,\bar{x}}) \simeq \mathscr{Q}_{b_i}, \tag{10.6}$$

where the map

$$\mathscr{Q}_{a_i} \to \operatorname{Spec}(\mathscr{O}_{X,\bar{x}}) \simeq \operatorname{Spec}((k[x_1, \dots, x_n]/(x_1 \cdots x_s))^{\mathrm{sh}})$$

is induced by the map

$$(k[x_1, \dots, x_n]/(x_1 \cdots x_s))^{\mathrm{sh}} \to (k[z_1, \dots, z_n]/(z_1 \cdots z_s))^{\mathrm{sh}}$$

sending x_i to $z_i^{a_i}$ for $i \leqslant s$ and x_i to z_i for $i > s$. The map

$$\mathscr{Q}_{b_i} \to \operatorname{Spec}(\mathscr{O}_{X,\bar{x}}) \simeq \operatorname{Spec}((k[x_1, \dots, x_n]/(x_1 \cdots x_s))^{\mathrm{sh}})$$

is defined similarly with a_i replaced by b_i. The isomorphisms (10.6) identify the map (10.5) with the natural map

$$H_{a_i} \to H_{b_i}.$$

Note also that we have a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & H_{a_i} \arrow[r] \arrow[d] & \mu_{a_i}^s \arrow[r] \arrow[d] & \mu_{a_i} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H_{b_i} \arrow[r] & \mu_{b_i}^s \arrow[r] & \mu_{b_i} \arrow[r] & 0, \end{tikzcd}$$

which implies that the natural isomorphism

$$\mu_{f_i}^s \simeq \operatorname{Ker}(\mu_{a_i}^s \to \mu_{b_i}^s)$$

induces an isomorphism

$$H_{f_i} \simeq \operatorname{Ker}(H_{a_i} \to H_{b_i}).$$

This isomorphism identifies $K_{\bar{x}}$ with H_{f_i}, so we need to show that H_{f_i} acts trivially on $P_{\bar{x}}$.

10.6. For any $1 \leqslant p < q \leqslant s$ there is an inclusion

$$\sigma_{pq} : \mu_{f_i} \hookrightarrow H_{f_i}$$

induced by the inclusion $\mu_{f_i} \hookrightarrow \mu_{f_i}^s$ sending ζ to the element $(\alpha_k) \in \mu_{f_i}^s$ with $\alpha_p = \zeta$, $\alpha_q = \zeta^{-1}$, and $\alpha_k = 1$ for $k \neq p, q$.

Note that the images of the σ_{pq} generate H_{f_i}. In fact the map

$$\sigma_{12} \times \sigma_{23} \cdots \times \sigma_{(s-1)s} : \mu_{f_i} \times \cdots \times \mu_{f_i} \to H_{f_i}$$

is an isomorphism.

As before, write

$$\mathscr{Q}_{a_i} = [\operatorname{Spec}((k[z_1, \dots, z_n]/(z_1 \cdots z_s))^{\mathrm{sh}})/H_{a_i}],$$

and let

$$P_{a_i} \to \mathrm{Spec}((k[z_1, \dots, z_n]/(z_1 \cdots z_s))^{\mathrm{sh}})$$

be the restriction of P, which comes equipped with an action of H_{a_i} lifting the action on $\mathrm{Spec}((k[z_1, \dots, z_n]/(z_1 \cdots z_s))^{\mathrm{sh}})$.

For $1 \leqslant p < q \leqslant s$ let $T_{pq} \subset \mathrm{Spec}((k[z_1, \dots, z_n]/(z_1 \cdots z_s))^{\mathrm{sh}})$ be the closed subscheme defined by $z_p = z_q = 0$. Note that the action of the subgroup scheme $\sigma_{pq}(\mu_{a_i}) \subset H_{a_i}$ on T_{pq} is trivial, therefore the restriction $P|_{T_{pq}}$ induces a conjugacy class of homomorphisms $\sigma_{pq}(\mu_{a_i}) \to G$. This conjugacy class of homomorphisms has a well-defined kernel $\Sigma_{pq} \subset \sigma_{pq}(\mu_{a_i})$, which is a diagonalizable group scheme being the kernel of a morphism from a diagonalizable group to a tame group scheme [1, 2.5]. The restriction of Σ_{pq} to the generic point of T_{pq} contains $\sigma_{pq}(\mu_{f_i})$ by construction, and therefore Σ_{pq} contains $\sigma_{pq}(\mu_{f_i})$ over all of T_{pq}. This implies that $\sigma_{pq}(\mu_{f_i})$ acts trivially on $P_{\bar{x}}$, and therefore H_{f_i} also acts trivially on $P_{\bar{x}}$.

This completes the proof that P descends to $\mathscr{X}_{j'}$.

10.7. For general S, note first that the subextension $N'_S \hookrightarrow N_S$ is determined by the induced map of constructible sheaves $\overline{N}'_S \hookrightarrow \overline{N}_S$, and hence the preceding case of a separably closed field implies that the factorization (10.1) and the descended stable torsor $P' \to \mathscr{X}_{j'}$ are unique if they exist. It therefore suffices to construct them étale locally on S.

10.8. Let $\bar{s} \to S$ be a geometric point and fix a commutative diagram

$$\begin{tikzcd} \mathbb{N}^r \arrow[r, "\underline{a}"] \arrow[d, "\simeq"] & \mathbb{N}^r \arrow[d, "\simeq"] \\ \overline{M}_{S,\bar{s}} \arrow[r] & \overline{N}_{S,\bar{s}} \end{tikzcd}$$

for some integers $a_1, \dots, a_r$. The case of a separably closed field then defines a sequence of integers $b_1, \dots, b_r$ such that $b_i | a_i$ for all i. As above we write $f_i := a_i/b_i$. Let $N'_{S,\bar{s}} \subset N_{S,\bar{s}}$ be the fiber product of the diagram

$$\begin{tikzcd} & & N_{S,\bar{s}} \arrow[d] \\ \mathbb{N}^r \arrow[r, "\times \underline{f}"] & \mathbb{N}^r \arrow[r, "\simeq"] & \overline{N}_{S,\bar{s}}. \end{tikzcd}$$

We then have a diagram of simple extensions

$$M_{S,\bar{s}} \hookrightarrow N'_{S,\bar{s}} \hookrightarrow N_{S,\bar{s}}.$$

By a standard limit argument we can after replacing S by an étale neighborhood extend this to a diagram of simple extensions of log structures

$$M_S \overset{j'}{\hookrightarrow} N'_S \hookrightarrow N_S.$$

Consider the resulting map $\mathscr{X}_j \to \mathscr{X}_{j'}$ (see 10.1). By Propositions 6.5 and 6.6 there exists an open substack $\mathscr{U} \subset \mathscr{X}_j$ containing the fiber over $\bar{s}$ such that $P|_{\mathscr{U}}$ descends to the image of $\mathscr{U}$ in $\mathscr{X}_{j'}$. Let $\mathscr{Z} \subset \mathscr{X}_j$ be the complement of $\mathscr{U}$. Replacing S by the complement of the image of $\mathscr{Z}$ (which is closed since $\mathscr{X}_j$ is proper over S) we obtain an étale neighborhood of $\bar{s}$ such that P descends to a torsor $P' \to \mathscr{X}_{j'}$. Now by construction the fiber over $\bar{s}$ of P' is stable, so by Corollary 8.4 there exists an étale neighborhood of $\bar{s}$ over which P' is stable.

Remark 10.3. — With notation as in paragraph 10.1, suppose $s \in S$ is a point such that the restriction of (j, P) to (X_s, M_{X_s}) is stable. Then the maps

$$\pi : \mathscr{X}_j \to \mathscr{X}_{j'}, \quad N'_S \to N_S$$

restrict to isomorphisms over s. This follows from the preceding construction.

11. Verification of the valuative criterion for properness

We conclude the proof of Theorem 1.2 (i) by verifying the valuative criterion for properness for ${}_G\mathscr{K}_{X/S}$.

11.1. Let $S = \mathrm{Spec}(V)$ be the spectrum of a discrete valuation ring V with field of fractions K. We write η (resp. s) for the generic (resp. closed) point of S, and denote the log structure M_S by M_V. Fix a uniformizer $\pi \in V$.

Assume given an object

$$(j_\eta : M_{V,\eta} \hookrightarrow N_\eta, P_\eta \to \mathscr{X}_{j_\eta}) \in {}_G\mathscr{K}_{X/S}(\mathrm{Spec}(K)).$$

We must show that this object extends to an object of ${}_G\mathscr{K}_{X/S}(\mathrm{Spec}(V))$, after possibly replacing V by a finite extension.

11.2. After possibly making an extension of V, we may assume that $\overline{M}_{V,s}$ is constant (note that $\overline{M}_{V,s}$ is a locally constant sheaf on s_{et}). Choose an isomorphism

$$\mathbb{N}^I \oplus \mathbb{N}^J \simeq \overline{M}_V \tag{11.1}$$

such that the induced map

$$\mathbb{N}^I \to \overline{M}_{V,\bar\eta}$$

is an isomorphism. We then have a canonical bijection between J (resp. I) and the connected components of the singular locus which do not meet (resp. meet) the generic fiber. Fix a lifting

$$\beta : \mathbb{N}^I \oplus \mathbb{N}^J \to M_V$$

of the isomorphism (11.1). Then there exists a unique set of natural numbers $(e_i)_{i\in I}$ and isomorphism $\bar\gamma$ such that the following diagram commutes

$$\begin{tikzcd} \mathbb{N}^I \arrow[r, "(e_i)"] \arrow[d] & \mathbb{N}^I \arrow[d, "\bar\gamma"] \\ \overline{M}_{V,\bar\eta} \arrow[r, "j_\eta"] & \overline{N}_{\bar\eta}. \end{tikzcd}$$

After possibly replacing V by a finite extension we may assume that this diagram can be lifted to a diagram

$$\begin{tikzcd} \mathbb{N}^I \arrow[r, "(e_i)"] \arrow[d, "\beta"] & \mathbb{N}^I \arrow[d, "\gamma"] \\ M_{V,\eta} \arrow[r, "j_\eta"] & N_\eta. \end{tikzcd}$$

Note that the map γ is a chart. Let $f_i \in V$ be the image of the i-th standard generator of the composite

$$\mathbb{N}^I \xrightarrow{\beta} \Gamma(\mathrm{Spec}(V), M_V) \longrightarrow V,$$

and let $g_i \in K$ be the image of the i-th standard generator under the composite

$$\mathbb{N}^I \xrightarrow{\gamma} \Gamma(\mathrm{Spec}(K), N_\eta) \longrightarrow K.$$

Then by construction we have $g_i^{e_i} \in V$ which implies that in fact $g_i \in V$. Let $N^{(0)}$ be the log structure on $\mathrm{Spec}(V)$ associated to the chart

$$\mathbb{N}^I \oplus \mathbb{N}^J \to V$$

which restricts to β on $\mathbb{N}^J$ and sends the i-th standard generator of $\mathbb{N}^I$ to g_i. We then have a simple extension

$$j^{(0)} : M_V \hookrightarrow N^{(0)}$$

over $\mathrm{Spec}(V)$ restricting to j_η over the generic point.

We therefore get a stack

$$\mathscr{X}_{j^{(0)}} \to \operatorname{Spec}(V)$$

with a G-torsor

$$P_\eta \to \mathscr{X}_{j^{(0)},\eta}.$$

11.3. Let $\mathscr{Z} \subset \mathscr{X}_{j^{(0)}}$ be the union of the connected components of the singular locus of $\mathscr{X}_{j^{(0)}}$ which are contained in the closed fiber, and let $Z \subset X$ denote the image in X. Then by Proposition 5.1 and Corollary 4.3 P_η extends to a G-torsor

$$P' \to \mathscr{X}'_{j^{(0)}} := \mathscr{X}_{j^{(0)}} \backslash \mathscr{Z},$$

after possibly making a base change $V \to V'$.

11.4. For each $j \in J$ write the image of the j-th standard generator under the map

$$\mathbb{N}^J \longrightarrow \Gamma(V, M_V) \longrightarrow V$$

as $u_j \pi^{k_j}$ where $u_j \in V^*$ and $k_j \in \mathbb{N}$. Let N be the log structure on $\operatorname{Spec}(V)$ induced by the map

$$\mathbb{N}^I \oplus \mathbb{N}^J \to V$$

whose restriction to $\mathbb{N}^I$ is the map defined by $N^{(0)}$ and which sends the j-th standard generator of $\mathbb{N}^J$ to π. The map

$$\mathbb{N}^I \oplus \mathbb{N}^J \to V^* \oplus \mathbb{N}^I \oplus \mathbb{N}^J, \quad e_i \mapsto e_i \quad (i \in I), \quad e_j \mapsto (u_j, 0, k_j e_j) \quad (j \in J)$$

induces a morphism of log structures $N^{(0)} \to N$. Let $j : M_V \to N$ be the composite

$$M_V \to N^{(0)} \to N.$$

We then obtain a morphism of stacks

$$\mathscr{X}_j \to \mathscr{X}_{j^{(0)}}$$

which is an isomorphism away from $\mathscr{Z}$. In particular, the torsor P' can be viewed as a torsor over

$$\mathscr{X}'_j := \mathscr{X}_j \times_X (X \backslash Z).$$

Now observe that for any geometric point $\bar{z} \to Z$ there exists an étale neighborhood U of $\bar{z}$ in X such that

$$\mathscr{X}_j \times_X U \simeq [\operatorname{Spec}(V[x_1, \dots, x_n]/(x_1 \cdots x_s - \pi))/\Delta]$$

for some $s \leqslant n$ (here Δ and the action is as in paragraph 3.3). It follows that P' extends to a torsor $P \to \mathscr{X}_j$ (by Corollary 3.3).

11.5. We therefore obtain a twisted G-torsor $(j : M_V \hookrightarrow N, P)$ over X whose restriction to X_η is our given object of ${}_G\mathscr{K}_{X/S}(\mathrm{Spec}(K))$. Applying the stabilization construction of section 10, which does not change the generic fiber of (j, P) (see remark 10.3), we then obtain the desired object of ${}_G\mathscr{K}_{X/S}(\mathrm{Spec}(V))$.

This completes the proof of Theorem 1.2 (i). □

12. The stack ${}_G\mathscr{K}_{X/S}$ is tame.

12.1. Let $(X, M_X)/(S, M_S)$ be as in paragraph 7.1, and let $(j : M_S \to N_S, P)$ be an object of ${}_G\mathscr{K}_{X/S}(S)$. It suffices to show that the geometric fibers of the automorphism group scheme of (j, P) is tame. So we assume that $S = \mathrm{Spec}(k)$ is the spectrum of an algebraically closed field, and let $\mathscr{A}$ denote the automorphism group scheme of (j, P). The group scheme $\mathscr{B}$ of automorphisms of N_S which restrict to the identity on M_S is isomorphic to a product of group schemes of the form μ_a, and there is a natural homomorphism

$$\mathscr{A} \to \mathscr{B}.$$

Let $K \subset \mathscr{A}$ be the kernel of this homomorphism. Since $\mathscr{B}$ is diagonalizable, to prove that $\mathscr{A}$ is tame it suffices to show that K is tame.

12.2. The group scheme K is the group scheme of automorphisms of the G-torsor P over $\mathscr{X}_j$. Write G as an extension

$$1 \to \Delta \to G \to H \to 1,$$

where Δ is diagonalizable and H is tame and étale. Let $\overline{P}$ denote the quotient P/Δ, which is an H-torsor over $\mathscr{X}_j$. The automorphism group scheme of $\overline{P}$ over $\mathscr{X}_j$ is a twisted form $\mathscr{H}$ of H. Let $\mathscr{C}$ denote the tame étale group scheme over $\mathrm{Spec}(k)$ corresponding to the group $\Gamma(\mathscr{X}_j, \mathscr{H})$, which is finite by the same argument as in the proof of Lemma 9.1. By the proper base change theorem, $\mathscr{C}$ represents the functor which to any k-scheme T associates the group of automorphisms of the base change of $\overline{P}$ to T. We therefore have a natural homomorphism

$$K \to \mathscr{C},$$

whose kernel K' is the subgroup scheme of automorphisms of P over $\overline{P}$. To prove that K is tame it then suffices to show that K' is tame, and for this in turn it suffices to show that the group scheme of automorphisms of P over $\overline{P}$ is tame.

12.3. Let Z/k denote $\mathrm{Spec}(\Gamma(\overline{P}, \mathscr{O}_{\overline{P}}))$. Since $\overline{P}$ is reduced the scheme Z is equal to a finite disjoint union of copies of $\mathrm{Spec}(k)$. Since Δ is abelian the group scheme of automorphisms of P over $\overline{P}$ is canonically isomorphic to the scheme

$$\underline{\mathrm{Hom}}_k(Z, \Delta),$$

which is isomorphic to a finite product of copies of Δ. In particular, this is a tame group scheme.

This completes the proof of Theorem 1.2 (ii). □

BIBLIOGRAPHY

[1] D. Abramovich, M. Olsson & A. Vistoli, "Tame stacks in positive characteristic", *Annales de l'Institut Fourier* **57** (2008), p. 1057-1091.

[2] ———, "Twisted stable maps to tame Artin stacks", to appear in J. Alg. Geometry.

[3] D. Abramovich & A. Vistoli, "Compactifying the space of stable maps", *J. Amer. Math. Soc.* **15** (2002), p. 27-75.

[4] M. Artin, "Algebraic approximation of structures over complete local rings", *Publications Mathématiques de l'IHÉS* **36** (1969), p. 23-58.

[5] ———, "Algebraization of formal moduli: I", in *Global Analysis (Papers in Honor of K. Kodaira)*, Univ. Tokyo Press, Tokyo, 1969, p. 21-71.

[6] M. Artin, A. Grothendieck & J.-L. Verdier, *Théorie des topos et cohomologie étale des schémas*, Lecture Notes in Mathematics, vol. 269, 270, 305, Springer-Verlag, Berlin, 1972.

[7] N. Borne & A. Vistoli, "Parabolic sheaves on logarithmic schemes", preprint, 2010.

[8] C. Cadman, "Using stacks to impose tangency conditions on curves", *American J. of Math.* **129** (2007), p. 405-427.

[9] P. Deligne, "Théorie de Hodge: II", *Inst. Hautes Études Sci. Publ. Math.* **40** (1971), p. 5-57.

[10] ———, "Cohomologie étale", in *Séminaire de Géométrie Algébrique* 4 1/2, Lecture Notes in Math, vol. 569, Springer, 1977.

[11] P. Deligne & D. Mumford, "The irreducibility of the space of curves of given genus", *Inst. Hautes Études Sci. Publ. Math.* **36** (1969), p. 75-109.

[12] J. Dieudonné & A. Grothendieck, *Éléments de géométrie algébrique*, vol. 4, 8, 11, 17, 20, 24, 28, 32, Inst. Hautes Études Sci. Publ. Math., 1961–1967.

[13] R. Friedman, "Global smoothings of varieties with normal crossings", *Ann. of Math.* **118** (1983), p. 75-114.

[14] A. Grothendieck, *Revêtements étales et groupe fondamental*, Lectures Notes in Math, vol. 224, Springer, 1971.

[15] F. Kato, "Log smooth deformation theory", *Tohoku Math. J.* **48** (1996), p. 317-354.

[16] K. Kato, "Logarithmic structures of Fontaine-Illusie", in *Algebraic analysis, geometry, and number theory* ((Baltimore, MD, 1988)), Johns Hopkins Univ. Press, Baltimore, MD, 1989, p. 191-224.

[17] K. Matsuki & M. Olsson, "Kawamata-Viehweg vanishing as Kodaira vanishing for stacks", *Math. Res. Letters* **12** (2005), p. 207-217.

[18] A. Ogus, "Logarithmic geometry and algebraic stacks", book in preparation, 2008.

[19] M. Olsson, "Logarithmic geometry and algebraic stacks", *Ann. Sci. d'ENS* **36** (2003), p. 747-791.

[20] ———, "Universal log structures on semi-stable varieties", *Tohoku Math. J.* **55** (2003), p. 397-438.

[21] ———, "On proper coverings of Artin stacks", *Adv. Math.* **198** (2005), p. 93-106.

[22] ———, "On (log) twisted curves", *Comp. Math.* **143** (2007), p. 476-494.

Manuscrit reçu le 2 octobre 2009,
accepté le 31 août 2011.

Martin OLSSON
University of California
Department of Mathematics
970 Evans Hall #3840
Berkeley, CA 94720-3840
molsson@math.berkeley.edu