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ON THE MULTIPLICITY OF EIGENVALUES OF CONFORMALLY COVARIANT OPERATORS

by Yaiza CANZANI (*)

Abstract. — Let (M, g) be a compact Riemannian manifold and P_g an elliptic, formally self-adjoint, conformally covariant operator of order m acting on smooth sections of a bundle over M. We prove that if P_g has no rigid eigenspaces (see Definition 2.2), the set of functions $f \in C^\infty(M, \mathbb{R})$ for which $P_{e^f g}$ has only simple non-zero eigenvalues is a residual set in $C^\infty(M, \mathbb{R})$. As a consequence we prove that if P_g has no rigid eigenspaces for a dense set of metrics, then all non-zero eigenvalues are simple for a residual set of metrics in the C^∞-topology. We also prove that the eigenvalues of P_g depend continuously on g in the C^∞-topology, provided P_g is strongly elliptic. As an application of our work, we show that if P_g acts on $C^\infty(M)$ (e.g. GJMS operators), its non-zero eigenvalues are generically simple.

Résumé. — Soit (M, g) une variété riemannienne et P_g un opérateur elliptique, auto-adjoint, covariant conforme d'ordre m agissant sur les sections lisses d'un fibré sur M. Nous montrons que si P_g n'admet pas d'espaces propres rigides (voir Définition 2.2), l'ensemble des fonctions$f \in C^\infty(M, \mathbb{R})$ pour lesquelles $P_{e^f g}$ n'admet que des valeurs propres non nulles est un ensemble résiduel dans $C^\infty(M, \mathbb{R})$. Ce résultat a comme conséquence que si P_g n'admet pas d'espaces propres rigides pour un ensemble dense de métriques, alors toutes les valeurs propres non nulles sont simples pour un ensemble résiduel de métriques dans la topologie C^∞. Nous montrons également que les valeurs propres de P_g dependent continûment de g dans la topologie C^∞ si P_g est fortement elliptique. Comme applications de nos résultats, nous montrons que si P_g agit sur $C^\infty(M)$, comme dans le cas des opérateurs GJMS, alors les valeurs propres non-nulles de cet opérateur sont génériquement simples.

1. Introduction

Conformally covariant operators (see Definition 3.1) are known to play a key role in Physics and Spectral Geometry. In the past few years, much work has been done on their systematic construction, understanding, and

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classification [3, 8, 6, 7, 17, 20, 22, 29, 33]. In Physics, the interest for conformally covariant operators started when Bateman [4] discovered that the classical field equations describing massless particles (like Maxwell and Dirac equations) depend only on the conformal structure. These operators are also important tools in String Theory and Quantum Gravity, used to relate scattering matrices on conformally compact Einstein manifolds with conformal objects on their boundaries at infinity [23]. In Spectral Geometry, the purpose is to relate global geometry to the spectrum of some natural operators on the manifold. For example, the nice behavior of conformally invariant operators with respect to conformal deformations of a metric yields a closed expression for the conformal variation of the determinants leading to important progress in the lines of [8, 9, 15].

When it comes to perturbing a metric to deal with any of the problems described above, it is often very helpful and simplifying to work under the assumption that the eigenvalues of a given operator are a smooth, or even continuous, function of a metric perturbation parameter. But reality is much more complicated, and usually, when possible, one has to find indirect ways of arriving to the desired results without such assumption. However, it is generally believed that eigenvalues of formally self-adjoint operators with positive leading symbol on $SO(m)$ or $Spin(m)$ irreducible bundles are generically simple. And, as Branson and Ørsted point out in [14, pag 22], since many of the quantities of interest are universal expressions, the generic case is often all that one needs. In many cases, it has been shown that the eigenvalues of *metric dependent*, formally self-adjoint, elliptic operators are generically simple. The main example is the *Laplace* operator on smooth functions on a compact manifold, see [25, 31, 2, 5]. The simplicity of eigenvalues has also been shown, generically, for the *Hodge-Laplace* operator on forms on a compact manifold of dimension 3 (see [18]). Besides, in 2002, Dahl proved such result for the Dirac operator on spinors of a compact spin manifold of dimension 3; see [16]. It seems to be the case that in the class of conformally covariant operators the latter is the only situation for which the simplicity of the eigenvalues has generically been established. In this note we hope to shed some light in this direction.

A summary of the main results follows. Let (M, g) be a compact Riemannian manifold and E_g a smooth bundle over M. Consider $P_g : \Gamma(E_g) \to \Gamma(E_g)$ to be an elliptic, formally self-adjoint, conformally covariant operator of order m acting on smooth sections of the bundle E_g. Endow the space $\mathcal{M}$ of Riemannian metrics over M with the C^∞-topology. Among the main results are:

- Suppose $P_g : \Gamma(E_g) \to \Gamma(E_g)$ *has no rigid eigenspaces* (see Definition 2.2). Then the set of functions $f \in C^\infty(M, \mathbb{R})$ for which $P_{e^f g}$ has only simple non-zero eigenvalues is a residual set in $C^\infty(M, \mathbb{R})$. As a corollary we prove that if P_g has no rigid eigenspaces for a dense set of metrics, then all non-zero eigenvalues are simple for a residual set of metrics in $\mathcal{M}$.
- Suppose $P_g : \Gamma(E_g) \to \Gamma(E_g)$ satisfies the unique continuation principle for a dense set of metrics in $\mathcal{M}$. Then the multiplicity of all non-zero eigenvalues is smaller than the rank of the bundle for a residual set of metrics in $\mathcal{M}$.
- As an application, if P_g acts on $C^\infty(M)$ (e.g. GJMS operator), its non-zero eigenvalues are simple for a residual set of metrics in $\mathcal{M}$.
- If $P_g : \Gamma(E_g) \to \Gamma(E_g)$ is strongly elliptic, then the eigenvalues of P_g depend continuously on g in the C^∞-topology of metrics.

Not many statements can be proved simultaneously for all conformally covariant operators, even if self-adjointness and ellipticity are enforced. Some of these operators act on functions, others act on bundles. For some of them the maximum principle is satisfied, whereas for others is not. Some of them are bounded below while others are not. We would therefore like to emphasize that we find remarkable that our techniques work for the whole class of conformally covariant operators.

Acknowledgements

One of the mains results of the paper, Corollary 2.4, is a generalization to the whole class of conformally covariant operators of the results presented by M. Dahl in [16], for the Dirac operator in 3-manifolds. In an earlier version of this manuscript, part of the argument in M. Dahl's paper was being reproduced (namely, the first two lines of the proof of Proposition 3.2 in [16]). At the end of June 2012, the author started working with Raphaël Ponge on an extension of the results in this paper to the class of pseudodifferential operators. While doing so, R. Ponge realized there was a mistake in M. Dahl's argument which was reproduced in an earlier version of this manuscript. The author is grateful to Raphaël Ponge for pointing out the mistake and for providing useful suggestions on the manuscript.

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2. Statement of the results

In order to provide a precise description of our results, we introduce the following

Conventions. Let (M,g) denote a compact Riemannian manifold and consider a smooth bundle E_g over M with product on the fibers $(\, , \,)_x$. Write $\Gamma(E_g)$ for the space of smooth sections and denote by $\langle \, , \, \rangle_g$ the global inner product $\langle u,v\rangle_g = \int_M (u(x),v(x))_x \mathrm{dvol_g}$, for $u,v \in \Gamma(E_g)$.
A differential operator $P_g : \Gamma(E_g) \to \Gamma(E_g)$ is said to be *formally self-adjoint* if for all $u,v \in \Gamma(E_g)$ we have $\langle P_g u,v\rangle_g = \langle u, P_g v\rangle_g$. Let σ_{P_g} denote the principal symbol of P_g and let m be the order of P_g. We say that P_g is *elliptic* if $\sigma_{P_g}(\xi) : (E_g)_x \to (E_g)_x$ is an invertible map for all $(x,\xi) \in T^*M$, $\xi \neq 0$. For a definition of a conformally covariant operator see Definition 3.1.

Throughout this paper we work under the following assumptions:

- M is a compact differentiable manifold, g is a Riemannian metric over M and E_g denotes a smooth bundle over M as described above.
- $P_g : \Gamma(E_g) \to \Gamma(E_g)$ **is an elliptic, formally self-adjoint, conformally covariant operator of order** m.
- The space $\mathcal{M}$ of Riemannian metrics over M, is endowed with the C^∞-topology: Fix a background metric g, and define the distance d_g^m between two metrics g_1, g_2 by
$$d_g^m(g_1,g_2) := \max_{k=0,\dots,m} \|\nabla_g^k (g_1 - g_2)\|_\infty.$$
The topology induced by d_g^m is independent of the background metric and it is called the C^m-topology of metrics on M.

There are many ways of splitting the spectrum of an operator. The main ideas in this paper are inspired by the constructive methods of Bleecker and Wilson [5]. In what follows the main results of this paper are stated.

Theorem 2.1. — *For $P_g : C^\infty(M) \to C^\infty(M)$, the set of functions $f \in C^\infty(M,\mathbb{R})$ for which all the non-zero eigenvalues of $P_{e^f g}$ are simple is a residual set in $C^\infty(M,\mathbb{R})$.*

To obtain a generalization of Theorem 2.1 for operators acting on bundles we introduce the following

Definition 2.2. — *An eigenspace of $P_g : \Gamma(E_g) \to \Gamma(E_g)$ is said to be* a rigid eigenspace *if it has dimension greater or equal than two, and for any two eigensections u,v with $\|u\|_g = \|v\|_g = 1$ then*
$$\|u(x)\|_x = \|v(x)\|_x \qquad \forall x \in M.$$

Remark. — By the polarization identity this condition is equivalent to the existence of a function c_g on M so that for all u, v in the eigenspace

$$(u(x), v(x))_x = c_g(x)\langle u, v\rangle_g \qquad \forall x \in M.$$

In this setting, we establish the following

THEOREM 2.3. — *If $P_g : \Gamma(E_g) \to \Gamma(E_g)$ has* no *rigid eigenspaces, the set of functions $f \in C^\infty(M, \mathbb{R})$ for which all the non-zero eigenvalues of $P_{e^f g}$ are simple is a residual set in $C^\infty(M, \mathbb{R})$.*

As a consequence of Theorem 2.3 (or Theorem 2.1) we prove

COROLLARY 2.4. — *Suppose $P_g : \Gamma(E_g) \to \Gamma(E_g)$ has no rigid eigenspaces for a dense set of metrics in $\mathcal{M}$, or that it acts on $C^\infty(M)$. Then, the set of metrics $g \in \mathcal{M}$ for which all non-zero eigenvalues of P_g are simple is a residual subset of $\mathcal{M}$.*

Of course, one would like to get rid of the "non rigidity" assumption. Probably, this assumption cannot be dropped if we restrict ourselves to work with conformal deformations only. However, we believe that a generic set of deformations should break the rigidity condition. We thereby make the following

CONJECTURE. — *P_g has no rigid eigenspaces for a dense set of metrics in $\mathcal{M}$.*

If we remove the "non rigidity" condition and ask the operator to satisfy the unique continuation principle we obtain

THEOREM 2.5. — *If $P_g : \Gamma(E_g) \to \Gamma(E_g)$ satisfies the unique continuation principle, the set of functions $f \in C^\infty(M, \mathbb{R})$ for which all the non-zero eigenvalues of $P_{e^f g}$ have multiplicity smaller than $rank(E_g)$ is a residual set in $C^\infty(M, \mathbb{R})$.*

We note that for line bundles the unique continuation principle gives simplicity of eigenvalues, for a generic set of conformal deformations.

COROLLARY 2.6. — *Suppose $P_g : \Gamma(E_g) \to \Gamma(E_g)$ satisfies the unique continuation principle for a dense set of metrics in $\mathcal{M}$. Then, the set of metrics $g \in \mathcal{M}$ for which all non-zero eigenvalues of P_g have multiplicity smaller than the rank of the bundle is a residual subset of $\mathcal{M}$.*

For $c \in \mathbb{R}$, consider the set $\mathcal{M}_c := \{g \in \mathcal{M} :\ c \notin \mathrm{Spec(P_g)}\}$. For $g \in \mathcal{M}_c$, let

$$\mu_1(g) \leqslant \mu_2(g) \leqslant \mu_3(g) \leqslant \dots$$

be all the eigenvalues of P_g in $(c, +\infty)$ counted with multiplicity. Note that it may happen that there are only finitely many $\mu_i(g)$'s. We prove

THEOREM 2.7. — *The set $\mathcal{M}_c$ is open, and the maps*

$$\mu_i : \mathcal{M}_c \to \mathbb{R} \qquad g \mapsto \mu_i(g)$$

are continuous in the C^∞-topology of metrics.

If P_g is strongly elliptic, its spectrum is bounded below. It can be shown [26, (7.14) Appendix] that for a fixed metric g_0 there exists $c \in \mathbb{R}$ and a neighborhood $\mathcal{V}$ of g_0 so that $\mathrm{Spec}(\mathrm{P_g}) \subset (\mathrm{c}, +\infty)$ for all $g \in \mathcal{V}$. An immediate consequence is

COROLLARY 2.8. — *If $P_g : \Gamma(E_g) \to \Gamma(E_g)$ is strongly elliptic, then its eigenvalues are continuous for $g \in \mathcal{M}$ in the C^∞-topology.*

Two important remarks:

- If $P : \Gamma(E) \to \Gamma(E)$ is an elliptic, formally self-adjoint operator acting on a smooth bundle over compact manifold, its eigenvalues are real and discrete. In addition, there is an orthonormal basis of $\Gamma_{L^2}(E)$ of eigensections of P.
- All the results stated above hold for *non-zero* eigenvalues. Given a non-zero eigenvalue of multiplicity greater than 1, we use conformal transformations of the reference metric to reduce its multiplicity. This cannot be done for zero eigenvalues for their multiplicity, $\dim\ker(P_g)$, is a conformal invariant.

The rest of the paper is organized as follows. In Section 3 we define conformally covariant operators and provide examples of operators to which our results can be applied. In Section 4 we introduce the tools of perturbation theory that we shall need to split non-zero eigenvalues when they are not simple. In Section 5 we adapt the results in perturbation theory to our class of operators and find necessary conditions to split the non-zero eigenvalues. In Section 6 we prove Theorems 2.1, 2.3 and 2.5. In Section 7 we prove Corollary 2.4, Corollary 2.6, and Theorem 2.7.

3. Conformally Covariant Operators: definition and examples

Next we provide examples of well known operators to which our results can be applied. Let g be a Riemannian metric over M and $P_g : C^\infty(M) \to C^\infty(M)$ a (metric dependent) differential operator of order m.

We say that P_g is *conformally covariant of bidegree* (a,b) if for any conformal perturbation of the reference metric, $g \to e^f g$ with $f \in C^\infty(M,\mathbb{R})$, the operators $P_{e^f g}$ and P_g are related by the formula

$$P_{e^f g} = e^{-\frac{bf}{2}} \circ P_g \circ e^{\frac{af}{2}}.$$

If we want to consider operators acting on vector bundles the definition becomes more involved. Let M be a compact manifold (possibly with orientation and spin structure), and E_g a vector bundle over M equipped with a bundle metric.

Definition 3.1. — *Let* $a,b \in \mathbb{R}$*. A* conformally covariant operator P of order m and bidegree (a,b) *is a map that to every Riemannian metric* g *over* M *associates a differential operator* $P_g : \Gamma(E_g) \to \Gamma(E_g)$ *of order* m*, in such a way that*

A) *For any two conformally related metrics,* g *and* $e^f g$ *with* $f \in C^\infty(M,\mathbb{R})$*, there exists a bundle isomorphism*

$$\kappa : E_{e^f g} \to E_g$$

that preserves length fiberwise and for which

$$P_{e^f g} = \kappa^{-1} \circ e^{-\frac{bf}{2}} \circ P_g \circ e^{\frac{af}{2}} \circ \kappa, \tag{3.1}$$

B) *The coefficients of* P_g *depend continuously on* g *in the* C^∞*-topology of metrics (see Definition 3.2).*

For a more general formulation see [1, pag. 4]*. It is well known that for all these operators one always has* $a \neq b$*.*

Definition 3.2. — *The coefficients of a differential operator* $P_g : \Gamma(E_g) \to \Gamma(E_g)$ *are said to* depend continuously *on* g *in the* C^k*-topology of metrics if the following is satisfied: every metric* g_0 *has a neighborhood* $\mathcal{W}$ *in the* C^k*-topology of metrics so that for all* $g \in \mathcal{W}$ *there is an isomorphism of vector bundles* $\tau_g : E_g \to E_{g_0}$ *with the property that the coefficients of the differential operator*

$$\tau_g \circ P_g \circ \tau_g^{-1} : \Gamma(E_{g_0}) \to \Gamma(E_{g_0})$$

depend continuously on g*.*

We proceed to introduce some examples of operators to which our results can be applied; see [1, pag 5], [13, pag 253], and [32, pag 285] for more.

Conformal Laplacian. On surfaces, the most common example is the Laplace operator Δ_g having bidegree $(0,2)$. In higher dimensions its generalization is the second order, elliptic operator, named Conformal Laplacian, $P_{1,g} = \Delta_g + \frac{n-2}{4(n-1)} R_g$ acting on $C^\infty(M)$. Here $\Delta_g = \delta_g d$ and R_g is the scalar curvature. $P_{1,g}$ is a conformally covariant operator of bidegree $\left(\frac{n-2}{2}, \frac{n+2}{2}\right)$.

Paneitz Operator. On compact 4 dimensional manifolds, Paneitz discovered the 4th order, elliptic operator $P_{2,g} = \Delta_g^2 + \delta_g(\frac{2}{3} R_g\, g - 2Ric_g)d$ acting on $C^\infty(M)$. Here Ric_g is the Ricci tensor of the metric g and both Ric_g and g are acting as $(1,1)$ tensors on 1-forms. $P_{2,g}$ is a formally self-adjoint, conformally covariant operator of bidegree $(0,4)$. See [29].

GJMS Operators. In general, on compact manifolds of dimension n even, if m is a positive integer such that $2m \leqslant n$, Graham-Jenne-Mason-Sparling constructed formally self-adjoint, elliptic, conformally covariant operators $P_{m,g}$, acting on $C^\infty(M)$ with leading order term given by Δ^m. $P_{m,g}$ is a conformally covariant operator of order $2m$ and bidegree $\left(\frac{n-2m}{2}, \frac{n+2m}{2}\right)$ that generalizes the Conformal Laplacian and the Paneitz operator to higher even orders. See [22].

Dirac Operator. Let (M,g) be a compact Riemannian spin manifold. Denote its spinor bundle by E_g and write γ for the fundamental tensor-spinor. Let ∇ be the connection defined as the natural extension of the Levi-Civita connection on TM to tensor-spinors of arbitrary type. The Dirac Operator $\nabla\!\!\!\!/\,_g$ is, up to normalization, the operator on $\Gamma(E_g)$ defined by $\nabla\!\!\!\!/\,_g = \gamma^\alpha \nabla_\alpha$. The Dirac operator is formally self-adjoint, conformally covariant, elliptic operator of order 1 and bidegree $\left(\frac{n-1}{2}, \frac{n+1}{2}\right)$. See [19, pag. 9] or [24].

Rarita-Schwinger Operator. In the setting of the previous example, let T_g denote the twistor bundle. The Rarita-Schwinger operator $\mathcal{S}_g^0$ acting on $\Gamma(T_g)$ is defined by $u \to \gamma^\beta \nabla_\beta u_\alpha - \frac{2}{n} \gamma_\alpha \nabla^\beta u_\beta$, where n is the dimension of M. $\mathcal{S}_g^0$ is an order 1, elliptic, formally self-adjoint, conformally covariant operator of bidegree $\left(\frac{n-1}{2}, \frac{n+1}{2}\right)$. See [11].

Conformally Covariant Operators on forms. In 1982 Branson introduced a general second order conformally covariant operator $D_{(2,k),g}$ on differential forms of arbitrary order k and leading order term $(n-2k+2)\delta_g d + (n-2k-2)d\delta_g$ for $n \neq 1,2$ being the dimension of the manifold. Later he generalized it to a four order operator $D_{(4,k)}, g$ with leading order term $(n-2k+4)(\delta_g d)^2 + (n-2k-4)(d\delta_g)^2$ for $n \neq 1,2,4$. Both $D_{(2,k)}, g$ and $D_{(4,k)}, g$ are formally self-adjoint, conformally covariant operators and their leading symbols are positive provided $k < \frac{n-2}{2}$ and $k < \frac{n-4}{2}$ respectively. On functions, $D_{(2,0)}, g = P_{1,g}$ and $D_{(4,k)}, g = P_{2,g}$. See [12, pag 276], [13, pag 253]. For recent results and higher order generalizations see [10] and [21].

4. Background on perturbation theory

In this section we introduce the definitions and tools we need to prove our main results. We follow the presentation in Rellich's book [30], and a proof for every result stated can be found there.

Let $\mathcal{H}$ be a Hilbert space with inner product $\langle\, , \rangle$ and $\mathcal{U}$ a dense subspace of $\mathcal{H}$. A linear operator A on $\mathcal{U}$ is said to be *formally self-adjoint*, if it satisfies $\langle Au, v \rangle = \langle u, Av \rangle$ for all $u, v \in \mathcal{U}$. A formally self-adjoint operator A is said to be *essentially self-adjoint* if the images of $A + i$ and $A - i$ are dense in $\mathcal{H}$; if these images are all of $\mathcal{H}$ we say that A is *self-adjoint*.

If A is a linear operator on $\mathcal{U}$, its *closure* is the operator $\bar{A}$ defined on $\overline{\mathcal{U}}$ as follows: $\overline{\mathcal{U}}$ is the set of elements $u \in \mathcal{H}$ for which there exists a sequence $\{u_n\} \subset \mathcal{U}$ with $\lim_n u_n = u$ and Au_n converges. Then $\bar{A}u := \lim_n Au_n$. We note that if A is formally self-adjoint, so is $\bar{A}$.

A family of linear operators $A(\varepsilon)$ on $\mathcal{U}$ indexed by $\varepsilon \in \mathbb{R}$ is said to be *regular* in a neighborhood of $\varepsilon = 0$ if there exists a bounded bijective operator $U : \mathcal{H} \to \mathcal{U}$ so that for all $v \in \mathcal{H}$, $A(\varepsilon)[U(v)]$ is a regular element, in the sense that it is a power series convergent in a neighborhood of $\varepsilon = 0$. Finding the operator U is usually very difficult. Under certain conditions on the operators, proving regularity turns out to be much simpler. To this end, we introduce the following criterion.

Criterion 4.1. — ([30, page 78]) *Suppose that $A(\varepsilon)$ on $\mathcal{U}$ is a family of formally self-adjoint operators in a neighborhood of $\varepsilon = 0$. Suppose that $A^{(0)} = A(0)$ is essentially self-adjoint, and there exist formally self-adjoint operators $A^{(1)}, A^{(2)}, \dots$ on $\mathcal{U}$ such that for all $u \in \mathcal{U}$*

$$A(\varepsilon)u = A^{(0)}u + \varepsilon A^{(1)}u + \varepsilon^2 A^{(2)}u + \dots$$

Assume in addition that there exists $a \geqslant 0$ so that

$$\|A^{(k)}u\| \leqslant a^k \|A^{(0)}u\|, \quad \text{for all } k = 1, 2, \dots$$

Then, on $\mathcal{U}$, $A(\varepsilon)$ is essentially self-adjoint and $\overline{A(\varepsilon)}$ on $\overline{\mathcal{U}}$ is regular in a neighborhood of $\varepsilon = 0$.

For the purpose of splitting non-zero eigenvalues, next proposition plays a key role.

Proposition 4.2. — ([30, page 74]) *Suppose that $B(\varepsilon)$ on $\mathcal{U}$ is a family of regular, formally self-adjoint operators in a neighborhood of $\varepsilon = 0$. Suppose that $B^{(0)} = B(0)$ is self-adjoint. Suppose that λ is an eigenvalue of finite multiplicity ℓ of the operator $B(0)$, and suppose there are positive numbers d_1, d_2 such that the spectrum of $B(0)$ in $(\lambda - d_1, \lambda + d_2)$ consists exactly of the eigenvalue λ.*

Then, there exist power series $\lambda_1(\varepsilon), \dots, \lambda_\ell(\varepsilon)$ convergent in a neighborhood of $\varepsilon = 0$ and power series $u_1(\varepsilon), \dots, u_\ell(\varepsilon)$, satisfying

(1) *$u_i(\varepsilon)$ converges for small ε in the sense that the partial sums converge in $\mathcal{H}$ to an element in $\mathcal{U}$, for $i = 1 \dots \ell$.*
(2) *$B(\varepsilon)u_i(\varepsilon) = \lambda_i(\varepsilon)u_i(\varepsilon)$ and $\lambda_i(0) = \lambda$, for $i = 1, \dots, \ell$.*
(3) *$\langle u_i(\varepsilon), u_j(\varepsilon)\rangle = \delta_{ij}$, for $i, j = 1, \dots, \ell$.*
(4) *For each pair of positive numbers d_1', d_2' with $d_1' < d_1$ and $d_2' < d_2$, there exists a positive number δ such that the spectrum of $B(\varepsilon)$ in $[\lambda - d_1', \lambda + d_2']$ consists exactly of the points $\lambda_1(\varepsilon), \dots, \lambda_\ell(\varepsilon)$, for $|\varepsilon| < \delta$.*

We note that since $B(\varepsilon)u_i(\varepsilon) = \lambda_i(\varepsilon)u_i(\varepsilon)$, differentiating with respect to ε both sides of the equality we obtain

$$\langle B^{(1)}(\varepsilon)u_i(\varepsilon), u_j(\varepsilon)\rangle + \langle u_i'(\varepsilon), B(\varepsilon)u_j(\varepsilon)\rangle \\ = \langle \lambda_i'(\varepsilon)u_i(\varepsilon), u_j(\varepsilon)\rangle + \langle u_i'(\varepsilon), \lambda_i(\varepsilon)u_j(\varepsilon)\rangle.$$

When $i = j$ the above equality translates to

$$\lambda_i'(\varepsilon) = \langle B^{(1)}(\varepsilon)u_i(\varepsilon), u_i(\varepsilon)\rangle. \tag{4.1}$$

Also, evaluating at $\varepsilon = 0$ we get

$$\lambda_i'(0) = \langle B^{(1)}u_i(0), u_j(0)\rangle\delta_{ij}. \tag{4.2}$$

5. Perturbation theory for Conformally Covariant operators

Consider a conformal change of the reference metric $g \to e^{\varepsilon f} g$ for $f \in C^\infty(M)$ and $\varepsilon \in \mathbb{R}$. Since $P_g : \Gamma(E_g) \to \Gamma(E_g)$ is a *conformally covariant operator* of bidegree (a, b), there exists $\kappa : E_{e^{\varepsilon f} g} \to E_g$, a bundle isomorphism that preserves the length fiberwise, so that

$$P_{e^{\varepsilon f} g} = \kappa^{-1} \circ e^{-\frac{b\varepsilon f}{2}} \circ P_g \circ e^{\frac{a\varepsilon f}{2}} \circ \kappa.$$

We work with a modified version of $P_{e^{\varepsilon f}g}$. For $c := \frac{a+b}{4}$ set

$$\eta := c - \frac{b}{2} = \frac{a}{2} - c$$

and define

$$A_f(\varepsilon) : \Gamma(E_g) \to \Gamma(E_g), \qquad A_f(\varepsilon) := e^{\eta\varepsilon f} \circ P_g \circ e^{\eta\varepsilon f}.$$

The advantage of working with these operators is that, unlike $P_{e^{\varepsilon f}g}$, they are formally self-adjoint with respect to $\langle\, ,\, \rangle_g$. Note that $\eta \neq 0$ for $a \neq b$, and observe that

$$\begin{aligned} A_f(\varepsilon) &= e^{\eta\varepsilon f} \circ P_g \circ e^{\eta\varepsilon f} \\ &= e^{c\varepsilon f} e^{-\frac{b\varepsilon f}{2}} \circ P_g \circ e^{\frac{a\varepsilon f}{2}} e^{-c\varepsilon f} \\ &= \kappa \circ e^{c\varepsilon f} \circ P_{e^{\varepsilon f}g} \circ e^{-c\varepsilon f} \circ \kappa^{-1}. \end{aligned}$$

Remark 5.1. — $A_f(\varepsilon)$ and $P_{e^{\varepsilon f}g}$ have the same eigenvalues. Indeed, $u(\varepsilon)$ is an eigensection of $P_{e^{\varepsilon f}g}$ with eigenvalue $\lambda(\varepsilon)$ if and only if $\kappa(e^{c\varepsilon f}u(\varepsilon))$ is an eigensection for $A_f(\varepsilon)$ with the same eigenvalue.

$A_f(\varepsilon)$ is a deformation of $P_g = A_f(0)$ that has the same spectrum as $P_{e^{\varepsilon f}g}$ and is formally self-adjoint with respect to $\langle,\rangle_g$. Note also that $A_f(\varepsilon)$ is elliptic so there exists a basis of $\Gamma_{L^2}(E_g)$ of eigensections of $A_f(\varepsilon)$.

Lemma 5.2. — *With the notation of Criterion 4.1, the operators* $A_f^{(k)}(\varepsilon) := \frac{1}{k!}\frac{d^k}{d\varepsilon^k}A_f(\varepsilon)$ *are formally self-adjoint and*

$$\left\| A_f^{(k)}(\varepsilon)u \right\|_g \leqslant \frac{\left(2\,|\eta|\,\|f\|_\infty\right)^k}{k!}\,\|A_f(\varepsilon)u\|_g \qquad \forall u \in \Gamma(E_g). \tag{5.1}$$

Proof. — Since $A_f(\varepsilon)$ is formally self-adjoint, so is $A_f^{(k)}(\varepsilon)$. Indeed, for $u, v \in \Gamma(E_g)$, $\langle A_f(\varepsilon)u, v\rangle_g - \langle u, A_f(\varepsilon)v\rangle_g = 0$. Hence, $0 = \frac{d^k}{d\varepsilon^k}(\langle A_f(\varepsilon)u, v\rangle_g - \langle u, A_f(\varepsilon)v\rangle_g)\big|_{\varepsilon=0} =$
$= k!(\langle A_f^{(k)}u, v\rangle_g - \langle u, A_f^{(k)}v\rangle_g)$. For the norm bound, observe that

$$\frac{d^k}{d\varepsilon^k}\Big[A_f(\varepsilon)(u)\Big] = \eta^k \sum_{l=0}^{k} \binom{k}{l} f^{k-l} A_f(\varepsilon)(f^l u), \tag{5.2}$$

and notice that from the fact that $A_f(\varepsilon)$ is formally self-adjoint it also follows that $\|A_f(\varepsilon)(hu)\|_g \leqslant \|h\|_\infty \|A_f(\varepsilon)u\|_g$ for all $h \in C^\infty(M)$. □

In the following Proposition we show how to split the multiple eigenvalues of P_g. From now on we write $A_f^{(k)} := A_f^{(k)}(0)$.

Proposition 5.3. — *Suppose* λ *is a non-zero eigenvalue of* P_g. *Let* V_λ *be the eigenspace of eigenvalue* λ *and* Π *the orthogonal projection onto it. With the notation of Proposition 4.2, if* $\Pi \circ A_f^{(1)}|_{V_\lambda}$ *is not a multiple of the identity, there exists* $\varepsilon_0 > 0$ *and a pair* (i, j) *for which* $\lambda_i(\varepsilon) \neq \lambda_j(\varepsilon)$ *for all* $0 < \varepsilon < \varepsilon_0$.

Proof. — Assume the results of Proposition 4.2 are true for $B(\varepsilon) = \overline{A_f(\varepsilon)}$, and note that for there is a basis of $\Gamma_{L^2}(E_g)$ of eigensections of $A_f(\varepsilon)$, the eigensections of $\overline{A_f(\varepsilon)}$ and $A_f(\varepsilon)$ coincide. By relation (4.2), $\lambda_1'(0), \dots, \lambda_\ell'(0)$ are the eigenvalues of $\Pi \circ A_f^{(1)}|_{V_\lambda}$. Since $\Pi \circ A_f^{(1)}|_{V_\lambda}$ is not a multiple of the identity, there exist i, j with $\lambda_i'(0) \neq \lambda_j'(0)$ and this implies that $\lambda_i(\varepsilon) \neq \lambda_j(\varepsilon)$ for small ε, which by Remark 5.1 is the desired result. We therefore proceed to show that all the assumptions in Proposition 4.2 are satisfied for $B(\varepsilon) = \overline{A_f(\varepsilon)}$, $U = \overline{\Gamma(E_g)}$ and $\mathcal{H} = \Gamma_{L^2}(E_g)$.

$\overline{A_f(0)} = \overline{P_g}$ *is self-adjoint:* This follows from the fact that P_g is essentially self-adjoint, and the closure of an essentially self-adjoint is a self-adjoint operator. To see that $A_f(0) = P_g$ is essentially self-adjoint, note that since there is a basis of $\Gamma_{L^2}(E_g)$ of eigensections of P_g, it is enough to show that for any eigensection ϕ of eigenvalue λ there exist $u, v \in \Gamma(E_g)$ for which $P_g u + iu = \phi$ and $P_g v - iv = \phi$. Thereby, it suffices to set $u = \frac{1}{\lambda+i}\phi$ and $v = \frac{1}{\lambda-i}\phi$.

$\overline{A_f(\varepsilon)}$ *is regular on* $\Gamma(E_g)$*:* From Lemma 5.2 and Criterion 4.1 applied to $A(\varepsilon) = A_f(\varepsilon)$, we obtain that $A_f(\varepsilon)$ is a family of operators on $\Gamma(E_g)$ which are essentially self-adjoint and their closure $\overline{A_f(\varepsilon)}$ on $\overline{\Gamma(E_g)}$ is regular. □

5.1. Splitting eigenvalues

Recall from Definition 2.2 that an eigenspace of P_g is said to be a *rigid eigenspace* if it has dimension greater or equal than two, and for any two eigensections u, v with $\|u\|_g = \|v\|_g = 1$ one has

$$\|u(x)\|_x = \|v(x)\|_x \qquad \forall x \in M.$$

Being an operator with *no* rigid eigenspaces is the condition that will allow us to split eigenvalues. For this reason, at the end of this section we show that operators acting on $C^\infty(M)$ have no rigid eigenspaces (see Proposition 5.6).

Our main tool is the following

Proposition 5.4. — *Suppose P_g has no rigid eigenspaces. Let λ be a non-zero eigenvalue of P_g of multiplicity $\ell \geqslant 2$. Then, there exists a function $f \in C^\infty(M,\mathbb{R})$ and $\varepsilon_0 > 0$ so that among the perturbed eigenvalues $\lambda_1(\varepsilon), \dots, \lambda_\ell(\varepsilon)$ of $P_{e^{\varepsilon f} g}$ there exists a pair (i,j) for which $\lambda_i(\varepsilon) \neq \lambda_j(\varepsilon)$ for all $0 < \varepsilon < \varepsilon_0$.*

Proof. — Since P_g has no rigid eigenspaces, there exist $u, v \in \Gamma(E_g)$ linearly independent normalized eigensections in the λ-eigenspace so that $\|u(x)\|_x^2 \neq \|v(x)\|_x^2$ for some $x \in M$. For such sections there exists $f \in C^\infty(M,\mathbb{R})$ so that $\langle fu, u\rangle_g \neq \langle fv, v\rangle_g$. To prove our result, by Proposition 5.3 it would suffice to show that

$$\langle A_f^{(1)} u, u\rangle_g \neq \langle A_f^{(1)} v, v\rangle_g.$$

Using that P_g is formally self-adjoint and evaluating equation (5.2) at $\varepsilon = 0$ (for k=1) we have

$$\langle A_f^{(1)} u, u\rangle_g = \eta\, \langle f P_g(u) + P_g(fu), u\rangle_g = 2\eta\, \lambda \langle fu, u\rangle_g,$$

and similarly, $\langle A_f^{(1)} v, v\rangle_g = 2\eta\, \lambda\langle fv, v\rangle_g$. The result follows. □

A weaker but more general result is the following

Proposition 5.5. — *Suppose $P_g : \Gamma(E_g) \to \Gamma(E_g)$ satisfies the unique continuation principle. Let λ be a non-zero eigenvalue of P_g of multiplicity $\ell > rank(E_g)$. Then, there exists $\varepsilon_0 > 0$ and a function $f \in C^\infty(M,\mathbb{R})$ so that among the perturbed eigenvalues $\lambda_1(\varepsilon), \dots, \lambda_\ell(\varepsilon)$ of $P_{e^{\varepsilon f} g}$ there is a pair (i,j) for which $\lambda_i(\varepsilon) \neq \lambda_j(\varepsilon)$ for all $0 < \varepsilon < \varepsilon_0$.*

Proof. — Let $\{u_1, \dots, u_\ell\}$ be an orthonormal basis of the λ-eigenspace. If for some $i \neq j$ there exists $x \in M$ for which $\|u_i(x)\|_x \neq \|u_j(x)\|_x$ we proceed as in Proposition 5.4 and find $f \in C^\infty(M,\mathbb{R})$ for which $\langle fu_i, u_i\rangle_g \neq \langle fu_j, u_j\rangle_g$. We show that under our assumptions this is the only possible situation.

If for any two normalized eigensections $u, v \in \Gamma(E_g)$ of eigenvalue λ we had $\|u(x)\|_x^2 = \|v(x)\|_x^2$ for all $x \in M$, then by the polarization identity (see remark in Definition 2.2) we would obtain $(u_i(x), u_j(x))_x = 0$ for all $i \neq j$ and $x \in M$. By the rank condition, for some $i = 1, \dots, \ell$ the section u_i has to vanish on an open set, and by the unique continuation principle it must vanish everywhere, which is a contradiction. □

We finish this section translating the previous results to the setting of smooth functions.

Proposition 5.6. — *Operators acting on $C^\infty(M)$ have no rigid eigenspaces.*

Proof. — Let $\tilde{u},\tilde{v}$ be two linearly independent, orthonormal eigenfunctions of P_g with eigenvalue λ. Set $D := \{x \in M : \tilde{u}(x) \neq \tilde{v}(x)\}$. If there is $x \in D$ with $\tilde{u}(x) \neq -\tilde{v}(x)$, the functions $u = \tilde{u}$ and $v = \tilde{v}$ break the rigidity condition. If for all $x \in D$ we have $\tilde{u}(x) = -\tilde{v}(x)$, the functions $u = \frac{\tilde{u}+\tilde{v}}{\|\tilde{u}+\tilde{v}\|_g}$ and $v = \frac{\tilde{u}-\tilde{v}}{\|\tilde{u}-\tilde{v}\|_g}$ do the job. Indeed, $v = 0$ on D^c and there exists $x \in D^c$ for which $u(x) \neq 0$ because otherwise $\tilde{u} \equiv -\tilde{v}$ and this contradicts the independence. □

6. Eigenvalue multiplicity for conformal deformations

In this section we address the proofs of Theorems 2.1, 2.3 and 2.5.

6.1. Proof of Theorems 2.1 and 2.3.

Given $\alpha \in \mathbb{N}$, and $g \in \mathcal{M}$ consider the set

$$F_{g,\alpha} := \Big\{ f \in C^\infty(M,\mathbb{R}) : \ \lambda \text{ is } \textit{simple} \quad \text{for all } \ \lambda \in \mathrm{Spec}(\mathrm{P}_{\mathrm{e^f g}}) \cap \big([-\alpha,0) \cup (0,\alpha]\big) \Big\}.$$

The set of functions $f \in C^\infty(M,\mathbb{R})$ for which all the non-zero eigenvalues of $P_{e^f g}$ are simple coincides with the set $\bigcap_{\alpha\in\mathbb{N}} F_{g,\alpha}$. To show that the latter is a residual subset of $C^\infty(M,\mathbb{R})$, we prove that the sets $F_{g,\alpha}$ are open and dense in $C^\infty(M,\mathbb{R})$.

We note that for conformal metric deformations, the multiplicity of the zero eigenvalue remains fixed. Indeed, according to (3.1), for $u \in \Gamma(E_g)$ and $f \in C^\infty(M,\mathbb{R})$, we know

$$P_g(u) = 0 \qquad \text{if and only if} \qquad P_{e^f g}(\kappa^{-1}(e^{-\frac{af}{2}}u)) = 0.$$

Throughout this subsection we assume the hypothesis of Theorems 2.1 or 2.3 hold.

6.1.1. $F_{g,\alpha}$ is dense in $C^\infty(M,\mathbb{R})$

Fix $f_0 \notin F_{g,\alpha}$ and let W be an open neighborhood of f_0. Since at least one of the eigenvalues in $[-\alpha,0) \cup (0,\alpha]$ has multiplicity greater than two, we proceed to split it. By Proposition 5.4 (and Proposition 5.6 when the operator acts on $C^\infty(M)$) there exists $f_1 \in C^\infty(M,\mathbb{R})$ for which at least two of the first α non-zero eigenvalues of $P_{e^{\varepsilon_1 f_1}(e^{f_0}g)}$ are different as long as ε_1 is small enough. Moreover, those eigenvalues that were simple would remain being simple for such ε_1. Also, for ε_1 small enough, we can assume that none of the eigenvalues that originally belonged to $[-\alpha,\alpha]^c$ will have perturbations belonging to $[-\alpha,\alpha]$. Let ε_1 be small as before and so that $\varepsilon_1 f_1 + f_0$ belongs to W. If $\varepsilon_1 f_1 + f_0$ belongs to $F_{g,\alpha}$ as well, we are done. If not, in finitely many steps, the repetition of this construction will lead us to a function $\varepsilon_N f_N + \cdots + \varepsilon_1 f_1 + f_0$ in $W \cap F_\alpha$. Hence, $F_{g,\alpha}$ is dense.

6.1.2. $F_{g,\alpha}$ is open in $C^\infty(M,\mathbb{R})$

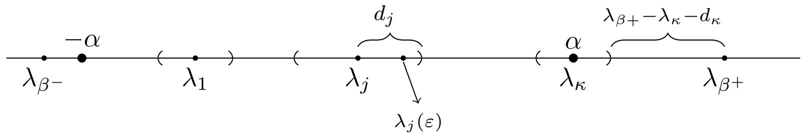
Fix $f_0 \in F_{g,\alpha}$. In order to show that $F_{g,\alpha}$ is open we need to establish how rapidly the eigenvalues of $A_f(\varepsilon)$ grow with ε. From now on we restrict ourselves to perturbations of the form $e^{\varepsilon f}(e^{f_0}g)$ for $f \in C^\infty(M,\mathbb{R})$ with $\|f\|_\infty < 1$. Let $u(\varepsilon)$ be an eigensection of $A_f(\varepsilon)$ with eigenvalue $\lambda(\varepsilon)$. Equation (4.1) gives $|\lambda'(\varepsilon)| \leqslant \|A_f^{(1)}(\varepsilon)u(\varepsilon)\|_g$ for $j = 1,\dots,\alpha$. Putting this together with inequality (5.1) for $k = 1$ we get

$$|\lambda'(\varepsilon)| \leqslant 2|\eta|\,\|A_f(\varepsilon)u(\varepsilon)\|_g = 2|\eta|\,|\lambda(\varepsilon)|.$$

The solution of the differential inequality leads to the following bound for the growth of the perturbed eigenvalues:

$$|\lambda(\varepsilon) - \lambda| \leqslant |\lambda|\left(e^{2|\eta|\,|\varepsilon|} - 1\right), \quad |\varepsilon| < \delta.$$

Write $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_\kappa$ for all the eigenvalues (repeated according to multiplicity) of $P_{e^{f_0}g}$ that belong to $[-\alpha,0) \cup (0,\alpha]$. Let $d_1,\dots,d_\kappa$ be so that the intervals $[\lambda_j - d_j, \lambda_j + d_j]$ for $j = 1,\dots,\kappa$, are disjoint. Write $\lambda_{\beta-}$ for the biggest eigenvalue in $(-\infty,-\alpha)$ and $\lambda_{\beta+}$ for the smallest eigenvalue in $(\alpha,+\infty)$. We further assume that $\lambda_{\beta-} \notin [\lambda_1 - d_1, \lambda_1 + d_1]$ and $\lambda_{\beta+} \notin [\lambda_\kappa - d_\kappa, \lambda_\kappa + d_\kappa]$.



In order to ensure that for each $j = 1,\dots,\alpha$ the perturbed eigenvalue $\lambda_j(\varepsilon)$ belongs to $[\lambda_j - d_j, \lambda_j + d_j]$, select $0 < \delta_1 \leqslant \delta$, so that whenever $|\varepsilon| < \delta_1$ we have that $|\lambda_j(\varepsilon) - \lambda_j| \leqslant |\lambda_j|\left(e^{2|\eta|\,|\varepsilon|} - 1\right) \leqslant d_j$ for all $j = 1,\dots,\kappa$.

To finish our argument, we need to make sure that none of the perturbations of the eigenvalues that initially belonged to $(-\infty,-\alpha) \cup (\alpha,+\infty)$

coincide with the perturbations corresponding to $\lambda_1, \dots, \lambda_\kappa$. To such end, it is enough to choose $0 < \delta_2 \leqslant \delta$ so that for $|\varepsilon| < \delta_2$,

$$|\lambda_{\beta+}| \left(e^{2|\eta|\,|\varepsilon|} - 1\right) < \min\{\lambda_{\beta+} - \lambda_\kappa - d_\kappa,\, \lambda_{\beta+} - \alpha\},$$

and

$$|\lambda_{\beta-}| \left(e^{2|\eta|\,|\varepsilon|} - 1\right) < \min\{\lambda_1 - d_1 - \lambda_{\beta-},\, -\alpha - \lambda_{\beta-}\}.$$

Summing up, if $\|f\|_\infty < 1$ and $|\varepsilon| < \min\{\delta_1, \delta_2\}$, then $\varepsilon f + f_0 \in F_{g,\alpha}$. Or in other words, $\{f_0 + f : \|f\|_\infty < \varepsilon\} \subset F_{g,\alpha}$, so $F_{g,\alpha}$ is open.

6.2. Proof of Theorem 2.5

The set of functions $f \in C^\infty(M, \mathbb{R})$ for which all the eigenvalues of $P_{e^f g}$ have multiplicity smaller than $rank(E_g)$ can be written as $\cap_{\alpha\in\mathbb{N}} \hat{F}_{g,\alpha}$ where

$$\hat{F}_{g,\alpha} := \Big\{ f \in C^\infty(M,\mathbb{R}) : \quad \dim \mathrm{Ker}(\mathrm{P_{e^f g}} - \lambda) \leqslant \mathrm{rank}(\mathrm{E_g}) \\ \text{for all } \lambda \in \mathrm{Spec}(\mathrm{P_{e^f g}}) \cap \big([-\alpha, 0) \cup (0, \alpha]\big) \Big\}.$$

$\hat{F}_{g,\alpha}$ is dense in $C^\infty(M,\mathbb{R})$ by the same argument presented in 6.1.1, using Proposition 5.5 to find the f_i's. The proof for $\hat{F}_{g,\alpha}$ being open in $C^\infty(M,\mathbb{R})$ is analogue to that in 6.1.2.

7. Local continuity of eigenvalues

The arguments we present in this section are an adaptation of the proof of Theorem 2 in [27] by Kodaira and Spencer; they prove similar results to Theorem 2.7 for strongly elliptic operators that have coefficients that depend continuously on a parameter $t \in \mathbb{R}^n$ in the C^∞-topology.

From now on fix a Riemannian metric g_0. Let $\{X_i\}_{i\in I}$ be a finite covering of M with local coordinates $(x_i^1, \dots, x_i^n)$ on each neighborhood X_i and let $u \in \Gamma(E_g)$ be represented in the form $(u_i^1(x), \dots, u_i^\mu(x))$ for $x \in X_i$ and $\mu = rank(E_g)$. For each integer k define the k- norm

$$\|u\|_k^2 := \sum_{\nu=1}^{\mu} \sum_{\substack{\alpha \\ |\alpha|\leqslant k}} \sum_{i\in I} \int_{X_i} |\partial_i^\alpha u_i^\nu(x)|^2 \, dvol_{g_0},$$

where $\partial_i^\alpha := \partial_{x_i^1}^{\alpha_1} \dots \partial_{x_i^n}^{\alpha_n}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We note that $\|\cdot\|_0 = \|\cdot\|_{g_0}$ and that the k-norm just introduced is equivalent to the k-Sobolev norm.

By the continuity of the coefficients of P_g (see Definition 3.2), there exists $\mathcal{W}_{g_0}$ neighborhood of g_0 in the C^∞-topology of metrics, so that for every metric $g \in \mathcal{W}_{g_0}$ there is an isomorphism of vector bundles $\tau_g : E_g \to E_{g_0}$ with the property that the coefficients of the differential operator

$$Q_g := \tau_g \circ P_g \circ \tau_g^{-1} : \Gamma(E_{g_0}) \to \Gamma(E_{g_0}) \tag{7.1}$$

depend continuously on $g \in \mathcal{W}_{g_0}$. The following analogue of Lemma 3 in [27] holds:

Lemma 7.1. — *There exists a neighborhood $\mathcal{W} \subset \mathcal{W}_{g_0}$ of g_0 so that for every integer $k \geqslant 0$ there is a constant c_k independent of $g \in \mathcal{W}$ for which*

$$\|u\|_{k+m}^2 \leqslant c_k \left(\|Q_g u\|_k^2 + \|u\|_0^2 \right),$$

for all $u \in \Gamma_{L^2}(E_{g_0})$ and $g \in \mathcal{W}$.

Proof. — Since P_g is elliptic, from relation (7.1) we deduce that Q_g is elliptic as well. By Theorem 5.2 part (*iii*) in [28, p.193], for every positive integer k there exists a constant c_k so that for all $u \in \Gamma(E_{g_0})$, $\|u\|_{k+m}^2 \leqslant c_k \left(\|Q_{g_0} u\|_k^2 + \|u\|_k^2 \right)$. By induction on k and the Sobolev embedding Theorem we obtain

$$\|u\|_{k+m}^2 \leqslant c_k \left(\|Q_{g_0} u\|_k^2 + \|u\|_0^2 \right).$$

The result follows from the continuity of the coefficients of Q_g for $g \in \mathcal{W}_{g_0}$. □

Since P_g is elliptic and formally self-adjoint, its spectrum $\mathrm{Spec}(\mathrm{P_g})$ is real and discrete. Note that the spectrum of P_g and Q_g coincide. Indeed, u is an eigensection of P_g with eigenvalue λ if and only if $\tau_g u$ is an eigensection of Q_g with eigenvalue λ. Fix $\xi \in \mathbb{C}$ and define

$$Q_g(\xi) := Q_g - \xi.$$

It is well known that $Q_g(\xi)$ is surjective provided ξ belongs to the resolvent set of Q_g (i.e. $\xi \notin \mathrm{Spec}(\mathrm{P_g})$). Furthermore, for ξ_0 in the resolvent set of P_{g_0}, set $b_{g_0} := \inf_{\lambda \in \mathrm{Spec}(\mathrm{P_{g_0}})} |\lambda - \xi_0|$. We then know

$$\|Q_{g_0}(\xi_0) u\|_0 \geqslant b_{g_0} \|u\|_0.$$

Lemma 7.2. — *There exists $\delta > 0$ and $\mathcal{V} \subset \mathcal{W}_{g_0}$ neighborhood of g_0 so that the resolvent operator $R_g(\xi) := Q_g(\xi)^{-1}$ exists for $g \in \mathcal{V}$ and $|\xi - \xi_0| < \delta$. In addition, for every $u \in \Gamma(E_{g_0})$ the section $R_g(\xi)u$ depends continuously on ξ and g in the $\|\cdot\|_0$ norm.*

Proof. — We first prove the injectivity of $Q_g(\xi)$. Let $\mathcal{W}$ be as in Lemma 7.1. It suffices to show that for all $\varepsilon > 0$ there exists $\delta > 0$ and $\mathcal{V} \subset \mathcal{W}$ so that

$$\|Q_g(\xi)u\|_0 \geqslant (b_{g_0} - \varepsilon)\|u\|_0$$

for $g \in \mathcal{V}$ and $|\xi - \xi_0| < \delta$. We proceed by contradiction. Suppose there exists $\varepsilon > 0$ together with a sequence $\{(\delta_i, \mathcal{V}_i, u_i)\}_i$, with $\delta_i \xrightarrow{i} 0$ and $\mathcal{V}_i$ shrinking around g_0, such that

$$\|Q_{g_i}(\xi_i)u_i\|_0 < (b_{g_0} - \varepsilon)\|u_i\|_0$$

for $g_i \in \mathcal{V}_i$ and $|\xi_i - \xi_0| < \delta_i$. Without loss of generality assume $\|u_i\|_0 = 1$.

By Lemma 7.1 we know $\|u_i\|_m \leqslant c_0(b_{g_0} - \varepsilon)$, and by the continuity in g of the coefficients of Q_g, it follows that $\|(Q_{g_i}(\xi_i) - Q_{g_0}(\xi_0))u_i\|_0 \to 0$. Since

$$\|(Q_{g_i}(\xi_i) - Q_{g_0}(\xi_0))u_i\|_0 \geqslant \|Q_{g_0}(\xi_0)u_i\|_0 - \|Q_{g_i}(\xi_i)u_i\|_0 \geqslant b_{g_0} - (b_{g_0} - \varepsilon) = \varepsilon,$$

we obtain the desired contradiction.

To prove the continuity statement notice that

$$\begin{aligned}
|R_g(\xi)u - R_{g_0}(\xi_0)u\|_0 \\
&\leqslant \frac{1}{b_{g_0}}\|Q_g(\xi)R_g(\xi)u - Q_g(\xi)R_{g_0}(\xi_0)u\|_0 \\
&= \frac{1}{b_{g_0}}\|Q_{g_0}(\xi_0)R_{g_0}(\xi_0)u - Q_g(\xi)R_{g_0}(\xi_0)u\|_0 \\
&= \frac{1}{b_{g_0}}\Big(\|\left(Q_{g_0} - Q_g\right)(R_{g_0}(\xi_0)u)\|_0 + |\xi - \xi_0|\|(R_{g_0}(\xi_0)u)\|_0\Big),
\end{aligned}$$

and use the continuity in g of the coefficients of Q_g. □

Let $g_0 \in \mathcal{M}$ and continue to write $\mathcal{W}_{g_0}$ for the neighborhood of g_0 for which the vector bundle isomorphism $\tau_g : E_g \to E_{g_0}$ is defined for all $g \in \mathcal{W}_{g_0}$. Let $\mathbb{C}$ be a differentiable curve with interior domain $D \subset \mathbb{C}$. For $g \in \mathcal{W}_{g_0}$, write $\mathbf{F}_g(C)$ for the linear subspace of $\Gamma(E_{g_0})$

$$\mathbf{F}_g(C) := \mathit{span}\left\{\tau_g u : \ \ u \in \mathrm{Ker(P_g - \lambda I)\ for\ \lambda \in D \cap Spec(P_g)}\right\}.$$

Note that

$$\dim \mathbf{F}_g(C) = \sum_{\lambda \in D\cap \mathrm{Spec}(\mathrm{P_g})} \dim \mathrm{Ker}(\mathrm{P_g} - \lambda \mathrm{I}). \tag{7.2}$$

Proposition 7.3. — *If C meets none of the eigenvalues of P_{g_0}, then there exists a neighborhood $\mathcal{V} \subset \mathcal{W}_{g_0}$ of g_0 so that for all $g \in \mathcal{V}$*

$$\dim \mathbf{F}_g(C) = \dim \mathbf{F}_{g_0}(C). \tag{7.3}$$

Proof.

Step 1. For $g \in \mathcal{W}_{g_0}$, define the spectral projection operator $F_g(C) : \Gamma(E_{g_0}) \to \Gamma(E_{g_0})$ to be the projection of $\Gamma(E_{g_0})$ onto $\mathbf{F}_g(C)$. Since C meets none of the eigenvalues of P_{g_0}, by Lemma 7.2 there exist a neighborhood C' of the curve C and a neighborhood $\mathcal{V}' \subset \mathcal{W}$ of g_0 so that none of the eigenvalues of P_g belong to C' for $g \in \mathcal{V}'$. By holomorphic functional calculus

$$F_g(C)\, u = -\frac{1}{2\pi i} \int_C R_g(\xi)\, u \, d\xi \qquad u \in \Gamma(E_g).$$

By Lemma 7.2 it follows that $F_g(C)\, u$ depends continuously for $\xi \in C_\delta$ and $g \in \mathcal{V}'$.

Step 2. Let $d = \dim \mathbf{F}_{g_0}(C)$ and $u_{\lambda_1(g_0)}, \dots, u_{\lambda_d(g_0)}$ be the eigenfunctions of P_{g_0} spanning $\mathbf{F}_{g_0}(C)$ with respective eigenvalues $\lambda_1(g_0) \leqslant \cdots \leqslant \lambda_d(g_0)$. Since $F_g(C)u$ depends continuously on $g \in \mathcal{V}'$, for all $u \in \Gamma(E_{g_0})$ we know that

$$\lim_{g \to g_0} \left\| F_g(C)\, [u_{\lambda_j(g_0)}] - u_{\lambda_j(g_0)} \right\|_0 = 0, \quad \text{for } j = 1, \dots, d,$$

and therefore there exists $\mathcal{V} \subset \mathcal{V}'$ neighborhood of g_0 so that

$$F_g(C)\, [u_{\lambda_1(g_0)}], \dots, F_g(C)\, [u_{\lambda_d(g_0)}]$$

are linearly independent for $g \in \mathcal{V}$. We thereby conclude,

$$\dim \mathbf{F}_g(C) \geqslant \dim \mathbf{F}_{g_0}(C) \qquad \text{for } \ g \in \mathcal{V}. \tag{7.4}$$

Step 3. Now let $l := \limsup_{g \to g_0} \dim \mathbf{F}_g(C)$. Consider $\{g_i\}_i \subset \mathcal{V}$ converging to g_0 so that $\dim \mathbf{F}_{g_i}(C) = l$ for $i = 1, 2, \dots$, and let $\tau_{g_i}(u_{\lambda_1(g_i)}), \dots, \tau_{g_i}(u_{\lambda_l(g_i)})$ be the eigensections that span $\mathbf{F}_{g_i}(C)$ with respective eigenvalues $\lambda_1(g_i) \leqslant \cdots \leqslant \lambda_l(g_i)$. By Lemma 7.1, $\|\tau_{g_i}(u_{\lambda_s(g_i)})\|_m \leqslant c_0(\lambda_s(g_i)+1)$ is bounded for all $s = 1, \dots, l$. Hence, since the Sobolev embedding is compact for $k > m$, we may choose a subsequence $\{g_{i_h}\}_h$ for which $\{\tau_{g_{i_h}}(u_{\lambda_s(g_{i_h})})\}_h$ converges in the Sobolev norm $\|\cdot\|_k$ for all $s = 1, \dots, l$.

Set $v_s := \lim_h \tau_{g_{i_h}}(u_{\lambda_s(g_{i_h})})$ and observe that since τ_{g_0} is the identity,

$$P_{g_0} v_s = Q_{g_0} v_s = \lim_h Q_{g_{i_h}} \tau_{g_{i_h}}(u_{\lambda_s(g_{i_h})}) = \lim_h \lambda_s(g_{i_h}) \tau_{g_{i_h}}(u_{\lambda_s(g_{i_h})}).$$

It follows that $v_1, \dots, v_l$ are linearly independent eigensections of P_{g_0} that belong to $\mathbf{F}_{g_0}(C)$. Thereby, for $g \in \mathcal{V}$, equality (7.3) follows from inequality (7.4) and

$$\dim \mathbf{F}_{g_0}(C) \geqslant l = \limsup_{g \to g_0} \ \dim \mathbf{F}_g(C). \qquad \square$$

7.1. Proof of Corollary 2.4

For $\delta \in (0,1)$ and $\alpha \in (0,+\infty)$, consider the sets

$$\mathcal{G}_{\delta,\alpha} := \Big\{ g \in \mathcal{M} : \ \lambda \text{ is } \textit{simple} \text{ for all } \lambda \in \mathrm{Spec}(\mathrm{P_g}) \cap \big([-\alpha,-\delta] \cup [\delta,\alpha]\big) \Big\}.$$

Assumming the hypothesis of Theorem 2.3 hold, we prove in Proposition 7.4 that the sets $\mathcal{G}_{\delta,\alpha}$ are open and dense in $\mathcal{M}$ with the C^∞-topology. Let $\{\delta_\ell\}_{\ell\in\mathbb{N}}$ be a sequence in $(0,1)$ satisfying $\lim_\ell \delta_\ell = 0$, and let $\{\alpha_k\}_{k\in\mathbb{N}}$ be a sequence in $(0,+\infty)$ satisfying $\lim_k \alpha_k = +\infty$. Then

$$\bigcap_{k=1}^{\infty} \bigcap_{\ell=1}^{\infty} \mathcal{G}_{\alpha_k,\delta_\ell}$$

is a residual set in $\mathcal{M}$ that coincides with the set of all Riemannian metrics for which all non-zero eigenvalues are simple. For the proof of Corollary 2.4 to be complete, it only remains to prove

Proposition 7.4. — *Suppose that $P_g : \Gamma(E_g) \to \Gamma(E_g)$ has no rigid eigenspaces for a dense set of metrics. Then, the sets $\mathcal{G}_{\delta,\alpha}$ are open and dense in the C^∞-topology.*

Proof. — We first show that $\mathcal{G}_{\delta,\alpha}$ is open. Let $g_0 \in \mathcal{G}_{\delta,\alpha}$ and write $\lambda_1(g_0),\dots,\lambda_d(g_0)$ for all the eigenvalues of P_{g_0} in $[-\alpha,-\delta]\cup[\delta,\alpha]$, which by definition of $\mathcal{G}_{\delta,\alpha}$ are simple. Assume further that the eigenvalues are labeled so that

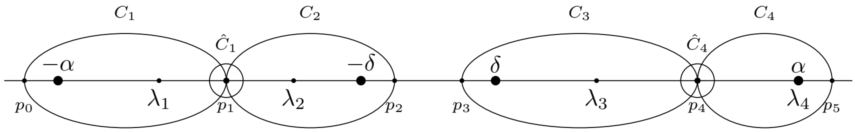
$$-\alpha \leqslant \lambda_1(g_0) < \ldots < \lambda_k(g_0) \leqslant -\delta \quad \text{and} \quad \delta \leqslant \lambda_{k+1}(g_0) < \ldots < \lambda_d(g_0) \leqslant \alpha.$$

Consider $\varepsilon_1 > 0$ small so that no eigenvalue of P_{g_0} belongs to

$$[-\alpha-\varepsilon_1,-\alpha] \cup [-\delta,-\delta+\varepsilon_1] \cup [\delta-\varepsilon_1,\delta] \cup [\alpha,\alpha+\varepsilon_1].$$

For all $1 \leqslant i \leqslant k-1$ let $p_i := \frac{1}{2}(\lambda_i(g_0)+\lambda_{i+1}(g_0))$, and for $k+2 \leqslant i \leqslant d$ let $p_i := \frac{1}{2}(\lambda_{i-1}(g_0)+\lambda_i(g_0))$. We also set $p_0 := -\alpha-\varepsilon_1$, $p_k := \delta+\varepsilon_1$, $p_{k+1} := \delta-\varepsilon_1$ and $p_{d+1} := \alpha+\varepsilon_1$.

For all $1 \leqslant i \leqslant k$ (resp. $k+1 \leqslant i \leqslant d$), let C_i be a differentiable curve that intersects the real axis transversally only at the points p_{i-1} and p_i (resp. p_i and p_{i+1}). In addition, let $\varepsilon_2 > 0$ be so that for each $1 \leqslant j \leqslant k-1$ and $k+2 \leqslant j \leqslant d$, the circle $\hat{C}_j$ centered at p_j of radius ε_2 does not contain any eigenvalue of P_{g_0}.



By Proposition 7.3, there exists an open neighborhood $\mathcal{V} \subset \mathcal{W}_{g_0}$ of g_0 so that for all $g \in \mathcal{V}$ and all i, j for which C_i and $\hat{C}_j$ were defined,

$$\dim \mathbf{F}_g(C_i) = \dim \mathbf{F}_{g_0}(C_i) = 1 \quad \text{and} \quad \dim \mathbf{F}_g(\hat{C}_j) = \dim \mathbf{F}_{g_0}(\hat{C}_j) = 0. \tag{7.5}$$

Since $[-\alpha, -\delta] \cup [\delta, \alpha]$ is contained in the union of all C_i's and $\hat{C}_j$'s, it then follows from (7.2) and (7.5) that for all $g \in \mathcal{V}$,

$$\dim \mathrm{Ker}(\mathrm{P_g} - \lambda \mathrm{I}) = 1 \qquad \forall \lambda \in \mathrm{Spec}(\mathrm{P_g}) \cap \big([-\alpha, -\delta] \cup [\delta, \alpha]\big)\,.$$

Since $\mathcal{V} \subset \mathcal{G}_{\delta,\alpha}$, it follows that $\mathcal{G}_{\delta,\alpha}$ is open.

We proceed to show that the sets $\mathcal{G}_{\delta,\alpha}$ are dense. Let $g_0 \notin \mathcal{G}_{\delta,\alpha}$ and $\mathcal{O}$ be an open neighborhood of g_0. Our assumptions imply that there exists $g \in \mathcal{O}$ so that the hypotheses of Theorem 2.3 are satisfied for P_g. It then follows that there exist a function $f \in C^\infty(M)$ so that the metric $e^f g \in \mathcal{O}$ and all non-zero eigenvalues of $P_{e^f g}$ are simple. Therefore, $e^f g \in \mathcal{O} \cap \mathcal{G}_{\delta,\alpha}$. □

7.2. Proof of Corollary 2.4

For $\delta \in (0,1)$ and $\alpha \in (0, +\infty)$, consider the sets

$$\hat{\mathcal{G}}_{\delta,\alpha} := \Big\{ g \in \mathcal{M} : \quad \dim \mathrm{Ker}(\mathrm{P_g} - \lambda \mathrm{I}) \leqslant \mathrm{rank}(\mathrm{E_g}) \\ \text{for all } \; \lambda \in \mathrm{Spec}(\mathrm{P_g}) \cap \big([-\alpha, -\delta] \cup [\delta, \alpha]\big) \Big\}$$

Using the same argument in Proposition 7.4 it can be shown that the sets $\hat{\mathcal{G}}_{\delta,\alpha}$ are open. To show that the sets $\hat{\mathcal{G}}_{\delta,\alpha}$ are dense, one carries again the same argument presented in Proposition 7.4, using the hypothesis of Theorem 2.5 to find the metric g. Let $\{\delta_\ell\}_{\ell \in \mathbb{N}}$ be a sequence in $(0,1)$ satisfying $\lim_\ell \delta_\ell = 0$, and let $\{\alpha_k\}_{k \in \mathbb{N}}$ be a sequence in $(0, +\infty)$ satisfying $\lim_k \alpha_k = +\infty$. Then $\cap_k \cap_\ell \hat{\mathcal{G}}_{\alpha_k, \delta_\ell}$ is a residual set in $\mathcal{M}$ that coincides with the set of all Riemannian metrics for which all non-zero eigenvalues of P_g have multiplicity smaller than the rank of the bundle E_g. This completes the proof of Corollary 2.4.

7.3. Proof of Theorem 2.7

For $c \in \mathbb{R}$, we continue to write $\mathcal{M}_c = \{g \in \mathcal{M} : \ c \notin \mathrm{Spec}(\mathrm{P_g})\}$. In addition, for $g \in \mathcal{M}_c$, we write

$$\mu_1(g) \leqslant \mu_2(g) \leqslant \mu_3(g) \leqslant \dots$$

for the eigenvalues of P_g in $(c, +\infty)$ counted with multiplicity. We recall that it may happen that there are only finitely many $\mu_j(g)$'s.
To see that $\mathcal{M}_c$ is open, fix $g_0 \in \mathcal{M}_c$. Let $\delta > 0$ be so that the circle C_δ centered at c of radius δ contains no eigenvalue of P_{g_0}. By Proposition 7.3 there exists $\mathcal{V}_1 \subset \mathcal{W}_{g_0}$, a neighborhood of g_0, so that for all $g \in \mathcal{V}_1$

$$\dim \mathbf{F}_g(C_\delta) = \dim \mathbf{F}_{g_0}(C_\delta) = 0.$$

It follows that $\mathcal{V}_1 \subset \mathcal{M}_c$.

We proceed to show the continuity of the maps

$$\mu_i : \mathcal{M}_c \to \mathbb{R}, \qquad\qquad g \mapsto \mu_i(g).$$

We first show the continuity of $g \mapsto \mu_1(g)$ at $g_0 \in \mathcal{M}_c$. Fix $\varepsilon_0 > 0$ and consider $0 < \varepsilon < \varepsilon_0$ so that the circle C_{ε_1} centered at $\mu_1(g_0)$ of radius ε contains only the eigenvalue $\mu_1(g_0)$ among all the eigenvalues of P_{g_0}. Let $\delta > 0$ be so that there is no eigenvalue of P_{g_0} in $[c - \delta, c]$. Consider a differentiable curve C that intersects transversally the x-axis only at the points $c - \delta$ and $\mu_1(g_0) - \varepsilon/2$.

By Proposition 7.3 there exists $\mathcal{V}_2 \subset \mathcal{W}_{g_0}$, a neighborhood of g_0, so that for all $g \in \mathcal{V}_2$

$$\dim \mathbf{F}_g(C) = \dim \mathbf{F}_{g_0}(C) \quad \text{and} \quad \dim \mathbf{F}_g(C_\varepsilon) = \dim \mathbf{F}_{g_0}(C_\varepsilon).$$

Since $\dim \mathbf{F}_{g_0}(C) = 0$, it follows that no perturbation $\mu_i(g)$, $i \geqslant 1$, belongs to $[c, \mu_1(g_0) - \varepsilon/2]$. Also, since the dimension of $\mathbf{F}_g(C_\varepsilon)$ is preserved, it follows that there exists j so that $|\mu_j(g) - \mu_1(g_0)| < \varepsilon$ for $j \neq 1$. Since

$$\mu_1(g_0) - \varepsilon < \mu_1(g) \leqslant \mu_j(g) \leqslant \mu_1(g_0) + \varepsilon,$$

it follows that for $g \in \mathcal{V}_2$ we have $|\mu_1(g) - \mu_1(g_0)| < \varepsilon$ as wanted.

The continuity of $g \mapsto \mu_i(g)$, for $i \geqslant 2$, follows by induction. One should consider a circle of radius ϵ centered at $\mu_i(g_0)$ and a differentiable curve C that intersects transversally the x-axis only at the points $c - \delta$ and $\mu_i(g_0) - \varepsilon/2$.

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