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BOUNDS ON THE DENOMINATORS IN THE CANONICAL BUNDLE FORMULA

by Enrica FLORIS (*)

Abstract. — In this work we study the moduli part in the canonical bundle formula of an lc-trivial fibration whose general fibre is a rational curve. If r is the Cartier index of the fibre, it was expected that $12r$ would provide a bound on the denominators of the moduli part. Here we prove that such a bound cannot even be polynomial in r, we provide a bound $N(r)$ and an example where the smallest integer that clears the denominators of the moduli part is $N(r)/r$. Moreover we prove that even locally the denominators depend quadratically on r.

Résumé. — Dans cet article on considère la partie modulaire dans la formule du fibré canonique pour une fibration lc-triviale dont la fibre générique est une courbe rationnelle. Soit r l'indice de Cartier de la fibre. Il avait été conjecturé que $12r$ est une borne sur les dénominateurs de la partie modulaire. Nous démontrons qu'une telle borne ne peut même pas être polynômiale en r, nous calculons une borne $N(r)$ et nous fournissons un exemple où la borne optimale sur les dénominateurs est $N(r)/r$. De plus nous montrons que même localement les dénominateurs dépendent quadratiquement de r.

1. Introduction

The canonical bundle formula is an important tool in classification theory to reduce the study of varieties of intermediate Kodaira dimension, that is $0 < \mathrm{kod}(X) < \dim X$, to the study of varieties, more precisely pairs, having Kodaira dimension 0 or equal to their dimension.

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To be precise, let (X, B) be a log canonical pair, where X is a normal variety of dimension n over the field $\mathbb{C}$ and B a $\mathbb{Q}$-divisor. We consider the canonical ring of (X, B)

$$R(X, B) = \oplus \Gamma(X, m(K_X + B))$$

where the sum runs over the m sufficiently divisible. If $R(X, B)$ is not the ring 0, then for m sufficiently large and divisible $|m(K_X + B)|$ defines a morphism

$$\phi\colon X' \to Z$$

where X' is some birational model of X. There are three cases.

(1) If $\dim Z = 0$ then $K_{X'} + B'$ is torsion.
(2) If $0 < \dim Z < n$ then ϕ is a fibration with general fibre F such that $K_F + B'|_F$ is torsion.
(3) If $\dim Z = n$ then (X, B) is *of log general type.*

If X is a smooth surface and $B = 0$ the three cases become the following.

(1) The canonical divisor K_X is torsion and more precisely $mK_X \cong \mathcal{O}_X$ for some $m \in \{1, 2, 3, 4, 6\}$. Smooth surfaces of this type are classified up to isomorphism.
(2) The morphism ϕ is a fibration with generic fibre an elliptic curve.
(3) If $\dim Z = 2$ then X is of general type.

In the second case we have Kodaira's canonical bundle formula for a minimal elliptic surface (see for instance [3, Chapter V, Theorem 12.1])

$$K_X = \phi^*(K_Z + \sum_{p \in Z}(1 - \frac{1}{m_p})p + L) \tag{1.1}$$

where L is of the form $R + j^*\mathcal{O}_{\mathbb{P}^1}(1)$, with R is supported on the singular locus of ϕ and $j\colon Z \to \mathbb{P}^1$ is the j-function. The sum in the formula is over the $p \in Z$ such that ϕ^*p is a multiple fibre and m_p is such that $\phi^*p = m_p S_p$ where S_p is the support of the fibre. Kawamata in [7, 8] pointed out that the divisor $R + \sum(1 - 1/m_p)p$ can be computed in terms of the pair (X, B). More precisely, if $R + \sum(1 - 1/m_p)p = \sum b_p p$ then $1 - b_p$ is the largest real number t such that the pair $(X, B + tf^*p)$ is log canonical. In the case where X has dimension n, the current generalization of the formula is due to Ambro [2] and reads as follows:

$$K_X + B + \frac{1}{r}(\varphi) = \phi^*(K_Z + B_Z + M_Z) \tag{1.2}$$

where $r \in \mathbb{N}$ is the Cartier index of the fibre, φ is a rational function, the divisor B_Z is called the *discriminant* and corresponds to $\sum(1 - \frac{1}{m_p})p + R$ in Kodaira's formula, while M_Z, called the *moduli part*, corresponds to

$j^*\mathcal{O}_{\mathbb{P}^1}(1)$ and measures the (birational) variation of the fibres. All the theory about the canonical bundle formula is developed for lc-trivial fibrations. The definition of this class of fibrations is quite technical and for it we refer to the second section. The following result is shown in [2] by Ambro, for (X, B) generically klt on the base, and in [4] by Kollár in the lc case.

Theorem 1.1 (Ambro, [2] Theorem 0.2, Kollár, [4]). — *Let $f\colon (X,B) \to Z$ be an lc-trivial fibration. Then there exists a proper birational morphism $Z' \to Z$ with the following properties:*

1. *$K_{Z'} + B_{Z'}$ is a $\mathbb{Q}$-Cartier divisor, and $\nu^*(K_{Z'} + B_{Z'}) = K_{Z''} + B_{Z''}$ for every proper birational morphism $\nu\colon Z' \to Z''$.*
2. *the divisor $M_{Z'}$ is $\mathbb{Q}$-Cartier and nef and $\nu^*(M_{Z'}) = M_{Z''}$ for every proper birational morphism $\nu\colon Z' \to Z''$.*

The regularity of the pair (Z, B_Z) depends on the regularity of (X, B), more precisely (Z, B_Z) is klt (resp. lc) if and only if (X, B) is (see [1, Proposition 3.4]).

Furthermore the following properties are conjectured for M_Z.

Conjecture 1.2 (Prokhorov-Shokurov, [10] Conjecture 7.13). — *Let $f\colon (X, B) \to Z$ be an lc-trivial fibration.*

1. *(Log Canonical Adjunction) There exists a proper birational morphism $Z' \to Z$ such that $M_{Z'}$ is semiample.*
2. *(Particular Case of Effective Log Abundance Conjecture) Let X_η be the generic fibre of f. Then $I_0(K_{X_\eta} + B_\eta) \sim 0$, where I_0 depends only on $\dim X_\eta$ and the multiplicities of the horizontal part of B.*
3. *(Effective Adjunction) The divisor M_Z is effectively semiample, that is, there exists a positive integer I depending only on the dimension of X and the horizontal multiplicities of B (a finite set of rational numbers) such that IM_Z is the pullback of M, where M is a base point free divisor on some model Z'/Z.*

The relevance of the above conjecture is well illustrated for instance by a remark due to X. Jiang, who observed recently [6, Remark 7.3] that Conjecture 1.2(3) implies a uniformity statement for the Iitaka fibration of *any* variety of positive Iitaka dimension under the assumption that the fibres have a good minimal model.

These conjectures are proved in the case where the fibres have dimension one.

Theorem 1.3 (Prokhorov-Shokurov, [10]). — *Conjecture 1.2 holds in the case* $\dim X = \dim Z + 1$.

It is important to remark that the proof of Theorem 1.3 strongly uses the existence of the moduli space $\mathcal{M}_{0,n}$. Moreover the constant I that appears in Theorem 1.3 is not explicitely determined. In [10, Remark 8.2] the authors expect that a sharp result might be $I = 12r$ where r is as in Formula (1.2). In particular this would imply that the denominators of the $\mathbb{Q}$-divisor M are bounded by r. In the case of one-dimensional fibre, if $B = 0$ the general fibre is an elliptic curve and the result follows from Kodaira's Formula (1.1). If $B \neq 0$ then the generic fibre F is a rational curve and B is effective and such that $\deg B|_F = 2$. In this case the situation is more complicated.

In this work we prove that in the case where the generic fibre is a rational curve the expectation of Prokhorov and Shokurov cannot be true. Indeed we can prove that there are examples in which $12rM$ has not even integer coefficients.

Counterexample 1.4. — *There exists an lc-trivial fibration $f\colon (X,B) \to Z$ whose generic fibre is a rational curve such that $12rB_Z$ has not integer coefficients. More precisely for any positive and odd $r \in \mathbb{N}$ there exists an lc-trivial fibration $f\colon (X,B) \to Z$ such that (1.2) holds and with moduli divisor $B_Z = \sum \beta_p p$ and there exists a point $o \in Z$ such that the minimal integer m such that $m\beta_o \in \mathbb{Z}$ is greater or equal to $2r^2 - r$.*

Neverthless we can show the following local result, which is not far from being sharp by the previous example:

Theorem 1.5. — *Let $f\colon (X,B) \to Z$ be an lc-trivial fibration whose generic fibre is a rational curve. Let $B_Z = \sum \beta_i p_i$ be the discriminant. Then for every i there exists $l_i \leqslant 2r$ such that $rl_i\beta_i \in \mathbb{Z}$.*

An important remark is that for an lc-trivial fibration whose general fibre is a rational curve, for every $I \in \mathbb{Z}$, IrM_Z has integer coefficients if and only if IrB_Z has integer coefficients. To prove Theorem 1.5 we give an expression of the log canonical threshold of a fibre with respect to (X,B) in terms of the pull back of the canonical divisor of X, the pull back of the fibre and the pull back of B.

An interesting question is to determine the best possible *global* bound on the denominators of M_Z. Theorem 1.5 implies that $(2r)!M_Z$ has integer coefficients, but it is certainly not the best bound. Using techniques from Theorem 1.5 we can prove that a polynomial global bound cannot exist and determine a bound.

Theorem 1.6. —

(1) *A polynomial global bound on the denominators of M_Z cannot exist. Precisely for all N there exists an lc-trivial*

fibration

$$f\colon (X,B)\to Z$$

such that if V is the smallest integer such that VM_Z has integer coefficients then

$$V\geqslant r^{N+1}.$$

(2) *Let $f\colon (X,B)\to Z$ be an lc-trivial fibration whose generic fibre is a rational curve. Then there exists an integer $N(r)$ that depends only on r such that $N(r)M_Z$ has integer coefficients. More precisely if we set $s(q)=\max\{s\mid q^s\leqslant 2r\}$ then*

$$N(r)=r\prod_{\substack{q\leqslant 2r\\ q\text{ prime}}} q^{s(q)}.$$

(3) *For all r odd there exists an lc-trivial fibration*

$$f\colon (X,B)\to Z$$

such that if V is the smallest integer such that VB_Z has integer coefficients then $V=N(r)/r$.

In [11] G. T. Todorov proves, in the case where the pair (X,B) is klt over the generic point of Z, the existence of an explicitely computable integer $I(r)$ such that $I(r)M_Z$ has integer coefficients using techniques from [5] where the existence of such an integer is proved in the case $B=0$. Todorov's bound is considerably greater than the bound provided by Theorem 1.6:

r	I(r)	N(r)
3	120	60
4	5040	420
5	1441440	2520
6	160626866400	27720
7	288807105787200	360360
8	619808900849199341280 0	360360
9	7093601304616933605068169600	12252240
10	194603155528763897469736633833782400	232792560

An explicit global bound on the denominators of M_Z is important in order to obtain effective results for the pluri-log-canonical maps of pairs with positive Kodaira dimension. For instance the bounds in [5, Theorem 6.1] and [11, Theorem 4.2] can be immediately improved by using Theorem 1.6. One of the difficulties of studying the moduli part of lc-trivial fibrations with fibres of dimension greater than one is the lack of a moduli space for the fibres. It is therefore worth noticing that our arguments make no use of $\mathcal{M}_{0,n}$. We hope that our more elementary approach could lead to a better understanding of the moduli divisor for fibrations with higher dimensional fibres.

2. Notations and preliminaries

2.1. Notations, definitions and known results

We will work over $\mathbb{C}$. In the following $\equiv$, $\sim$ and $\sim_{\mathbb{Q}}$ will respectively indicate numerical, linear and $\mathbb{Q}$-linear equivalence of divisors. The following definitions are taken from [9].

Definition 2.1. — *Let (X, B) be a pair, $B = \sum b_i B_i$ with $b_i \in \mathbb{Q}$. Suppose that $K_X + B$ is $\mathbb{Q}$-Cartier. Let $\nu\colon Y \to X$ be a birational morphism, Y normal. We can write*

$$K_Y \equiv \nu^*(K_X + B) + \sum a(E_i, X, B)E_i.$$

where $E_i \subseteq Y$ are distinct prime divisors and $a(E_i, X, B) \in \mathbb{R}$. Furthermore we adopt the convention that a nonexceptional divisor E appears in the sum if and only if $E = \nu_*^{-1} B_i$ for some i and then with coefficient $a(E, X, B) = -b_i$.
The $a(E_i, X, B)$ are called discrepancies.

Definition 2.2. — *Let (X, B) be a pair and $f\colon X \to Z$ be a morphism. Let $o \in Z$ be a point (possibly of positive dimension). A log resolution of (X, B) over o is a birational morphism $\nu\colon X' \to X$ such that for all $x \in f^{-1}o$ the divisor $\nu^*(K_X + B)$ is simple normal crossing at x.*

Definition 2.3. — *We set*

$$\mathrm{discrep(X, B) = \inf\{a(E, X, B) \mid E\, exceptional\, divisor\, over\, X\}}.$$

A pair (X, B) is defined to be

- *klt (kawamata log terminal) if* $\mathrm{discrep(X, B)} > -1$,
- *lc (log canonical) if* $\mathrm{discrep(X, B)} \geqslant -1$.

Definition 2.4. — *Let $f\colon (X, B) \to Z$ be a morphism and $o \in Z$ a point. For an exceptional divisor E over X we set $c(E)$ its image in X. We set*

$$\mathrm{discrep_o(X, B) = \inf\{a(E, X, B) \mid E\, exceptional\, divisor\, over\, X,\ f(c(E)) = o\}}.$$

A pair (X, B) is defined to be

- *klt over o (kawamata log terminal) if* $\mathrm{discrep_o(X, B)} > -1$,
- *lc over o (log canonical) if* $\mathrm{discrep_o(X, B)} \geqslant -1$.

Definition 2.5. — *Let (X, B) be an lc pair, D an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. The log canonical threshold of D for (X, B) is*

$$\gamma = \sup\{t \in \mathbb{R}^+ \mid (X, B + tD) \text{ is lc}\}.$$

Definition 2.6. — *Let (X, B) be a lc pair, $\nu\colon X' \to X$ a log resolution. Let $E \subseteq X'$ be a divisor on X' of discrepancy -1. Such a divisor is called a log canonical place. The image $\nu(E)$ is called center of log canonicity of the pair. If we write*

$$K_{X'} \equiv \nu^*(K_X + B) + E,$$

we can equivalently define a place as an irreducible component of $\lfloor -E \rfloor$.

Definition 2.7. — *Let (X, B) be a pair and $\nu\colon X' \to X$ a log resolution of the pair. We set*

$$A(X, B) = K_{X'} - \nu^*(K_X + B)$$

and

$$A(X, B)^* = A(X, B) + \sum_{E \text{ place}} E.$$

Definition 2.8. — *A lc-trivial fibration $f\colon (X, B) \to Z$ consists of a contraction of normal varieties $f\colon X \to Z$ and of a log pair (X, B) satisfying the following properties:*

1. *(X, B) has log canonical singularities over a big open subset $U \subseteq Z$;*
2. $\operatorname{rank} f'_* \mathcal{O}_X(\lceil A^*(X, B) \rceil) = 1$ *where $f' = f \circ \nu$ and ν is a given log resolution of the pair (X, B);*
3. *there exists a positive integer r, a rational function $\varphi \in k(X)$ and a $\mathbb{Q}$-Cartier divisor D on Z such that*

$$K_X + B + \frac{1}{r}(\varphi) = f^*D.$$

Remark 2.9. — The smallest possible r is the minimum of the set

$$\{m \in \mathbb{N} \mid m(K_X + B)|_F \sim 0\}$$

that is the Cartier index of the fibre. We will always assume that the r that appears in the formula is the smallest.

Definition 2.10. — *Let $p \subseteq Z$ be a codimension one point. The log canonical threshold of $f^*(p)$ with respect to the pair (X, B) is*

$$\gamma_p = \sup\{t \in \mathbb{R} \mid (X, B + tf^*(p)) \text{ is lc over } p\}.$$

We define the discriminant *of* $f\colon (X,B)\to Z$ *as*

$$B_Z=\sum_p(1-\gamma_p)p. \tag{2.1}$$

We remark that, since the above sum is finite, B_Z is a $\mathbb{Q}$-Weil divisor.

Remark 2.11. — In what follows we will treat the case where $f\colon X\to Z$ is a $\mathbb{P}^1$-bundle over a smooth curve. We write B as the sum of its vertical part and its horizontal part, $B=B^h+B^v$. Since every fibre of f is irreducible there exists a $\mathbb{Q}$-divisor Δ on Z such that $B^v=f^*\Delta$. This implies that also $f\colon (X,B^h)\to Z$ is an lc-trivial fibration and let B'_Z and M'_Z be its discriminant and moduli part. Then by [2, Remark 3.3] $B_Z=B'_Z+\Delta$ and $M_Z=M'_Z$. Thus we can suppose $B=B^h$. In this case, if we write $B=\sum b_iB_i$, the smallest possible r is the least common multiple of the denominators of the b_i's and for all i

$$b_i\in\frac{1}{r}\mathbb{Z}.$$

Remark 2.12. — Let $f\colon (X,B)\to Z$ be an lc-trivial fibration on a smooth curve and let $o\in Z$ be a point. Let $F=f^*o$ be its fibre. Let $\delta\colon \hat{X}\to X$ be a log resolution of $(X,B+f^*o)$ over o, that is, if E is an exceptional curve of δ then $f(\delta(E))=o$. Then we have

$$\delta^*K_X=K_{\hat{X}}-\sum e_iE_i$$
$$\delta^*F=\tilde{F}+\sum a_iE_i$$
$$\delta^*B=\tilde{B}+\sum \alpha_iE_i$$

The resolution δ is a log-resolution over o also for the pair $(X,B+tF)$ for all t. If $(X,B+tF)$ is lc then by definition for all i

$$-e_i+ta_i+\alpha_i\leqslant 1.$$

Since the coefficient of F has to be less or equal than one, we also have $t\leqslant 1$. Therefore

$$t\leqslant\min\left\{1,\min_i\left\{\frac{1}{a_i}(1+e_i-\alpha_i)\right\}\right\}.$$

Definition 2.13. — *Fix* $\varphi\in\mathbb{C}(X)$ *such that* $K_X+B+\frac{1}{r}(\varphi)=f^*D$. *Then there exists a unique divisor* M_Z *such that we have*

$$K_X+B+\frac{1}{r}(\varphi)=f^*(K_Z+B_Z+M_Z) \tag{2.2}$$

where B_Z *is as in (2.1). The* $\mathbb{Q}$*-Weil divisor* M_Z *is called the* moduli part.

We have the two following results.

THEOREM 2.14. — [2, Theorem 2.5], [4] *Let $f\colon (X,B)\to Z$ be a lc-trivial fibration. Then there exists a proper birational morphism $Z'\to Z$ with the following properties:*

- **(i):** *$K_{Z'}+B_{Z'}$ is a $\mathbb{Q}$-Cartier divisor, and $\nu^*(K_{Z'}+B_{Z'})=K_{Z''}+B_{Z''}$ for every proper birational morphism $\nu\colon Z''\to Z'$.*
- **(ii):** *$M_{Z'}$ is a nef $\mathbb{Q}$-Cartier divisor and $\nu^*(M_{Z'})=M_{Z''}$ for every proper birational morphism $\nu\colon Z''\to Z'$.*

THEOREM 2.15 (Inverse of adjunction). — [1, Proposition 3.4] *Let $f\colon (X,B)\to Z$ be a lc-trivial fibration. Then (Z,B_Z) has klt (lc) singularities in a neighborhood of a point $p\in Z$ if and only if (X,B) has klt (lc) singularities in a neighborhood of $f^{-1}p$.*

The Formula (2.2), with the properties stated in Theorem 2.14 and Theorem 2.15 is called *canonical bundle formula.*

2.2. A useful result on blow-ups on surfaces

Let X be a smooth surface. Let $\delta\colon \hat{X}\to X$ be a sequence of blow-ups, $\delta=\varepsilon_h\circ\ldots\circ\varepsilon_1$ and denote p_i the point blown-up by ε_i. In what follows by abuse of notation we will denote with E_i the exceptional curve of ε_i as well as its birational transform in further blow-ups. **In what follows we will suppose that in** $\mathrm{Exc}(\delta)$ **there is just one** (-1)**-curve.** Since the exceptional curve E_h of ε_h is a (-1)-curve it is the only exceptional curve of $\mathrm{Exc}(\delta)$. Suppose that the first point p_1 that is blown-up belongs to a smooth curve F. We will denote by $\tilde{F}$ the strict transform of F by $\varepsilon_i\circ\ldots\circ\varepsilon_1$ for all i.

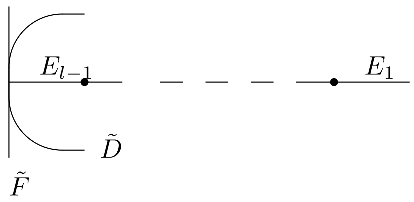
LEMMA 2.16. — *Let $f\colon (X,B)\to Z$ be a $\mathbb{P}^1$-bundle on a smooth curve Z and suppose that $B=(2/d)D$ where D is a reduced divisor such that $DF=d$. Suppose moreover that there is a point $o\in Z$ such that D is tangent to $F=f^*o$ at a smooth point of D with multiplicity $d/2\leqslant l<d$. Then the log canonical threshold*

$$\gamma:=\gamma_o=\sup\{t\in\mathbb{R}|\,((X,B),tf^*o)\text{ is lc over }o\}$$

has the following expression

$$\gamma=1+\frac{1}{l}-\frac{2}{d}.$$

Proof. — A log resolution for the pair $(X, 2/dD + \gamma_o F)$ over o is a sequence of blow-ups $\delta = \varepsilon_l \circ \ldots \circ \varepsilon_1$ such that a picture of the $(l-1)$-th step is



Then

$$\delta^* D = \tilde{D} + \sum_{j=1}^{l} jE_j$$

and we have

$$\delta^*(\frac{2}{d}D) = \frac{2}{d}\tilde{D} + \frac{2}{d}\sum_{j=1}^{l} jE_j.$$

By definition α_l is the coefficient of $\delta^*(2/dD)$ at E_l, and by our computation it is $2l/d$. Since

$$\begin{aligned}\gamma &= \min\{1, \min_{i=1\ldots l}\{1 + \frac{1}{i} - \frac{2}{d}\}\} \\ &= \min\{1, 1 + \frac{1}{l} - \frac{2}{d}\}\end{aligned}$$

we obtain

$$\gamma = 1 + \frac{1}{l} - \frac{2}{d}.$$

□

3. Local results

In this section we will be always in the situation where the fibres have dimension 1. In this case, if $B = 0$ the condition that K_F is torsion implies the generic fibre is an elliptic curve. If $B \neq 0$ then F has to be a rational curve and the second condition in the definition of the lc-trivial fibration implies that the horizontal part of B is effective.

Thanks to the following lemma, studying the denominators of M_Z is the same thing as studying the denominators of B_Z.

Lemma 3.1. — *Let $f\colon (X, B) \to Z$ be an lc-trivial fibration whose general fibre is a rational curve. Then for all $I \in \mathbb{N}$ IrB_Z has integer coefficients if and only if IrM_Z has integer coefficients.*

Proof. — By cutting with sufficiently general hyperplane sections we can assume that $\dim Z = 1$.
We write the canonical bundle formula for $f\colon (X,B) \to Z$:

$$K_X + B + \frac{1}{r}(\varphi) = f^*(K_Z + B_Z + M_Z).$$

Let $\nu\colon \hat{X} \to X$ be a desingularization of X, let $\hat{B}$ be the divisor defined by

$$K_{\hat{X}} + \hat{B} = \nu^*(K_X + B)$$

and $\hat{f} = f \circ \nu$. Then $\hat{f}\colon (\hat{X}, \hat{B}) \to Z$ is lc-trivial and has the same discriminant as f. Moreover it has the same moduli divisor, since

$$K_{\hat{X}} + \hat{B} + \frac{1}{r}(\varphi) = \nu^*(K_X + B) + \frac{1}{r}(\varphi) = \hat{f}^*(K_Z + B_Z + M_Z).$$

The surface $\hat{X}$ is smooth and $\hat{X} \to Z$ has generic fibre $\mathbb{P}^1$ then there exists a birational morphism defined over Z

$$\begin{array}{ccc} \hat{X} & \longrightarrow & X' \\ {\scriptstyle \hat{f}}\downarrow & \swarrow {\scriptstyle f'} & \\ Z & & \end{array}$$

where $f'\colon X' \to Z$ is a $\mathbb{P}^1$-fibration. It follows that each fibre of $\hat{f}$ has an irreducible component with coeffient one. Then the statement follows from the equality

$$r(K_{\hat{X}} + \hat{B}) + (\varphi) = r\hat{f}^*(K_Z + B_Z + M_Z).$$

□

Theorem 3.2. — *Let $f\colon X \to Z$ be a $\mathbb{P}^1$-bundle with $\dim X = 2$. Let $o \in Z$ be a point and γ be the log canonical threshold of f^*o with respect to (X,B). Then there is a constant $m \leqslant 2r^2$ such that $m\gamma$ is integer. Such an m is of the form lr where $l \leqslant 2r$.*

Proof. — The pair $(X, B + \gamma F)$ is lc and not klt, that is, it has an lc centre. There are now two cases.

The centre has dimension one.

If the centre has dimension one, then it is the whole fibre because all the fibres are irreducible. In this case we have

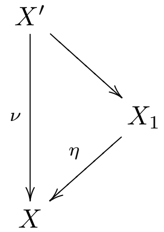
$$1 = \mathrm{mult}_F(B + \gamma F) = \mathrm{mult}_F(B) + \gamma$$

and since $r\mathrm{mult}_F(B) \in \mathbb{Z}$ also $r\gamma \in \mathbb{Z}$.

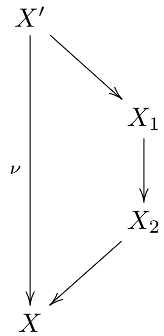
The centre has dimension zero.

Step 1 Take $\nu\colon X' \to X$ a log resolution of $(X, B + \gamma F)$. Notice that the fibre over o is a tree of $\mathbb{P}^1$'s.

Since $(X, B + \gamma F)$ is lc and not klt there is a place appearing between the leaves of the tree. Write ν as a composition of blow-ups, set $\nu = \varepsilon_N \circ \ldots \circ \varepsilon_1$ and let k be the minimum of the indices such that the exceptional curve of ε_k is a place for $(X, B + \gamma F)$, $P = E_k$. Let η be the composition $\varepsilon_k \circ \ldots \circ \varepsilon_1 \colon X_1 \to X$. We have:

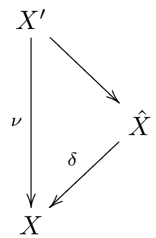


If the only (-1)-curve in X_1 is P then we set $\hat{X} = X_1$ and $\delta := \eta$. Otherwise, if there is another (-1)-curve, by the Castelnuovo's theorem we can contract it in a smooth way:



This process ends because in X' there were finitely many ν-exceptional curves. Then we obtain a smooth surface $\hat{X}$ such that the only (-1)-curve in X is P. We set $\delta \colon \hat{X} \to X$ and write $\delta = \varepsilon_h \circ \ldots \circ \varepsilon_1$.

Step 2 We have obtained $\hat{X}$ smooth with a diagram



where $\hat{X} \to X$ is minimal in order to obtain a log canonical place P which has to be a -1-curve and $\delta = \varepsilon_h \circ \ldots \circ \varepsilon_1$ is a sequence of blow ups. Let p_i be the point blown up by ε_i. Let $\tilde{B}_i^j$ be the strict transform of the component B_i of B at the step j and $\tilde{B}^j$ be the strict transform of B. By abuse of notation we will denote by $\tilde{F}$ the strict transform of F by every ε_i and by E_i the exceptional curve of ε_i as well as its strict transform in the further blow-ups. Notice that $P = E_h$. In what follows we will adopt the following notation:

$$B = \sum b_i B_i;$$

$$\delta^* K_X = K_{\hat{X}} - \sum e_i E_i; \quad \delta^* B = \tilde{B} + \sum \alpha_i E_i; \quad \delta^* F = \tilde{F} + \sum a_i E_i.$$

Here $\tilde{B}$ and $\tilde{F}$ denote the strict transform of B and F. Remark that for all i we have

$$\alpha_i \in \frac{1}{r}\mathbb{Z}. \tag{3.1}$$

Indeed $b_i \in 1/r\mathbb{Z}$ for all i by Remark 2.9. Equation (3.1) follows from the fact that

$$\alpha_1 = \sum_{B_i \ni p_1} b_i \mathrm{mult}_{p_1} B_i$$

and, for $l > 1$, that α_l is a linear combination of the α_j's with $j < l$ plus $\sum_{\tilde{B}_i^{l-1} \ni p_l} b_i \mathrm{mult}_{p_l} \tilde{B}_i^{l-1}$.

Since E_h is a place we have

$$1 = \mathrm{mult}_{E_h}(\delta^*(K_X + B + \gamma F) - K_{\hat{X}}) = -e_h + \alpha_h + \gamma a_h.$$

Since e_h is an integer and $\alpha_h \in 1/r\mathbb{Z}$, if we prove that $a_h \leqslant 2r$ we are done. By the minimality of δ there exists a component B_1 of B such that the strict transform $\tilde{B}_1^h$ of B_1 meets E_h, that is $\tilde{B}_1^h E_h > 0$. Then

$$\begin{aligned} 2r \geqslant B_1 F = \delta^* B_1 \delta^* F = \tilde{B}_1^h \delta^* F = \tilde{B}_1^h (\tilde{F} + \sum a_i E_i) \\ \geqslant a_h \tilde{B}_1^h E_h \geqslant a_h. \end{aligned}$$

□

We can finally prove the main result.

Proof of Theorem 1.5. — The statement in dimension 2 follows from Theorem 3.2 and [2, Lemma 2.6]. Indeed if $X \to Z$ is a fibration whose general fibre is a $\mathbb{P}^1$ and X is smooth, then by the general theory of smooth surfaces there exists a birational morphism $\sigma\colon X \to X'$ where X' is a $\mathbb{P}^1$-bundle. More precisely X' is a minimal model of X that is unique if the genus of Z is positive.

The general result follows from the one in dimension 2 by induction on the dimension of the base. Suppose now that the statement is true in dimension $n-1$ and let $X \to Z$ be a fibration of dimension n. The set

$$\mathcal{S} = \left\{ \begin{array}{l} o \text{ point of } Z \text{ of codimension 1 such that the log canonical} \\ \text{threshold of } f^*o \text{ with respect to } (X,B) \text{ is different from 1} \end{array} \right\}$$

is a finite set.
We fix then a point $o \in \mathcal{S}$. By the Bertini theorem, since Z is smooth, we can find a hyperplane section $H \subseteq Z$ such that

(1) H is smooth;
(2) H intersects o transversally;
(3) H does not contain any intersection $o \cap o'$ where $o' \in \mathcal{S} \backslash \{o\}$.

Set

$$X_H = f^{-1}(H); \;\; f_H = f|_{X_H}; \;\; B_H = B|_{X_H}; \;\; o_H = o \cap H.$$

The restriction $f_H \colon (X_H, B_H) \to H$ is again an lc-trivial fibration. Then the log canonical threshold of $f_H^* o_H$ with respect to (X_H, B_H) is equal to the log canonical threshold of f^*o with respect to (X, B) and the theorem follows from the inductive hypothesis. □

Notice that even if in many cases $m = r$ is sufficient to have that mM_Z has integer coefficients there exist cases in which a greater coefficient is needed.

Example 3.3. — Let $\pi \colon X \to C$ be a $\mathbb{P}^1$-bundle on a curve C. Let $X^0 \to U$ be a local trivialization, where $U \subseteq C$ is an open subset and $X^0 = \pi^{-1}U$. This means that there is a commutative diagram

$$\begin{array}{ccc} X^0 & \xrightarrow{\sim} & U \times \mathbb{P}^1 \\ {\scriptstyle \pi}\downarrow & \swarrow {\scriptstyle p_1} & \\ U. & & \end{array}$$

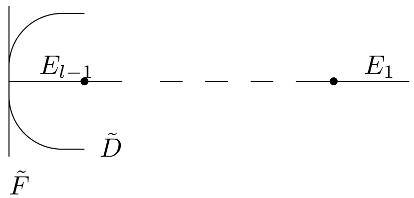
We can furthermore suppose that we have a local coordinate t on U. Let $[x : y]$ be coordinates on $\mathbb{P}^1$. Set

$$D = \{ty^d - x^l y^{d-l} - x^d = 0\} \subseteq U \times \mathbb{P}^1$$

and let $\bar{D}$ be the Zariski closure of D in X.
Consider the pair $(X, 2/d\bar{D})$. Then we have $\deg(K_X + 2/d\bar{D})|_F = 0$ and there exists a rational function φ such that we can write

$$K_X + 2/d\bar{D} + \frac{1}{r}(\varphi) = f^*(K_C + B_C + M_C)$$

where $r = d$ if d is odd and $r = d/2$ if d is even. We want to compute now the coefficient of the divisor B_C at the point $t = 0$. Its coefficient is $1 - \gamma$ where γ is the log canonical threshold of $((X, 2/d\bar{D}), F)$. A log resolution for the pair $(X, 2/d\bar{D})$ over the point $t = 0$ is given by the composition of l blow-ups. At the $(l-1)$-th step the picture is as follows



We call $\delta\colon \hat{X} \to X$ this composition of blow-ups. We have

$$\delta^* K_X = K_{\hat{X}} - \sum_{i=1}^{l} iE_i \quad \delta^* \bar{D} = \tilde{D} + \sum_{i=1}^{l} iE_i \quad \delta^* F = \tilde{F} + \sum_{i=1}^{l} iE_i,$$

where by abuse of notation we denote by E_i the exceptional divisor of the i-th blow-up as well as its strict transforms after the following blow-ups. Thus

$$\delta^*(K_X + 2/d\bar{D} + \gamma F) = K_{\hat{X}} + 2/d\tilde{D} + \gamma\tilde{F} + \sum_{i=1}^{l} i(-1 + \gamma + 2/d)E_i.$$

By Lemma 2.16 we have

$$\gamma = 1 + \frac{1}{l} - \frac{2}{d}.$$

So if we chose $l < d$ and such that $2l > d$, we obtain $\gamma = 1 - \frac{2l-d}{ld}$. For $l = 5$ and $d = 9$ we have $\gamma = 1 - \frac{1}{45} \notin \frac{1}{12r}\mathbb{Z}$ contrary to the Prokhorov and Shokurov expectation.
Notice that this gives us an example also if we take l to be any prime greater or equal to 13 and $d = 2l - 1$.

To prove that the bound stated in Theorem 3.2 is not far from being sharp, we take d even such that $d/2$ is odd and $l = d - 1$. Then $r = d/2$ and

$$\gamma = 1 - \frac{2l-d}{ld} = 1 - \frac{2(2r-1)-2r}{2r^2-r} = 1 - \frac{2(2r-1)-2r}{2r^2-r} = 1 - \frac{2(r-1)}{(2r-1)r}.$$

Since $2(r-1)$ and $(2r-1)r$ are coprime, the smallest integer m such that $m\gamma$ is integer is $m = 2r^2 - r$.

4. Global results

LEMMA 4.1. — *Let $f\colon X \to Z$ be a $\mathbb{P}^1$-bundle on a smooth curve Z. Let $D \subseteq X$ be a reduced divisor such that $f|_D\colon D \to Z$ is a ramified covering of degree d with at least N ramification points $p_1 \dots p_N$ that are smooth points for D. Suppose that d is even. Suppose moreover that the ramification indices $l_1, \dots, l_N$ at $p_1, \dots, p_N$ satisfy the following properties:*

(1) *$2l_i \geqslant d$ for all i;*
(2) *l_i and l_j are coprime for all $i \neq j$;*
(3) *l_i and d are coprime for all i.*

Then

(i): *the fibration*
$$f\colon (X, 2/dD) \to Z$$
is an lc-trivial fibration, in particular there exists a rational function φ such that
$$K_X + \frac{2}{d}D + \frac{1}{r}(\varphi) = f^*(K_Z + M_Z + B_Z).$$
(ii): *The Cartier index of the fibre is $r = d/2$.*
(iii): *Let V be the smallest integer such that VM_Z has integer coefficients.*
Then $V \geqslant r^{N+1}$.

Proof. — The first part of the statement follows easily from the fact the degree of $(K_X + 2/dD)|_F$ is 0. The Cartier index of the fibre is
$$r = \min\{m |\, m(K_X + 2/dD)|_F \text{ is a Cartier divisor}\}.$$
But since F is a smooth rational curve this is
$$r = \min\{m |\, m(K_X + 2/dD)|_F \text{ has integer coefficients}\} = \frac{d}{2}$$
and the second part of the statement is proved. In order to prove the third part of the statement we remark that since D is smooth at p_i and $f|_D$ ramifies at p_i the only possibility is that D is tangent to F at p_i with order of tangency exactly l_i.
Then we can apply Lemma 2.16 and by Equation (3.1) an expression for γ is
$$\gamma = 1 + \frac{1}{l_i} - \frac{2}{d}.$$
Since l_i and d are coprime, $l_i d$ divides V for all i. Again since l_i and l_j are coprime for all $i \neq j$
$$l_1 \dots l_N d \mid V.$$

Since $l_i \geqslant d/2 = r$ for all i we have

$$V \geqslant l_1 \dots l_N d \geqslant 2r^{N+1}.$$

□

Proof of Theorem 1.6 (1). — Let N be a positive integer and $f\colon X \to Z$ be a $\mathbb{P}^1$-bundle on a smooth curve. Let $U \subseteq Z$ be an open set that trivializes the $\mathbb{P}^1$-bundle and such that we have a local coordinate t on it. Take $d, l_1, \dots, l_N \in \mathbb{N}$ be such that

$$l_0 := 0 < l_1 < \dots < l_N < l_{N+1} := d$$

and such that they verify conditions (1)(2)(3) of Lemma 4.1. Let $o_1, \dots, o_N$ be distinct points in U. Let $[u : v]$ be the coordinates on the fibre and $x = u/v$ the local coordinate on the open set $\{v \neq 0\}$. Let D be the Zariski closure in X of

$$D_0 = \left\{ \sum_{k=1}^{N+1} \left((x^{l_{k-1}} + \dots + x^{l_k - 1}) \prod_{i=k}^{N} (t - o_i) \right) \right\}.$$

The restriction of D to the fibre over o_i is the zero locus of a polynomial of the form

$$h_i(x) = x^{l_i} q_i(x)$$

such that x does not divide q_i. Notice that D is smooth at the points $p_i = (0, o_i)$ because the derivative with respect to t of the polynomial that defines D_0 is non-zero at those points. This insures that D is tangent to the fibre $F = f^* o_i$ with multiplicity exactly l_i and then that

$$f|_D\colon D \to Z$$

has ramification index exactely l_i at p_i. The fibration $f\colon (X, 2/dD) \to Z$ satisfies all the hypotheses of Lemma 4.1. Therefore if V is the minimum positive integer such that VM_Z has integer coefficients we have $V \geqslant r^{N+1}$.

□

Proof of Theorem 1.6 (2). — Let $B_Z = \sum b_i o_i$ be the discriminant divisor. Let V be the minimum integer number such that VB_Z has integer coefficients. If we write $b_i = u_i/v_i$ with $u_i, v_i \in \mathbb{N}$ and coprime it is clear that $V = lcm\{v_i\}$. We have seen in the proof of Theorem 3.2 that v_i divides $l_i r$ for some $l_i \leqslant 2r$. Then

$$V = lcm\{v_i\} \mid lcm\{l_i r\}.$$

Let us remark that if q is a prime number such that q^k divides V then there exists a point p such that q^k divides $l_p r$. Let $r = \prod q_i^{k(q_i)}$ be the decomposition of r into prime factors and suppose that q is equal to some prime q_1. We have then that

$$q_1^{k-k(q_1)} \mid l_p \leqslant 2r.$$

Set

$$s(q) = \max\{s \mid q^s \leqslant 2r\}.$$

□

The bound of Theorem 1.6 is not far from being sharp thanks to the following example.

Proof of Theorem 1.6 (3). — Let r be an odd integer number. Let $s(q)$ be the integer defined above. Set

$$h(q) = \max\{h \mid r \leqslant 2^h q^{s(q)} \leqslant 2r\}$$

and set

$$\{l_1 < \ldots < l_N\} = \{2^{h(q)} q^{s(q)} | \; q < 2r, \; q \text{ prime}\},$$
$$l_0 = 0, l_{N+1} = d = 2r.$$

Consider the divisor $\bar{D}$ defined as the Zariski closure of

$$D_0 = \left\{ \sum_{k=1}^{N+1} \left((x^{l_{k-1}} + \ldots + x^{l_k - 1}) \prod_{i=k}^{N} (t - o_i) \right) \right\}.$$

Consider now $B = 1/r\bar{D}$. The fibration $f\colon (X,B) \to Z$ is lc-trivial. Let V be the minimum integer such that VM_Z has integer coefficients.

Then for each $i = 1 \ldots N$ by Lemma 2.16 we have the following expression for γ_i:

$$\gamma_i = 1 - \frac{2l_i - d}{l_i d} = 1 + \frac{r - l_i}{l_i r}.$$

For every i we have $l_i = 2^{h(q)} q^{s(q)}$ for a suitable q. Since r is odd

$$gcd\{2^{h(q)} q^{s(q)}, r\} = q^{s'(q)}$$

for some $s'(q)$, then

$$\gamma_i = 1 - \frac{l_i - r}{l_i r} = 1 + \frac{r/q^{s'(q)} - 2^{h(q)} q^{s(q)-s'(q)}}{2^{h(q)} q^{s(q)-s'(q)} r}.$$

Then for all q such that $q \leqslant 2r$ we have

$$2^{h(q)} q^{s(q)-s'(q)} r | V$$

that implies that

$$lcm\{2^{h(q)} q^{s(q)-s'(q)} r\} | V.$$

But

$$lcm\{2^{h(q)} q^{s(q)-s'(q)} r\} = \frac{N(r)}{r}.$$

□

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