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ON THE UNIQUENESS OF ELLIPTIC $K3$ SURFACES WITH MAXIMAL SINGULAR FIBRE

by Matthias SCHÜTT & Andreas SCHWEIZER (*)

ABSTRACT. — We explicitly determine the elliptic $K3$ surfaces with section and maximal singular fibre. If the characteristic of the ground field is different from 2, for each of the two possible maximal fibre types, I_{19} and I_{14}^* , the surface is unique. In characteristic 2 the maximal fibre types are I_{18} and I_{13}^* , and there exist two (resp. one) one-parameter families of such surfaces.

RÉSUMÉ. — Nous déterminons des équations explicites pour les surfaces elliptiques de type $K3$ qui ont une section et une fibre singulière maximale. Si la caractéristique du corps sous-jacent est différente de 2, pour chacun des deux types de fibre maximale, I_{19} et I_{14}^* , la surface est unique. En caractéristique 2 les fibres maximales sont de type I_{18} ou I_{13}^* , et il y a deux, respectivement une, familles 1-dimensionales de telles surfaces.

1. Introduction

The possible singular fibres of complex elliptic surfaces have been classified by Kodaira in [9]. Tate proved in [27] that the same classification persists over any perfect field. A singular fibre always consists of a finite number of rational curves arranged in compatibility with an extended Dynkin diagram of ADE type. For instance, a fibre of type I_n ($n > 2$) in Kodaira's notation consists of a cycle of n rational curves, each meeting its two neighbours transversally. A singular fibre of an elliptic surface is called maximal if the number of its components attains the maximum possible within the specified deformation class of elliptic surfaces.

Keywords: elliptic surface, $K3$ surface, maximal singular fibre, wild ramification.

Math. classification: 14J27, 14J28, 11G05.

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The problem of maximal singular fibres is classically solved for rational elliptic surfaces (types I_9, I_4^*, II^*). Shioda in [21] treated the case of elliptic K3 surfaces with section in characteristic 0 where the maximal fibres have type I_{19} and I_{14}^* . Using the Artin invariant, the first author proved in [16] that these are also the maximal fibre types in characteristic $p > 0$ if p is odd. Meanwhile in characteristic 2, types I_{18} and I_{13}^* were shown to be maximal.

Then we considered in [18] the maximal fibres of general elliptic surfaces with section over \mathbb{P}^1 . We proved that in general the maximal fibres are strictly larger in positive characteristic than in characteristic zero. Moreover, we also derived partial uniqueness results.

In this paper, we prove the uniqueness in the K3 case, a problem suggested to one of us by Shioda.

THEOREM 1.1. — *Let p be an odd prime and k an algebraically closed field of characteristic p . For each of the maximal fibre types I_{19} and I_{14}^* , there exists an elliptic K3 surface with section over k , unique up to isomorphism, with a singular fibre of that type.*

For both fibre types we have an obvious candidate at hand: the mod p reduction of the corresponding elliptic K3 surface over \mathbb{Q} (cf. [21], [19]). This approach fails only if $p = 2$; in this case we will determine the families of elliptic K3 surfaces with section that realise the respective maximal fibres I_{18} and I_{13}^* (Propositions 6.2, 7.2).

Although we are primarily interested in positive characteristic, our method at the same time re-proves the maximality and uniqueness in characteristic 0.

We note that the K3 case is special not only in the sense that the results concerning maximal fibres types from characteristic zero hold. Even over \mathbb{C} , the uniqueness breaks down as soon as we consider honestly elliptic surfaces, i.e. with Euler number $e \geq 36$ (cf. [22]).

For characteristic different from 2 and 3 the result on maximality and uniqueness can be reformulated in the following elementary way.

THEOREM 1.2. — *Let k be an algebraically closed field of characteristic different from 2 and 3. Fix $M = 3$ or 4. Let $f, g \in k[T]$ with $\deg(f) = 2M$, $\deg(g) = 3M$. Then $f^3 \neq g^2$ implies $\deg(f^3 - g^2) \geq M + 1$. Moreover, up to affine transformation of T and scaling there exists exactly one pair (f, g) with $\deg(f^3 - g^2) = M + 1$.*

Proof. — If there is no polynomial a with $a^4|f$ and $a^6|g$, then the equation

$$Y^2 = X^3 - 3fX - 2g$$

defines an elliptic $K3$ surface with discriminant $\Delta = -108(f^3 - g^2)$. The fibre at ∞ has type I_n if $M = 4$, and I_n^* if $M = 3$, with $n = 6M - \deg(\Delta)$. Hence the claim follows from Theorem 1.1. On the other hand, if there is a polynomial a with $a^2|f$ and $a^3|g$, then $a^6|(f^3 - g^2)$. In particular, $\deg(f^3 - g^2) \geq 6 > M + 1$. \square

The degree of $f^3 - g^2$ (for general M) is subject to Hall’s conjecture, see [8]. In characteristic zero, this degree is minimal if and only if the associated Belyi function f^3/g^2 is branched above only three points of \mathbb{P}^1 (cf. [25]). Through the j -map, this problem connects to elliptic surfaces, see [22] for characteristic 0 and [18] for positive characteristic. The semi-stable elliptic $K3$ case where the j -map has degree 24 has been solved for characteristic zero in [3].

Indeed, one can try to prove the uniqueness of the $K3$ surfaces in characteristic $p \geq 5$ by spelling out the equation $\deg(f^3 - g^2) = M + 1$ and solving for the coefficients of f and g . In Sections 3 and 4 we will use a different Weierstrass equation that makes the calculations much less involved and that works in characteristic 3 as well. In characteristic 2 a somewhat more structural approach applies, based on results by the second author in [15]. Here we also make use of a classification of wild ramification of singular fibres. Although this might be known to the experts, we could not find a reference for it, so we include it in Section 5 (Proposition 5.1).

Conventions: Throughout the paper, an elliptic surface is assumed to have a section and a singular fibre. The former condition is mild, since we can always consider the jacobian surface. The latter rules out products.

For a configuration of singular fibres $I_{n_1}, \dots, I_{n_r}^*$, we are going to use the shorthand notation $[n_1, \dots, n_r^*]$.

2. The configurations in odd characteristic

We call an elliptic surface S (in)separable if its j -map is (in)separable. Associated to a non-isotrivial elliptic surface S , there is a unique separable elliptic surface $X \rightarrow \mathbb{P}^1$ such that S arises from X by purely inseparable base change of degree p^d . This is the inseparability degree of the j -invariant of S . By construction the conductors of S and X have the same degree. Hence we can apply the following result to X in order to reveal additional information about S .

THEOREM 2.1 (Pesenti-Szpiro [13, special case of Theorem 0.1]). — *Let $S \rightarrow \mathbb{P}^1$ be a non-isotrivial elliptic surface with conductor \mathcal{N} and Euler number $e(S)$. Then*

$$e(S) \leq 6p^d(\deg \mathcal{N} - 2).$$

where p^d is the inseparability degree of the j -invariant of S .

Remark 2.2. — In [16] the Artin invariant [1] was used to determine the maximal singular fibres of elliptic K3 surfaces. We mention that the inequality from Theorem 2.1 allows an approach without the Artin invariant. We briefly sketch the main idea.

First of all, if S has a fibre of type I_n , then necessarily $n \leq 21$ since there are no elliptic surfaces with conductor of degree less than 4. In the non-isotrivial case, this is a consequence of Theorem 2.1 since the Euler number of an elliptic surface equals the degree of the discriminant divisor which is a positive multiple of 12. Alternatively one can argue with the Shioda-Tate formula for the Picard number [20, Corollary 5.3]. Analogously, $m \leq 16$ for fibre type I_m^* (at least in characteristic different from 2).

If there is a fibre of type I_{21} , I_{20} , I_{16}^* or I_{15}^* and the characteristic is different from 2, 3, 5, 7, the surface is separable. Moreover, the degree of the conductor is at most 5, which by the inequality of Pesenti and Szpiro contradicts $e(S) = 24$. In small characteristic some more fine-tuning is required.

The remaining results of this section would follow immediately from the explicit determination of the elliptic surfaces in the next two sections. However, since the calculations of the equations are long, in particular for a fibre of type I_{19} , we have decided to also include the following structural proofs.

LEMMA 2.3. — *Let S be an elliptic K3-surface in characteristic $p \geq 0$ with a fibre of type I_{14}^* . If $p \neq 2, 7$ the configuration of S is $[14^*, 1, 1, 1, 1]$.*

Proof. — Since $p \neq 2$, the j -invariant has a pole of order 14, so S is non-isotrivial. Moreover, S is separable, since $p \neq 2, 7$. The degree of the conductor of S is at most 6. Actually, it has to be exactly 6, since otherwise by the bound of Pesenti-Szpiro the surface would be rational.

So if the fibres outside I_{14}^* are all multiplicative, the configuration must be as stated. To finish the proof, it suffices to show that there is no additive fibre outside I_{14}^* .

Assume on the contrary that there is another additive fibre. Then we can apply a quadratic twist that ramifies exactly at that fibre and at I_{14}^* . The twisted surface has conductor of degree 5 while being separable, so by

Theorem 2.1 it is rational. However, the j -invariant still has a pole of order 14, contradiction. \square

LEMMA 2.4. — *Let S be an elliptic K3-surface in characteristic $p \geq 0$ with a fibre of type I_{19} . If $p \neq 2, 19$ the configuration of S is $[19, 1, 1, 1, 1]$.*

Proof. — The proof is similar to that of Lemma 2.3. The degree of the conductor of S can only be 6 by Theorem 2.1. If there were an additive fibre, we could apply a quadratic twist that ramifies exactly at this additive fibre and I_{19} . The conductor of the resulting surface would have degree 7. By Theorem 2.1, the twisted surface is rational or K3. But this is impossible with a fibre of type I_{19}^* . \square

LEMMA 2.5. — *An elliptic K3 surface in characteristic 19 with a fibre of type I_{19} is necessarily inseparable and its configuration must be $[I_{19}, II, III]$. Moreover, the surface is unique. If we place the fibres of type I_{19} , II and III at $\infty, 0$ and 1, its equation is*

$$Y^2 = X^3 - 3T^7(T - 1)X + 2T(T - 1)^{11}.$$

Proof. — If the surface is separable, then by the same proof as for Lemma 2.4 the configuration must be $[19, 1, 1, 1, 1]$. The j -invariant of this surface gives a map of degree 24 from \mathbb{P}^1 to \mathbb{P}^1 . By the Hurwitz formula we obtain $-1 = -24 + \frac{1}{2} \deg(\mathfrak{D})$ where \mathfrak{D} is the different. The points above 0 have ramification indices that are divisible by 3, so they contribute at least $8(3 - 1) = 16$ to the degree of the different. Similarly the points above 1 contribute at least $12(2 - 1) = 12$. Finally the j -invariant has a pole of order 19 which, since the ramification is wild, contributes at least 19. Summing up we obtain the contradiction

$$46 = \deg(\mathfrak{D}) \geq 16 + 12 + 19 = 47.$$

We have seen that the surface is inseparable. In consequence its configuration can only be $[19, II, III]$. Hence the surface is the Frobenius base change of a (rational) elliptic surface with configuration $[1, II, III^*]$. Up to isomorphism, this surface is unique (in any characteristic $\neq 2, 3$) by [18, Lemma 8.2]. It can be given in Weierstrass equation

$$Y^2 = X^3 - 3T(T - 1)^3 X - 2T(T - 1)^5.$$

Frobenius base change gives the claimed equation after minimalising. \square

Remark 2.6. — By a similar argument one can easily show that in characteristic 7 an elliptic K3 surface with a fibre of type I_{14}^* is inseparable and

must have configuration $[14^*, II, III]$. Placing these fibres at ∞ , 1 and -1 one obtains the equation

$$Y^2 = X^3 - 3(T^2 - 1)^3 X - 2T^7(T^2 - 1),$$

which is exactly the surface from [21, Theorem 3.2].

3. Fibre type I_{14}^* in characteristic $\neq 2$

In this section, we prove Theorem 1.1 for the maximal fibre type I_{14}^* . In characteristic zero, the uniqueness was proven in [25].

Let k be an algebraically closed field of characteristic different from 2. Let S be an elliptic K3 surface over k with a section. By assumption, we can work with an extended Weierstrass form

$$(3.1) \quad S: Y^2 = X^3 + A(T)X^2 + B(T)X + C(T).$$

In this setting, the discriminant is given by

$$\Delta = -27C^2 + 18ABC + A^2B^2 - 4A^3C - 4B^3.$$

Note that we have four normalisations available: Möbius transformation in T (e.g. to fix the images of the singular fibres) as well as rescaling by $(X, Y) \mapsto (\alpha^2 X, \alpha^3 Y)$. Note that over non-algebraically closed fields, this variable change yields the quadratic twist over $k(\sqrt{\alpha})$. In the following, we will use these normalisations in a convenient way.

PROPOSITION 3.1. — *There is a unique elliptic K3-surface over k with a fibre of type I_{14}^* . Its equation can be given as*

$$Y^2 = X^3 + (T^3 + 2T)X^2 - 2(T^2 + 1)X + T.$$

This surface has discriminant $\Delta = 4T^4 + 13T^2 + 32$.

Proof. — Assume that S has a fibre of type I_{14}^* . We locate the image of the special fibre at ∞ . If the fibre type is I_n^* with $n \geq 8$, then we can assume after a translation $X \mapsto X + \alpha(T)$ that

$$A(T) = a_3T^3 + a_2T^2 + a_1T + a_0, \quad B(T) = b_2T^2 + b_1T + b_0, \quad C(T) = c_1T + c_0.$$

Here $a_3 \neq 0$, since otherwise S would be rational. Hence we can scale such that $a_3 = 1$, e.g. via $(X, Y, T) \mapsto (a_3^4 X, a_3^6 Y, a_3 T)$. By construction, Δ has degree at most 10:

$$\Delta = \sum_{l=0}^{10} d_l T^l.$$

We ask for all solutions to the system of equations

$$d_5 = \dots = d_{10} = 0.$$

In the first instance,

$$d_{10} = -4c_1 + b_2^2, \quad d_9 = -4c_0 + 2b_1b_2 + 2b_2^2a_2 - 12a_2c_1$$

give

$$c_1 = b_2^2/4, \quad c_0 = b_1b_2/2 - a_2b_2^2/4.$$

Then we claim that $b_2 \neq 0$. Otherwise, $d_8 = 0$ would imply $b_1 = 0$. Thus $b_0 = 0$, since $d_6 = 0$, so $B \equiv C \equiv 0$ which gives a contradiction.

Since $b_2 \neq 0$, we can rescale $(X, Y, T) \mapsto (\beta^6 X, \beta^9 Y, \beta^2 T)$ for a root of $\beta^8 = (-b_2/2)$ to set $b_2 = -2$ while preserving the normalisation $a_3 = 1$. By a translation in T , we then achieve $b_1 = 0$. This uses up our final normalisation. The vanishing of

$$d_8 = -4(a_1 + b_0 - a_2^2), \quad d_7 = -4(a_0 + a_1a_2 + 2b_0a_2 - 2a_2^3)$$

then gives a_1 and a_0 . It follows that

$$d_5 = -2a_2(b_0 - 2a_2^2), \quad d_6 = (b_0 - 2a_2^2 + 2)(b_0 - 2a_2^2 - 2).$$

The vanishing of d_6 corresponds to two choices of b_0 which give rise to isomorphic elliptic curves. To see this, apply the scaling $(a_2, T, x, y) \mapsto (ia_2, iT, -ix, \zeta y)$ with $\zeta^2 = i = \sqrt{-1}$. Let $a_2 = \mu, b_0 = 2\mu^2 - 2$. We obtain a one dimensional family parametrising elliptic K3 surfaces with a fibre of type at least I_{13}^* :

$$(3.2) \quad Y^2 = X^3 + ((2 - \mu^2)(\mu + T) + \mu T^2 + T^3)X^2 + 2(\mu^2 - 1 - T^2)X - \mu + T.$$

This family has discriminant

$$(3.3) \quad \Delta = 8\mu T^5 + (4 + 8\mu^2)T^4 + (32\mu - 16\mu^3)T^3 + (13 + 24\mu^2 - 16\mu^4)T^2 + (86\mu - 32\mu^3 + 8\mu^5)T + 32 - 35\mu^2 - 28\mu^4 + 8\mu^6.$$

Hence there is a unique specialisation with a fibre of type I_{14}^* at $\mu = 0$. This gives the claimed equation and discriminant for S in any characteristic $p \neq 2$. □

Remark 3.2. — It is immediate from the shape of the coefficients A, B, C that S arises from a rational elliptic surface by quadratic base change and twisting. It follows that this rational elliptic surface is uniquely determined by its configuration of singular fibres. The configuration is $[7,1,1,III]$ if $p \neq 7$, and $[7,II,III]$ if $p = 7$.

Remark 3.3. — The model in Proposition 3.1 relates to the Weierstrass equation in [21] as follows: The first author exhibited in [16, §6] a quadratic twist of the latter model which has good reduction at 3. Then use the coordinate $T = \frac{1}{s}$ and twist over $\mathbb{Q}(\sqrt{-1})$.

We can also compute the Picard number $\rho(S)$ of the K3 surface from Proposition 3.1. It turns out that only two cases occur: $\rho(S) = 20$ and 22. After Artin [1], K3 surfaces with the latter property are called supersingular since $\rho(S) = b_2(S)$.

COROLLARY 3.4. — *Let S be an elliptic K3 surface over k with a fibre of type I_{14}^* . Then S is supersingular if and only if $p \equiv 3 \pmod{4}$.*

Proof. — By Proposition 3.1, S arises from the corresponding elliptic K3 surface S_0 over \mathbb{Q} by reduction. Since S_0 is a singular K3 surface (i.e. $\rho(S_0) = 20$), it is modular by a result of Livné [10, Remark 1.6]. The associated newform of weight 3 has CM by $\mathbb{Q}(\sqrt{-1})$. (In fact, it has level 16 (cf. [17, Table 1]).) This implies that the characteristic polynomial of Frobenius on $H_{\text{ét}}^2(S, \mathbb{Q}_\ell)$ has all zeroes of the shape ζp for some root of unity ζ if and only if $p \equiv 3 \pmod{4}$. Hence the Corollary follows from the Tate conjecture [26] which is known for elliptic K3 surfaces by [2]. \square

4. Fibre type I_{19} in characteristic $\neq 2$

In this section, we sketch the proof of Theorem 1.1 for the maximal fibre type I_{19} in characteristic $p \neq 2$. In characteristic zero, this was again included in [25].

We use the same approach and notation as in Sect. 3. In particular, the proof first determines the elliptic K3 surfaces with a fibre of type I_{18} . We find two one-dimensional families (cases 1 and 2 in the proof of Proposition 4.1). The family with the simpler equation is distinguished by the existence of a 3-torsion section. Then we determine the unique specialisation within the torsion-free family with a fibre of type I_{19} . Since the proof heavily relies on the help of a machine to factor polynomials, we omit some of the details. They can be obtained from the authors upon request.

PROPOSITION 4.1. — *The elliptic K3-surface over k with a fibre of type I_{19} is unique. Its equation can be given as*

$$Y^2 = X^3 + (T^4 + T^3 + 3T^2 + 1)X^2 + 2(T^3 + T^2 + 2T)X + T^2 + T + 1.$$

This surface has discriminant $\Delta = 4T^5 + 5T^4 + 18T^3 + 3T^2 + 14T - 31$.

Proof. — Let S be an elliptic K3 surface with a fibre of type I_{19} . Let S be given in extended Weierstrass form (3.1). Locating the special fibre at ∞ , we can assume that

$$\begin{aligned} A(T) &= a_4T^4 + a_3T^3 + a_2T^2 + a_1T + a_0, \\ B(T) &= b_3T^3 + b_2T^2 + b_1T + b_0, \\ C(T) &= c_2T^2 + c_1T + c_0. \end{aligned}$$

Since the special fibre is multiplicative, $a_4 \neq 0$. Since k is algebraically closed, we can scale such that $a_4 = 1$. In this setting, Δ has degree at most 14:

$$\Delta = \sum_{l=0}^{14} d_l T^l.$$

We ask for the solutions to the system of equations

$$d_6 = \dots = d_{14} = 0.$$

In the first instance, we shall ignore d_6 , thus investigating the special fibre type I_{18} . The vanishing of the polynomials d_{14}, d_{13}, d_{12} determines c_2, c_1, c_0 . Then

$$d_{11} = -b_3^2 a_1 + \dots, \quad d_{10} = -b_3^2 a_0 + \dots,$$

so we have to distinguish whether $b_3 = 0$.

1st case: $b_3 = 0$. In this case, $d_{11} = b_2(2b_1 - b_2a_3)$. The choice $b_2 = 0$ successively implies $b_1 = b_0 = 0$. In consequence, $B \equiv C \equiv 0$, and equation (3.1) becomes singular. Hence we can assume $b_2 = 1$ after rescaling and obtain b_1 from $d_{11} = 0$. The successive factorisations of d_{10}, \dots directly give b_0, a_1 and a_0 . Then we normalise by a translation in T to assume $a_3 = 0$. We obtain a family of elliptic K3 surfaces with a fibre of type I_{18} and a 3-torsion point.

$$Y^2 = X^3 + \left(T^2 + \frac{a_2}{2}\right)^2 X^2 + \left(T^2 + \frac{a_2}{2}\right) X + \frac{1}{4}.$$

The translation $Y \mapsto Y + (T^2 + \frac{a_2}{2})X + \frac{1}{2}$ moves the 3-torsion point to $(0, 0)$ so that the family reads

$$(4.1) \quad Y^2 + (2T^2 + a_2)XY + Y = X^3.$$

There are several notable properties of this family:

- It arises from the (unique) rational elliptic surface with a fibre of type I_9 ,

$$(4.2) \quad E : Y^2 + TXY + Y = X^3$$

by the family of quadratic base changes

$$T \mapsto 2T^2 + a_2.$$

- There is no specialisation with a fibre of type I_{19} , since on an elliptic K3 surface, the existence of a 3-torsion section predicts that $3 \mid n$ for all singular fibres of type I_n with $n > 6$. (Confer [11] where the quotient by translation by the section is considered. This quotient is again K3, so the Euler number yields the divisibility property.) Another way to deduce the property is to consider the discriminant

$$\Delta = (2T^2 + a_2 - 3)(4T^4 + (6 + 4a_2)T^2 + 9 + 3a_2 + a_2^2).$$

- It admits a model over \mathbb{Q} with good reduction at 2: Instead of the base change $T \mapsto 2T^2 + a_2$ simply apply a base change which is equivalent up to Möbius transformation, e.g. $T \mapsto T^2 + \lambda T$.

2nd case: $b_3 \neq 0$. In this case, the vanishing of d_{11} and d_{10} determines a_1 and a_0 . In consequence,

$$(4.3) \quad d_9 = -\frac{2}{b_3} \underbrace{(b_1b_3 - b_2^2 + b_2b_3a_3 - b_3^2a_2)}_h b_0 + \dots$$

The coefficient h of b_0 does not vanish because otherwise $d_9 = \frac{1}{2}b_3^3 \neq 0$. Hence we obtain b_0 from $d_9 = 0$. This leaves us with polynomials d_6, d_7, d_8 in five variables where we have still two normalisations left. In the next instance, we note that

$$d_7 - 2a_3d_8 = \frac{1}{2}b_3(3b_1b_3 + 3b_2^2 - 6b_2b_3a_3 - 2b_3^2a_2 + 3b_3^2a_3^2).$$

This factorisation provides us with a_2 . At this point, we want to analyse the vanishing of d_8 independently from d_6 , i.e. we first make sure that there is a fibre of type I_{17} and then promote it successively to type I_{18} and I_{19} . The numerator of d_8 is a complicated polynomial. We shall sketch two ways to solve it. The first normalisation is ad-hoc, while the second will use some extra knowledge.

The first normalisation is a linear transformation in T such that $a_3 = 1, b_1 = 0$. (This is possible unless $\text{char}(k) = 3$ and $b_2 = 0$. In this special case, we find five single solutions to d_8 which are not roots of d_6 .) Then the numerator of d_8 is a polynomial of degree 12 in b_2, b_3 . Since every summand has degree at least 10, this polynomial has degree 2 in the homogenising variable. Hence it can be solved explicitly. We obtain a one-dimensional rational parametrisation of elliptic K3 surfaces with I_{18} fibres. No K3 surface in this family has a torsion section other than the zero section.

The second solution is much more efficient. It was motivated by a private correspondence with N. Elkies who kindly informed us about an explicit 1-parameter family of elliptic K3 surfaces with an I_{18} fibre and generically trivial group of sections which he had found independently. We therefore decided to choose the normalisation in such a way that it would meet his example after a change of variables.

Claim: For any solution, there are two linear transformations $T \mapsto \alpha T + \beta$ such that after rescaling:

$$a_4 = 1, \quad b_2 = a_3 b_3, \quad b_1 = 2b_3.$$

Proof. — After the transformation in T , we rescale $(X, Y) \mapsto (\alpha^4 X, \alpha^6 Y)$ to retain our first normalisation $a_4 = 1$. In consequence, the new coefficients read

$$a'_3 = \frac{a_3 + 4\beta}{\alpha}, \quad b'_3 = \frac{b_3}{\alpha^5}, \quad b'_2 = \frac{b_2 + 3b_3\beta}{\alpha^6}, \quad b'_1 = \frac{b_1 + 2b_2\beta + 3b_3\beta^2}{\alpha^7}.$$

The first requirement

$$b'_2 = a'_3 b'_3 \Leftrightarrow b_2 + 3b_3\beta = (a_3 + 4\beta)b_3$$

gives $\beta = \frac{b_2}{b_3} - a_3$. Then the second condition $b'_1 = 2b'_3$ implies

$$\alpha^2 = -\frac{h(a_3, b_1, b_2, b_3)}{b_3^2}$$

with the polynomial h as in (4.3). By assumption, $h \neq 0$, so the claim follows. □

Applying one of the above linear transformations to our elliptic surface, we obtain

$$d_8 = b_3^9(2a_3 + b_3 + 4)(2a_3 + b_3 - 4)/16.$$

It is easily checked that both solutions are identified under the scaling $(T, a_3) \mapsto (-T, -a_3)$ (which interchanges the two choices of α). Setting $b_3 = 2\lambda, a_3 = 2 - \lambda$, we obtain the second family of elliptic K3 surfaces with a fibre of type I_{18} , but without a 3-torsion section:

$$(4.4) \quad Y^2 = X^3 + (T^4 - (\lambda - 2)T^3 + 3T^2 - (2\lambda - 2)T + 1) X^2 + 2\lambda(T^3 - (\lambda - 2)T^2 + 2T - \lambda + 1) X + \lambda^2((T + 1)^2 - \lambda T).$$

This has discriminant

$$\begin{aligned} \Delta = & \lambda^3[(4\lambda - 4) T^6 - (8\lambda^2 - 24\lambda + 12) T^5 + (4\lambda^3 - 20\lambda^2 + 45\lambda - 24) T^4 \\ & - (30\lambda^2 - 76\lambda + 28) T^3 + (13\lambda^3 - 52\lambda^2 + 66\lambda - 24) T^2 \\ & - (46\lambda^2 - 72\lambda + 12) T + 32\lambda^3 - 96\lambda^2 + 37\lambda - 4]. \end{aligned}$$

Hence the elliptic K3 surface with a fibre of type I_{19} is uniquely obtained as the specialisation at $\lambda = 1$. This gives the claimed equation and discriminant. \square

Remark 4.2. — To relate the models in Proposition 4.1 and in [21], we go through the twist in [19]. Here we have to correct the following typo: One summand of B in [19, §2] ought to be $15t^4$. Then the equation in Proposition 4.1 is obtained from the translation $x \mapsto x + t + t^2$ after the variable change $t \mapsto -T - 1$.

COROLLARY 4.3. — *Let S be an elliptic K3 surface over k with a fibre of type I_{19} . Then S is supersingular if and only if $p \neq 2$ is not a quadratic residue mod 19 or $p = 19$.*

Proof. — As before, S arises from a singular K3 surface S_0 over \mathbb{Q} by reduction by Theorem 1.1. Hence the claim follows as in the proof of Corollary 3.4. (See e.g. [19, Corollary 2.2 & §3].) \square

Remark 4.4. — In case of characteristic zero, the uniqueness of the underlying complex K3 surface X for the elliptic fibrations in Propositions 3.1 and 4.1 follows from the Torelli Theorem [14] since X is determined by its transcendental lattice which corresponds to the unique binary quadratic form of discriminant -4 resp. -19 (cf. [23]). However, it is a non-trivial problem to show that this guarantees the uniqueness of the respective elliptic fibrations.

5. Wild ramification of singular fibres

In the absence of wild ramification, the type of a (singular) fibre is determined by the vanishing orders of the j -invariant and the discriminant (cf. [24, p. 365, Table 4.1]). In characteristics 2 and 3, however, wild ramification makes it necessary to go through Tate's algorithm to determine the fibre type. Although there are papers investigating the case of wild ramification in wide generality (e.g. [12]), it seems that there is no reference which gives explicit lower bounds for the index of wild ramification. For convenience and future reference, we decided to include such a list.

PROPOSITION 5.1. — *Let E be an elliptic curve over a complete valuation ring R with residue characteristic $p = 2$ or 3 . Let π denote a uniformizer of R . Let w denote the index of wild ramification of the special fibre of E at π . Depending on the reduction type, the following table lists whether we*

always have $w = 0$ or in one case $w = 1$, or it gives a sharp lower bound for w :

fibres type	$p = 2$	$p = 3$
$I_n (n \geq 0)$	0	0
II	≥ 2	≥ 1
III	≥ 1	0
IV	0	≥ 1
$I_n^* (n = 1)$	1	0
$I_n^* (n \neq 1)$	≥ 2	0
IV^*	0	≥ 1
III^*	≥ 1	0
II^*	≥ 1	≥ 1

The fibre type I_1^* in characteristic 2 is exceptional due to the fact that it admits wild ramification, but only of index $w = 1$. For all other wild fibres, there is no general upper bound on the index of wild ramification. In characteristic 2, this is visible from the elliptic surface with Mordell-Weil rank 2^r that Elkies constructed in [6]. More recently, Gekeler investigated these questions in [7]. This behaviour is in contrast with the situation over a given p -adic field or number field. (Compare [24, Theorem 10.4].)

COROLLARY 5.2. — *In the notation of Proposition 5.1, let the special fibre at π be additive. Then the vanishing order of the discriminant Δ at π satisfies*

$$v_\pi(\Delta) \geq \begin{cases} 3, & \text{if } p = 3, \\ 4, & \text{if } p = 2. \end{cases}$$

Proposition 5.1 is easily verified with Tate’s algorithm [27]. Here we shall only give the proof in characteristic 2 (the case considered in the next two sections). We work with the general Weierstrass form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x^2 + a_6 \quad (a_i \in R).$$

In characteristic 2, the discriminant reads

$$(5.1) \quad \Delta = a_1^4(a_1^2a_6 + a_1a_3a_4 + a_2a_3^2 + a_4^2) + a_3^4 + a_1^3a_3^3.$$

Note that the sharpness of the given bounds will follow immediately from the proof. We will follow the exposition in [24] and employ the notation

$$a_{i,j} = a_i/\pi^j.$$

We let k denote the algebraic closure of the residue field of R . Note that the argument works over the residue field itself as well, but then one has

to take special care of fibre components which are only defined over some extension.

Proof of Proposition 5.1 in characteristic $p = 2$: Recall that at an additive fibre the vanishing order of Δ satisfies

$$v_\pi(\Delta) = 1 + \#(\text{connected components of the fibre}) + w.$$

Starting with reduced fibres (types *II, III, IV*), we have to prove that $\pi^4 | \Delta$ (with equality for type *IV*). By a change of variables, we can move the singular point to $(0, 0)$. Hence $\pi | a_3, a_4, a_6$. The singular fibre is additive if and only if $\pi | a_1$. By (5.1), this implies $\pi^4 | \Delta$. In fact, we obtain

$$(5.2) \quad \Delta = \pi^4 a_{3,1}^4 + O(\pi^5).$$

Fibre type *III* moreover requires $\pi^2 | a_6$, so still $v_\pi(\Delta) \geq 4$. For fibre type *IV*, there is another condition $\pi | (a_2 a_{3,1}^2 + a_{4,1}^2)$. The two exceptional components of the fibre are encoded in the distinct zeroes of the polynomial

$$U^2 + a_{3,1}U + a_{6,2} \in k[U].$$

In particular, this implies $v_\pi(a_{3,1}) = 0$, so $v_\pi(\Delta) = 4$ by (5.2). This completes the analysis of reduced fibres.

Assume now that the singular fibre is not reduced. Following [24], we can assume that $\pi | a_1, a_2, \pi^2 | a_3, a_4, \pi^4 | a_6$. In particular, we see that $\pi^8 | \Delta$. We distinguish three cases depending on the number of distinct roots of the polynomial

$$P(U) = U(U^2 + a_{2,1}U + a_{4,2}) \in k[U].$$

Case 1: If $P(U)$ has three distinct roots in k , then the singular fibre has type I_0^* , and $v_\pi(\Delta) \geq 8$, so $w \geq 2$.

Case 2: If $P(U)$ has a double root and a simple root in k , then the fibre has type I_n^* for some $n > 0$. Translate the root to 0 so that $\pi | a_{4,2}$. Then

$$(5.3) \quad \Delta = \pi^8 a_{3,2}^4 + O(\pi^9).$$

The singular fibre has type I_1^* if and only if the polynomial

$$Q(U) = U^2 + a_{3,2}U + a_{6,4}$$

has distinct roots in k . Equivalently $\pi \nmid a_{3,2}$, so $v_\pi(\Delta) = 8$ by (5.3) and $w = 1$ as claimed.

Generally, the integer $n > 0$ in the fibre type I_n^* is determined by the conditions that after further coordinate changes

$$(5.4) \quad \pi^{l+2} | a_4, \pi^{2l+2} | a_6, \pi^{l+1} \nmid a_3 \quad \text{if } n = 2l - 1 \text{ is odd,}$$

$$(5.5) \quad \pi^{l+2} | a_3, \pi^{2l+3} | a_6, \pi^{l+2} \nmid a_4 \quad \text{if } n = 2l \text{ is even.}$$

Here \parallel denotes exact divisibility. To prove the claim $w \geq 2$ for $n > 1$, we shall use two-step induction: To start the induction, we need that for $n = 2$, the a priori lowest order term in (5.1) is $a_1^4 a_4^2$, so $\pi^{10} \mid \Delta$. If $n = 3$, then we have to consider the term $a_1^4 a_2 a_3^2$ in (5.1), so $\pi^{11} \mid \Delta$. To complete the induction, we note that $n \rightarrow n + 2$ increases the π -divisibility of every summand in (5.1) by at least two by inspection of (5.4), (5.5). Hence $w \geq 2$ if $n \geq 2$.

Case 3: If $P(U)$ has a triple root, then we translate it to 0 again so that $\pi \mid a_{2,1}, a_{4,2}$. For the discriminant, we still have (5.3).

The singular fibre has type IV^* if and only if the polynomial $Q(U)$ has distinct roots in k . As before, this yields $v_\pi(\Delta) = 8$, so here $w = 0$.

Otherwise, we translate the double root to 0 so that $\pi \mid a_{3,2}, a_{6,4}$. Then

$$\Delta = \pi^{10} a_{1,1}^4 a_{4,3}^2 + O(\pi^{11}).$$

The singular fibre has type III^* if and only if $\pi \nmid a_{4,3}$. Equivalently $v_\pi(\Delta) \geq 10$ and thus $w \geq 1$. Otherwise we have fibre type II^* with $v_\pi(\Delta) \geq 11$ and again $w \geq 1$. □

The proof in characteristic 3 is similar, but simpler since we can work with an extended Weierstrass form (3.1) and I_n^* fibres do not admit wild ramification. It is left as an exercise to the reader.

An elliptic curve over a field K of characteristic 2 with $j(E) \neq 0$ can be given in normal form

$$(5.6) \quad Y^2 + XY = X^3 + a_2 X^2 + \frac{1}{j(E)}.$$

Note, however, that this form in general is not integral or minimal. A twist replaces the coefficient a_2 by $a_2 + D$ while preserving $j(E)$. Such a twist is trivial (i.e. the two curves are isomorphic over K) if D is of the form $\beta^2 + \beta$ with $\beta \in K$. If $a_2 = 0$, the above normal form has multiplicative fibres at all poles of $j(E)$.

In characteristic 2, a fibre of type I_ν^* does not imply that the j -invariant has a pole at the corresponding place. Actually, every twist with sufficiently wild ramification will produce a fibre of type I_ν^* . We describe this locally:

LEMMA 5.3. — *Let k be an algebraically closed field of characteristic 2. Consider the elliptic curve*

$$E : Y^2 + XY = X^3 + DX^2 + \frac{\pi^r Q}{\pi^{6e}}$$

over $k((\pi))$ where $e \in \mathbb{Z}$, $0 \leq r \leq 5$ and $Q = \sum_{i=0}^{\infty} \omega_i \pi^i$ is a unit in $k[[\pi]]$. Let

$$D = \frac{\delta}{\pi^{2d-1}} + \dots \in \frac{1}{\pi^{2d-1}} k[[\pi]]$$

with $\delta \in k^\times$ and $d > 0$.

If $d > e$ then E has a special fibre of type I_ν^* with

$$\nu = 8d - 4 - 6e + r = 8d - 4 - v(j(E))$$

and index of wild ramification $w = 4d - 2$.

Proof. — Essentially this is proved in [15, Lemma 2.1], although the type of the fibre is not stated explicitly there. \square

6. Type I_{13}^* in characteristic 2

The first approach towards maximal singular fibres on elliptic K3 surfaces in characteristic 2 might be to mimic the construction for odd characteristic: reduce the maximal surfaces over \mathbb{Q} from Propositions 3.1 and 4.1 mod 2. However, in characteristic 2, these equations do only define a quasi-elliptic fibration; i.e. every fibre is a curve of genus one, but the general fibre is singular (type *II*). Such surfaces can only exist in characteristic 2 and 3 (cf. [4] for instance).

The present quasi-elliptic fibrations have only one reducible singular fibre; namely it has type I_{16}^* . Both fibrations can be transformed to the Weierstrass form

$$S: Y^2 = X^3 + T^3 X^2 + T.$$

This surface (and some other models of it) has been studied extensively in [5]. In other words, the Néron-Severi lattice of S accomodates the fibre through the abstract decomposition $\text{NS}(S) = U + D_{20}$, but unlike the usual situation in characteristics different from 2, 3, the corresponding fibration is only quasi-elliptic.

Returning to fibre type I_{14}^* , it was shown in [16] that there is no elliptic K3 surface in characteristic 2 with such a singular fibre. It was immediate from the proof that the surfaces with a fibre of type I_{13}^* come in a family. In the notation of [16], this family was determined by the choice $c = \sqrt{e}$. To correct a typographical error in [16], we note that this is the exact case when $v_0(\Delta) = 21$.

In this section we shall re-prove this result. The motivation to do so is twofold. First, we use an argument which is structural and does not require assistance by a machine. Secondly, we will directly derive the explicit equation of the family.

We now recall some results from [15] that are helpful for our classification purposes in characteristic 2. The setup is an algebraically closed field k of characteristic 2 and an elliptic curve E over $k(T)$ which is Frobenius-minimal (i.e. $j(E)$ is separable).

Note that the places of supersingular reduction of E are exactly the zeroes of $j(E)$ that are places of good reduction.

LEMMA 6.1 ([15, Lemma 2.4 (a)]). — *Let $\text{char}(k) = 2$. Assume that E is Frobenius-minimal with $j(E) \notin k$. Let β_i denote the finite places of bad reduction of E and*

$$\begin{aligned} r_i \in \{0, 1, 2, 3\} \quad \text{such that} \quad r_i &\equiv -v_{\beta_i}(j(E)) \pmod{4}, \\ s_i \in \{0, 1\} \quad \text{such that} \quad s_i &\equiv -v_{\beta_i}(j(E)) \pmod{2}. \end{aligned}$$

Define

$$G(T) = \prod (T - \beta_i)^{r_i}, \quad H(T) = \prod (T - \beta_i)^{s_i}.$$

Let ϑ be a finite place of supersingular reduction of E and expand $G(T) = \sum \gamma_i (T - \vartheta)^i$. Let c_G denote be the smallest index i with $4 \nmid i$ and $\gamma_i \neq 0$. Then $v_\vartheta(j(E)) = 12e$ for some $e \in \mathbb{N}$ with

$$(T - \vartheta)^{3e-1} | H'(T) \quad \text{and} \quad 3e \leq c_G.$$

Now we are ready to prove our first main result in characteristic 2.

PROPOSITION 6.2. — *In characteristic 2, any elliptic K3-surface with a fibre of type I_{13}^* occurs in the following family:*

$$Y^2 + TXY + Y = X^3 + \lambda T^3 X^2 + \lambda T, \quad \lambda \in k^*.$$

All these surfaces have $\Delta = T^3 + 1$ and $j = \frac{T^{12}}{T^3 + 1}$.

Proof. — We locate the special fibre at ∞ . Proposition 5.1 prescribes the vanishing order $v_\infty(\Delta) \geq 21$. Hence it follows from Corollary 5.2 that all other singular fibres are multiplicative.

In contrast to other characteristics, the existence of a fibre of type I_n^* ($n > 0$) does not rule out the possibility of $j(E) = 0$ in characteristic 2. In that case we move the maximal fibre to 0 and have a model

$$Y^2 + T^6 Y = X^3 + a_2 X^2 + a_4 X + a_6.$$

As already explained in [16], the Tate algorithm then shows that $m \leq 9$ for fibres I_m^* (under the condition that we have a K3 surface). Hence $j(E) \neq 0$,

and since the fibre at ∞ is additive, there is at most one finite supersingular place.

If there is no finite supersingular place (i.e. $j(E) = \frac{1}{\Delta(T)}$), then Tate’s algorithm [27] for the integral model of the twisted normal form (5.6)

$$Y^2 + XY = X^3 + D(T)X^2 + \Delta(T)$$

shows that the fibre at ∞ can maximally have type I_{12}^* . (This was the case $a_1 = t^2$ in [16], where the maximal fibre had been located at 0.)

Otherwise, we locate the finite supersingular place at $T = 0$ by a Möbius transformation. It follows from Lemma 6.1 that either $v_\infty(\Delta) = 21$ with three distinct affine roots or the surface is not Frobenius-minimal. In the latter case, we have $j(E) = \frac{T^{12}}{(aT+b)^2}$ with $b \neq 0$. But then the associated separable surface $S(E)$ would have $j(S(E)) = \frac{T^6}{aT+b}$ and hence there would be an additive singular fibre at 0. (In characteristic 2 the j -invariant is a_1^{12}/Δ , so the multiplicity of a good place in the j -invariant must be divisible by 12.) Since a purely inseparable base change cannot replace an additive fibre by a smooth or multiplicative fibre, we obtain a contradiction.

We now consider the first case, $v_\infty(\Delta) = 21$. By Lemma 6.1, we obtain

$$j(E) = \frac{T^{12}}{\varepsilon(T^3 + c)}$$

with $\varepsilon, c \in k^*$. Rescaling, we achieve $c = 1$. The “untwisted” form

$$Y^2 + XY = X^3 + \frac{\varepsilon(T^3 + 1)}{T^{12}}$$

has fibres I_9 at ∞ , I_1 at the third roots of unity, and also a singular fibre at 0. Since in characteristic 2 every twist can be build from twists that ramify at only one place, we can change the fibres at 0 and at ∞ individually. First we apply a twist that makes the fibre at 0 smooth. In [15, Proposition 5.1] it was shown that this is only possible with $\varepsilon = 1$ and the following twist

$$Y^2 + XY = X^3 + \frac{1}{T^3}X^2 + \frac{T^3 + 1}{T^{12}}.$$

In other words, our K3 surface with fibre of type I_{13}^* must be a twist of the extremal elliptic surface (4.2), which has configuration [9,1,1,1].

Now we apply a twist that ramifies only at ∞ . Lemma 5.3 tells us that we will get a fibre of type I_{13}^* at ∞ if and only if this twist is

$$Y^2 + XY = X^3 + \left(\lambda T + \frac{1}{T^3}\right) X^2 + \frac{T^3 + 1}{T^{12}}.$$

The corresponding integral model can be minimalised after the translation $(X, Y) \mapsto (X + T, Y + 1)$. This gives the claimed equation. □

A similar approach, following [15, Lemma 2.4 (b)], can be carried out in characteristic 3, for fibre type I_{14}^* as well as for I_{19} . We have omitted this, since the proofs in Sections 3 and 4 settle all cases of odd characteristic with one calculation.

7. Type I_{18} in characteristic 2

In [16], it was shown that a fibre of type I_{19} is impossible for an elliptic K3 surface S in characteristic 2. We briefly sketch how this can be proven purely in terms of Lemma 6.1:

Assume that S has a fibre of type I_{19} at ∞ . Hence S is separable, and Δ has degree five. We first consider the case with an additive fibre, which we place at 0. Then, by Corollary 5.2 and Lemma 6.1, there is no other additive fibre and no finite supersingular place. Hence $a_1 = T^2$, and the argumentation in the proof of Lemma 7.1 rules this case out.

Otherwise, all fibres are multiplicative. Since Δ' has degree four, it follows from Lemma 6.1 that there are two distinct supersingular places (and the configuration is [19,1,1,1,1,1], which also follows directly from Theorem 2.1). Normalise so that $\vartheta_1 = 0, \vartheta_2 = 1$. Then $\Delta(T) = \varepsilon H(T)$, where by Lemma 6.1

$$H(T) = T^5 + aT^4 + T^3 + cT^2 + e, \quad (e \neq 0, a + c + e \neq 0).$$

We apply the criterion with the index c_G from Lemma 6.1: At ϑ_1 , it gives $c = 0$. But then the expansion of $H(T)$ at ϑ_2 is

$$H(T) = (T + 1)^5 + (a + 1)(T + 1)^4 + (T + 1)^3 + (T + 1)^2 + a + e,$$

so the criterion gives a contradiction. Hence there is no elliptic K3 surface in characteristic 2 with a fibre of type I_{19} .

We shall now study the next case of fibre type I_{18} . In Section 4, we have exhibited two families of elliptic surfaces over \mathbb{Q} with such a fibre. For the surface in (4.1), we already saw that it has good reduction at 2. In this section, we will prove that also the second family reduces nicely mod 2 (Remark 7.3), and that any elliptic K3 surface in characteristic 2 with a fibre of type I_{18} is a member of one of these families (cf. Proposition 7.2). We first determine the possible configurations:

LEMMA 7.1. — *Let S be an elliptic K3 surface in characteristic 2 with a fibre of type I_{18} . Then S is necessarily semistable. Moreover, if S is*

inseparable, it must be the Frobenius base change of the rational elliptic surface E in (4.2), i.e. we have

$$S : Y^2 + T^2XY + Y = X^3$$

with configuration [18,2,2,2]. On the other hand, if S is separable, the configuration is necessarily [18,1,1,1,1,1].

Proof. — We locate the fibre of type I_{18} at ∞ . Then Δ must have degree 6. Corollary 5.2 shows that there is at most one additive fibre.

We first assume that there is an additive fibre and locate it at 0. Hence $T|a_1$. Since $T^4|\Delta$ there are at most two more singular fibres. We claim that 0 is the only zero of $j(S)$. This can be seen as follows. If S is separable, then the polynomial $H(T)$ has degree at most 2. If S is inseparable, then it is the Frobenius base change of a separable elliptic surface with one fibre of type I_9 at ∞ , one additive fibre at 0, and possibly one I_1 fibre. Again the polynomial $H(T)$ for this surface has degree at most 2. In either case there cannot be a zero of $j(E)$ at a good place by Lemma 6.1. This is equivalent to the claim. So in the minimal model of S

$$Y^2 + a_1XY + a_3Y = X^3 + a_2X^2 + a_4X + a_6$$

we can assume $a_1 = T^2$. From the discriminant

$$\Delta = a_1^6a_6 + a_1^5a_3a_4 + a_1^4a_2a_3^2 + a_1^4a_4^2 + a_1^3a_3^3 + a_3^4$$

we see that T must divide a_3 . Hence Δ is congruent to a_3^4 modulo T^8 . In particular Δ cannot have degree 6. From this contradiction we conclude the semistability of S .

Now, if S is inseparable, its configuration has to be [18,2,2,2] by Theorem 2.1. Hence it is the Frobenius base change of the unique rational elliptic surface E with configuration [9,1,1,1] in (4.2).

If S is separable, the conductor must have degree at least 6 by Theorem 2.1. Hence the configuration can only be [18,1,1,1,1,1] or [18,2,1,1,1,1]. But for the second configuration, $\deg H(T) = 4$, so Lemma 6.1 would imply the contradiction $\deg(a_1) \leq 1$. □

PROPOSITION 7.2. — *Separable elliptic K3 surface in characteristic 2 with a fibre of type I_{18} come in two families:*

$$(7.1) \quad Y^2 + (T^2 + T + 1)XY + rY = X^3, \quad (r \neq 0)$$

with $\Delta = r^3(T^6 + T^5 + T^3 + T + 1 + r)$. This family with a 3-torsion section is obtained from the extremal elliptic surface (4.2) by a family of quadratic base changes.

$$(7.2) \quad Y^2 + (T^2 + T + 1)XY + r(T + 1)Y = X^3 + rT^3X^2 + r^3T, \quad (r \neq 0)$$

with $\Delta = r^3(T^6 + T^5 + rT^4 + T^3 + T + 1 + r)$.

Proof. — Locating the special fibre at ∞ and applying Lemma 6.1 we obtain

$$\Delta(T) = \varepsilon H(T), \quad H(T) = T^6 + aT^5 + bT^4 + cT^3 + dT^2 + eT + f.$$

We claim that $a \neq 0$. To prove this, assume on the contrary $a = 0$, so $\deg H' \leq 2$. It follows from Lemma 6.1 that there is at most one finite supersingular place, and that this place has multiplicity 1 as a root of a_1 . Hence the elliptic surface is either rational ($\deg a_1 = 1$), or the fibre at ∞ has additive type. This gives a contradiction.

Since $a \neq 0$, H' has degree four. By Lemma 6.1 and the above argument, there are two distinct supersingular places. For convenience we normalise them to be primitive third roots of unity ϱ, ϱ^2 :

$$A(T) = T^2 + T + 1.$$

By Lemma 6.1, this implies $e = c = a$:

$$H(T) = T^6 + aT^5 + bT^4 + aT^3 + dT^2 + aT + f.$$

Then the index criterion from Lemma 6.1 at the supersingular places implies

$$(1 + a)\varrho + d = (1 + a)\varrho + a + d + 1 = 0.$$

Hence $a = 1, d = 0$, and

$$H(T) = T^6 + T^5 + bT^4 + T^3 + T + f.$$

Note that the variable change $T \mapsto T + 1$ preserves $A(T)$ and the shape of $H(T)$ after replacing f by $b + f$. The surface

$$Y^2 + XY = X^3 + \frac{\varepsilon H(T)}{A(T)^{12}}$$

has multiplicative fibres at all poles of the j -invariant $\frac{A(T)^{12}}{\varepsilon H(T)}$ as desired, but it also has singular fibres at the zeroes of $A(T)$. To make these fibres smooth we apply a twist that only ramifies at ϱ and ϱ^2 . By Lemma 5.3 this twist can only be of the form

$$Y^2 + XY = X^3 + \left(\frac{\alpha_3}{(T - \varrho)^3} + \frac{\alpha_1}{T - \varrho} + \frac{\beta_3}{(T - \varrho^2)^3} + \frac{\beta_1}{T - \varrho^2} \right) X^2 + \frac{\varepsilon H(T)}{A(T)^{12}}.$$

We now consider the following integral model of S :

$$(7.3) \quad Y^2 + A(T)^2 XY = X^3 + A(T)D(T)X^2 + \varepsilon H(T).$$

Here, after a variable change $Y \mapsto Y + A(T)\alpha(T)X$, it suffices to allow the following twisting polynomials:

$$D(T) = d_3T^3 + d_2T^2 + d_1T + d_0.$$

In order for the fibres at the supersingular places to be smooth, we require that the integral model (7.3) is not minimal at the supersingular places. We pursue these issues simultaneously by going through Tate’s algorithm [27]. We apply several translations and, if necessary, substitutions to increase the vanishing orders of all coefficients of the Weierstrass form successively. For the first variable change, we introduce new parameters λ, μ with

$$\lambda^2 = \varepsilon(b + f + 1), \quad \mu^2 = \varepsilon b.$$

Then we transform $Y \mapsto Y + \mu T + \lambda$ to obtain

$$Y^2 + A(T)^2XY = X^3 + A(T)D(T)X^2 + A(T)^2(\mu T + \lambda)X + A(T)^2(\varepsilon A(T) + \mu^2).$$

Next, $Y \mapsto Y + A(T)\mu$ gives

$$Y^2 + A(T)^2XY = X^3 + A(T)D(T)X^2 + A(T)^2(\mu T + \lambda + \mu A(T))X + \varepsilon A(T)^3.$$

Here we have to rule out fibre type I_n^* at the supersingular places. In other words, the following two polynomials must have a triple zero each:

$$\begin{aligned} P(Z) &= Z^3 + (d_0 + \varrho d_1 + \varrho^2 d_2 + d_3)Z^2 + (\varrho\mu + \lambda)Z + \varepsilon, \\ Q(Z) &= Z^3 + (d_0 + \varrho^2 d_1 + \varrho d_2 + d_3)Z^2 + (\varrho^2\mu + \lambda)Z + \varepsilon. \end{aligned}$$

We distinguish whether these polynomials are equal or not:

$\mathbf{P(Z) = Q(Z)}$: Denoting the triple root by r , we obtain

$$d_1 = d_2, \quad d_0 + d_2 + d_3 = r, \quad \mu = 0, \quad \lambda = r^2, \quad \varepsilon = r^3.$$

Hence the variable change $X \mapsto X + rA(T)$ gives

$$Y^2 + A(T)^2XY + rA(T)^3Y = X^3 + A(T)^2(Td_3 + d_2 + d_3)X^2 + r^2A(T)^4(Td_3 + d_2 + d_3).$$

For the next transformation, we let $u^2 = d_2, v^2 = d_3$. Then the translation $Y \mapsto Y + rA(T)^2(Tv + u)$ gives

$$Y^2 + A(T)^2XY + rA(T)^3Y = X^3 + A(T)^2(Td_3 + d_2 + d_3)X^2 + rA(T)^4(Tv + u)X + r^2A(T)^5(Tv + u + v^2).$$

The final step for non-minimality requires that $A(T)^6$ divides the coefficient of the constant term. Since this has only degree 11, it has to be zero:

$$u = v^2, \quad v = 0.$$

After minimalising, we obtain the family (7.1).

$P(Z) \neq Q(Z)$: In this case, the triple roots differ by a third root of unity. Hence, given a choice of ϱ there are unique r, ω such that $\omega = \varrho$ or $\omega = \varrho^2$ and

$$P(Z) = (Z + \omega r)^3, \quad Q(Z) = (Z + \omega^2 r)^3.$$

Since the choices of ϱ, ϱ^2 are permuted by the variable change $T \mapsto T + 1$, we can assume that $\omega = \varrho^2$. This implies (by adding the coefficients of $P(Z), Q(Z)$) that

$$d_1 + d_2 = r, \quad d_0 + d_1 + d_3 = 0, \quad \mu = r^2, \quad \lambda = 0, \quad \varepsilon = r^3.$$

In consequence, we translate $X \mapsto X + rA(T)(T + 1)$. This results in

$$\begin{aligned} & Y^2 + A(T)^2XY + rA(T)^3(T + 1)Y \\ &= X^3 + A(T)^2(Td_3 + d_3 + d_2)X^2 + r^2A(T)^4(A(T)(Td_3 + d_2) + Td_1 + d_3). \end{aligned}$$

We let $d_1 = u^2, d_3 = v^2$ and translate $Y \mapsto Y + rA(T)^2(uT + u + v)$. This gives

$$\begin{aligned} & Y^2 + A(T)^2XY + rA(T)^3(T + 1)Y = X^3 + A(T)^2(Td_3 + d_3 + d_2)X^2 \\ & + rA(T)^4(uT + u + v)X + r^2A(T)^5(uA(T) + (u + v + v^2)T + v + r). \end{aligned}$$

Again non-minimality requires that the final summand is divisible by $A(T)^6$. Hence

$$v = r, \quad u = v + v^2.$$

After minimalising, we obtain the family of elliptic K3 surfaces in the parameter r as

$$\begin{aligned} & Y^2 + (T^2 + T + 1)XY + r(T + 1)Y = \\ & X^3 + r(r^3 + Tr + 1)X^2 + r^2(rT + T + r)X + r^3(r + 1) \end{aligned}$$

with $H(T) = T^6 + T^5 + rT^4 + T^3 + T + r + 1$. Finally the translation $Y \mapsto Y + r(T + 1 + r)X + r^2$ gives (7.2). □

Remark 7.3. — For the family (7.1), we have seen that it can be obtained from the first family in section 4 by reduction mod 2. For the second family, start with (4.4) and apply the variable change

$$(X, Y, \lambda) \mapsto (4X, 8Y, r/4).$$

Then the translation $Y \mapsto Y + \frac{1}{2}(T^2 + T + 1)X + \frac{1}{2}r(T + 1)$ gives

$$Y^2 + (T^2 + T + 1)XY + r(T + 1)Y = X^3 - rT(T^2 + 2)X^2 - 2r^2(T^2 + 1)X - r^3T.$$

This reduces mod 2 to the claimed equation (7.2).

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