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SEMICLASSICAL RESOLVENT ESTIMATES AT TRAPPED SETS

by Kiril DATCHEV & András VASY (*)

Abstract. — We extend our recent results on propagation of semiclassical resolvent estimates through trapped sets when a priori polynomial resolvent bounds hold. Previously we obtained non-trapping estimates in trapping situations when the resolvent was sandwiched between cutoffs χ microlocally supported away from the trapping: $\|\chi R_h(E+i0)\chi\| = \mathcal{O}(h^{-1})$, a microlocal version of a result of Burq and Cardoso-Vodev. We now allow one of the two cutoffs, $\tilde{\chi}$, to be supported at the trapped set, giving $\|\chi R_h(E+i0)\tilde{\chi}\| = \mathcal{O}(\sqrt{a(h)}h^{-1})$ when the a priori bound is $\|\tilde{\chi} R_h(E+i0)\tilde{\chi}\| = \mathcal{O}(a(h)h^{-1})$.

Résumé. — Nous étendons nos résultats récents sur la propagation d'estimations de résolvantes semi-classiques à travers des ensembles captifs sous des bornes a priori de type polynomial. Précédemment, nous obtenions des estimations non-captives dans des situations captives quand la résolvante est contrôlée par au dessus et en dessous par des fonctions cutoff χ dont le support microlocal est situé loin de l'ensemble captif : $\|\chi R_h(E+i0)\chi\| = \mathcal{O}(h^{-1})$ (version microlocale d'un résultat de Burq et Cardoso-Vodev). Nous considérons maintenant le cas où l'une des deux fonctions cutoff, $\tilde{\chi}$, est à support dans l'ensemble captif, obtenant $\|\chi R_h(E+i0)\tilde{\chi}\| = \mathcal{O}(\sqrt{a(h)}h^{-1})$ lorsque la borne a priori est $\|\tilde{\chi} R_h(E+i0)\tilde{\chi}\| = \mathcal{O}(a(h)h^{-1})$.

This short article is an addendum to the previous paper by K. Datchev and A. Vasy.

Let (X,g) be a Riemannian manifold which is asymptotically conic or asymptotically hyperbolic in the sense of [6], let $V \in C_0^\infty(X)$ be real valued, let $P = h^2\Delta_g + V(x)$, where $\Delta_g \geqslant 0$, and fix $E > 0$.

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Theorem 1. — [6, Theorem 1.2] *Suppose that for any* $\chi_0 \in C_0^\infty(X)$ *there exist* $C_0, k, h_0 > 0$ *such that for any* $\varepsilon > 0$, $h \in (0, h_0]$ *we have*

$$\|\chi_0(h^2\Delta_g + V - E - i\varepsilon)^{-1}\chi_0\|_{L^2(X)\to L^2(X)} \leqslant C_0 h^{-k}. \tag{1}$$

Let $K_E \subset T^*X$ *be the set of trapped bicharacteristics at energy* E, *and suppose that* $b \in C_0^\infty(T^*X)$ *is identically* 1 *near* K_E. *Then there exist* $C_1, h_1 > 0$ *such that for any* $\varepsilon > 0$, $h \in (0, h_1]$ *we have the following nontrapping estimate:*

$$\|\langle r\rangle^{-1/2-\delta}(1-\operatorname{Op}(b))(h^2\Delta_g + V - E - i\varepsilon)^{-1}(1-\operatorname{Op}(b))\langle r\rangle^{-1/2-\delta}\|_{L^2(X)\to L^2(X)} \leqslant C_1 h^{-1}. \tag{2}$$

Here by bicharacteristics at energy E we mean integral curves in $p^{-1}(E)$ of the Hamiltonian vector field H_p of the Hamiltonian $p = |\xi|^2 + V(x)$, and the trapped ones are those which remain in a compact set for all time. We use the notation $r = r(z) = d_g(z, z_0)$, where d_g is the distance function on X induced by g and $z_0 \in X$ is fixed but arbitrary.

If $K_E = \varnothing$ then (1) holds with $k = 1$. If $K_E \neq \varnothing$ but the trapping is sufficiently ‘mild’, then (1) holds for some $k > 1$: see [6] for details and examples. The point is that the losses in (1) due to trapping are removed when the resolvent is cutoff away from K_E. Theorem 1 is a more precise and microlocal version of an earlier result of Burq [1] and Cardoso and Vodev [3], but the assumption (1) is not needed in [1, 3]. See [6] for additional background and references for semiclassical resolvent estimates and trapping.

In this paper we prove that an improvement over the a priori estimate (1) holds even when one of the factors of $(1 - \operatorname{Op}(b))$ is removed:

Theorem 2. — *Suppose that there exist* $k > 0$ *and* $a(h) \leqslant h^{-k}$ *such that for any* $\chi_0 \in C_0^\infty(X)$ *there exists* $h_0 > 0$ *such that for any* $\varepsilon > 0$, $h \in (0, h_0]$ *we have*

$$\|\chi_0(h^2\Delta_g + V - E - i\varepsilon)^{-1}\chi_0\|_{L^2(X)\to L^2(X)} \leqslant a(h)/h. \tag{3}$$

Suppose that $b \in C_0^\infty(T^*X)$ *is identically* 1 *near* K_E. *Then there exist* $C_1, h_1 > 0$ *such that for any* $\varepsilon > 0$, $h \in (0, h_1]$,

$$\|\langle r\rangle^{-1/2-\delta}(1-\operatorname{Op}(b))(h^2\Delta_g + V - E - i\varepsilon)^{-1}\langle r\rangle^{-1/2-\delta}\|_{L^2(X)\to L^2(X)} \leqslant C_1\sqrt{a(h)}/h. \tag{4}$$

Note that by taking adjoints, analogous estimates follow if $1 - \operatorname{Op}(b)$ is placed to the other side of $(h^2\Delta_g + V - E - i\varepsilon)^{-1}$.

Such results were proved by Burq and Zworski [2, Theorem A] and Christianson [4, (1.6)] when K_E consists of a single hyperbolic orbit. Theorem 2 implies an optimal semiclassical resolvent estimate for the example operator of [6, §5.3]: it improves [6, (5.5)] to

$$\|\chi_0(P-\lambda)^{-1}\chi_0\| \leqslant C\log(1/h)/h.$$

Further, this improved estimate can be used to extend polynomial resolvent estimates from complex absorbing potentials to analogous estimates for damped wave equations; this is a result of Christianson, Schenk, Wunsch and the second author [5].

Theorems 1 and 2 follow from microlocal propagation estimates in a neighborhood of K_E, or more generally in a neighborhood of a suitable compact invariant subset of a bicharacteristic flow.

To state the general results, suppose X is a manifold, $P \in \Psi^{m,0}(X)$ a self adjoint, order $m > 0$, semiclassical pseudodifferential operator on X, with principal symbol p. For $I \subset \mathbb{R}$ compact and fixed, denote the characteristic set by $\Sigma = p^{-1}(I)$, and suppose that the projection to the base, $\pi\colon \Sigma \to X$, is proper (it is sufficient, for example, to have p classically elliptic). Suppose that $\Gamma \Subset T^*X$ is invariant under the bicharacteristic flow in Σ. Define the *forward, resp. backward flowout* Γ_+, resp. Γ_-, of Γ as the set of points $\rho \in \Sigma$, from which the backward, resp. forward bicharacteristic segments tend to Γ, i.e. for any neighborhood O of Γ there exists $T > 0$ such that $-t \geqslant T$, resp. $t \geqslant T$, implies $\gamma(t) \in O$, where γ is the bicharacteristic with $\gamma(0) = \rho$. Here we think of Γ as the trapped set or as part of the trapped set, hence points in Γ_-, resp. Γ_+ are backward, resp. forward, trapped. Suppose V, W are neighborhoods of Γ with $\overline{V} \subset W$, $\overline{W}$ compact. Suppose also that

$$\text{If } \rho \in W \setminus \Gamma_+, \text{ resp. } \rho \in W \setminus \Gamma_-,$$
$$\text{then the backward, resp. forward bicharacteristic from } \rho \text{ intersects } W\setminus\overline{V}. \tag{5}$$

This means that all bicharacterstics in V which stay in V for all time tend to Γ.

The main result of [6], from which the other results in the paper follow, is the following:

Theorem 3. — [6, Theorem 1.3] *Suppose that $\|u\|_{H_h^{-N}} \leqslant h^{-N}$ for some $N \in \mathbb{N}$ and $(P-\lambda)u = f$, $\operatorname{Re}\lambda \in I$ and $\operatorname{Im}\lambda \geqslant -\mathcal{O}(h^\infty)$. Suppose f is $\mathcal{O}(1)$ on W, $\mathrm{WF}_h(f) \cap \overline{V} = \emptyset$, and u is $\mathcal{O}(h^{-1})$ on $W \cap \Gamma_- \setminus \overline{V}$. Then u is $\mathcal{O}(h^{-1})$ on $W \cap \Gamma_+ \setminus \Gamma$.*

Here we say that u is $\mathcal{O}(a(h))$ at $\rho \in T^*X$ if there exists $B \in \Psi^{0,0}(X)$ elliptic at ρ with $\|Bu\|_{L^2} = \mathcal{O}(a(h))$. We say u is $\mathcal{O}(a(h))$ on a set $E \subset T^*X$ if it is $\mathcal{O}(a(h)))$ at each $\rho \in E$.

Note that there is no conclusion on u at Γ; typically it will be merely $\mathcal{O}(h^{-N})$ there. However, to obtain $\mathcal{O}(h^{-1})$ bounds for u on Γ_+ we only needed to assume $\mathcal{O}(h^{-1})$ bounds for u on Γ_- and nowhere else. Note also that by the propagation of singularities, if u is $\mathcal{O}(h^{-1})$ at one point on any bicharacteristic, then it is such on the whole forward bicharacteristic. If $|\operatorname{Im}\lambda| = \mathcal{O}(h^\infty)$ then the same is true for backward bicharacteristics.

In this paper we show that a (lesser) improvement on the a priori bound holds even when f is not assumed to vanish microlocally near Γ:

Theorem 4. — *Suppose that $\|u\|_{H_h^{-N}} \leqslant h^{-N}$ for some $N \in \mathbb{N}$ and $(P-\lambda)u = f$, $\operatorname{Re}\lambda \in I$ and $\operatorname{Im}\lambda \geqslant -\mathcal{O}(h^\infty)$. Suppose f is $\mathcal{O}(1)$ on W, u is $\mathcal{O}(a(h)h^{-1})$ on W, and u is $\mathcal{O}(h^{-1})$ on $W \cap \Gamma_- \setminus \overline{V}$. Then u is $\mathcal{O}(\sqrt{a(h)}h^{-1})$ on $W \cap \Gamma_+ \setminus \Gamma$.*

In [6] Theorem 1 is deduced from Theorem 3. Theorem 2 follows from Theorem 4 by the same argument.

Proof of Theorem 4. — The argument is a simple modification of the argument of [6, End of Section 4, Proof of Theorem 1.3]; we follow the notation of this proof. Recall first from [6, Lemma 4.1] that if U_- is a neighborhood of $(\Gamma_- \setminus \Gamma) \cap (\overline{W} \setminus V)$ then there is a neighborhood $U \subset V$ of Γ such that if $\alpha \in U \setminus \Gamma_+$ then the backward bicharacteristic from α enters U_-. Thus, if one assumes that u is $\mathcal{O}(h^{-1})$ on Γ_- and f is $\mathcal{O}(1)$ on $\overline{V}$, it follows that that u is $\mathcal{O}(h^{-1})$ on $U \setminus \Gamma_+$, provided U_- is chosen small enough that u is $\mathcal{O}(h^{-1})$ on U_-. Note also that, because $U \subset V$, f is $\mathcal{O}(1)$ on U. We will show that u is $\mathcal{O}(\sqrt{a(h)}h^{-1})$ on $U \cap \Gamma_+ \setminus \Gamma$: the conclusion on the larger set $W \cap \Gamma_+ \setminus \Gamma$ follows by propagation of singularities.

Next, [6, Lemma 4.3] states that if U_1 and U_0 are open sets with $\Gamma \subset U_1 \Subset U_0 \Subset U$ then there exists a nonnegative function $q \in C_0^\infty(U)$ such that

$$q = 1 \text{ near } \Gamma, \qquad H_p q \leqslant 0 \text{ near } \Gamma_+, \qquad H_p q < 0 \text{ on } \Gamma_+^{\overline{U_0}} \setminus U_1.$$

Moreover, we can take q such that both $\sqrt{q}$ and $\sqrt{-H_p q}$ are smooth near Γ_+.

Remark. — The last paragraph in the proof of [6, Lemma 4.3] should be replaced by the following: To make $\sqrt{-H_p\tilde{q}}$ smooth, let $\psi(s) = 0$ for $s \leqslant 0$, $\psi(s) = e^{-1/s}$ for $s > 0$, and assume as we may that $U_\rho \cap \mathcal{S}_\rho$ is a ball with respect to a Euclidean metric (in local coordinates near ρ) of radius $r_\rho > 0$ around ρ. We then choose φ_ρ to behave like $\psi({r'_\rho}^2 - |.|^2)$ with $r'_\rho < r_\rho$ for $|.|$ close to r'_ρ, bounded away from 0 for smaller values of $|.|$, and choose $-\chi'_\rho$ to vanish like ψ at the boundary of its support. That sums of products of such functions have smooth square roots follows from [7, Lemma 24.4.8].

The proof of Theorem 4 proceeds by induction: we show that if u is $\mathcal{O}(h^k)$ on a sufficiently large compact subset of $U \cap \Gamma_+ \setminus \Gamma$, then u is $\mathcal{O}(h^{k+1/2})$ on $\Gamma_+^{\overline{U_0}} \setminus U_1$, provided $\sqrt{a(h)}h^{-1} \leqslant Ch^{k+1/2}$.

Now let U_- be an open neighborhood of $\Gamma_+ \cap \operatorname{supp} q$ which is sufficiently small that $H_p q \leqslant 0$ on U_- and that $\sqrt{-H_p q}$ is smooth on U_-. Let U_+ be an open neighborhood of $\operatorname{supp} q \setminus U_-$ whose closure is disjoint from Γ_+ and from $T^*X \setminus \overline{U}$. Define $\phi_\pm \in C^\infty(U_+ \cup U_-)$ with $\operatorname{supp} \phi_\pm \subset U_\pm$ and with $\phi_+^2 + \phi_-^2 = 1$ near $\operatorname{supp} q$.

Put

$$b \stackrel{\text{def}}{=} \phi_- \sqrt{-H_p q^2}, \qquad e \stackrel{\text{def}}{=} \phi_+^2 H_p q^2.$$

Let $Q, B, E \in \Psi^{-\infty,0}(X)$ have principal symbols q, b, e, and microsupports $\operatorname{supp} q$, $\operatorname{supp} b$, $\operatorname{supp} e$, so that

$$\frac{i}{h}[P, Q^*Q] = -B^*B + E + hF,$$

with $F \in \Psi^{-\infty,0}(X)$ such that $\mathrm{WF}'_h F \subset \operatorname{supp} dq \subset U \setminus \Gamma$. But

$$\begin{aligned}\frac{i}{h}\langle [P, Q^*Q]u, u\rangle &= \frac{2}{h} \operatorname{Im}\langle Q^*Q(P-\lambda)u, u\rangle + \frac{2}{h}\langle Q^*Q \operatorname{Im}\lambda u, u\rangle \\ &\geqslant -2h^{-1}\|Q(P-\lambda)u\|\,\|Qu\| - \mathcal{O}(h^\infty)\|u\|^2 \geqslant -Ch^{-2}a(h) - \mathcal{O}(h^\infty),\end{aligned}$$

where we used $\operatorname{Im}\lambda \geqslant -\mathcal{O}(h^\infty)$ and that on $\operatorname{supp} q$, $(P-\lambda)u$ is $\mathcal{O}(1)$. So

$$\|Bu\|^2 \leqslant \langle Eu, u\rangle + h\langle Fu, u\rangle + Ch^{-2}a(h) + \mathcal{O}(h^\infty).$$

But $|\langle Eu, u\rangle| \leqslant Ch^{-2}$ because $\mathrm{WF}'_h E \cap \Gamma_+ = \emptyset$ gives that u is $\mathcal{O}(h^{-1})$ on $\mathrm{WF}'_h E$ by the first paragraph of the proof. Meanwhile $|\langle Fu, u\rangle| \leqslant C(h^{-2} + h^{2k})$ because all points of $\mathrm{WF}'_h F$ are either in $U \setminus \Gamma_+$, where we know u is $\mathcal{O}(h^{-1})$ from the first paragraph of the proof, or on a single compact subset of $U \cap \Gamma_+ \setminus \Gamma$, where we know that u is $\mathcal{O}(h^k)$ by inductive hypothesis. Since $b = \sqrt{-H_p q^2} > 0$ on $\Gamma_+^{\overline{U_0}} \setminus U_1$, we can use microlocal elliptic regularity to conclude that u is $\mathcal{O}(h^{k+1/2})$ on $\Gamma_+^{\overline{U_0}} \setminus U_1$, as desired. □

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