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NON-EMBEDDABLE 1-CONVEX MANIFOLDS

by Jan STEVENS

ABSTRACT. — We show that every small resolution of a 3-dimensional terminal hypersurface singularity can occur on a non-embeddable 1-convex manifold.

We give an explicit example of a non-embeddable manifold containing an irreducible exceptional rational curve with normal bundle of type $(1, -3)$. To this end we study small resolutions of cD_4-singularities.

RÉSUMÉ. — Nous montrons que chaque petite résolution d'une singularité de hypersurface 3-dimensionnelle terminale peut se produire sur une variété 1-convexe non plongeable.

Nous donnons un exemple explicite d'une variété non plongeable contenant une courbe exceptionnelle rationnelle irréductible avec fibré normal du type $(1, -3)$. À cette fin, nous étudions de petites résolutions des singularités cD_4.

Introduction

A 1-convex (or strongly pseudoconvex) complex manifold X with 1-dimensional exceptional set can be embedded in some $\mathbb{C}^M \times \mathbb{P}^N$, except possibly when $\dim X = 3$ and an irreducible component of the exceptional curve is a rational curve with normal bundle of type $(-1, -1)$, $(0, -2)$ or $(1, -3)$. Non-embeddable examples are known in the first two cases [18, 6, 4]. In this paper we show that last type also occurs.

An irreducible exceptional rational curve C on a 3-dimensional manifold X with normal bundle of type (a, b) with $a + b = -2$ blows down to a terminal Gorenstein singularity, that is, a cDV-singularity. This means that the general hyperplane section through the singular point is Du Val, or in other terminology, a rational double point. The simplest possibility is an ordinary double point (a 3-dimensional A_1-singularity). The first example of a non-embeddable 1-convex manifold [18, 6] is a variant of Moishezon's

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example of a non-projective Moishezon manifold [12]. Let $Y \subset \mathbb{C}^4$ be a general hypersurface of degree $d \geqslant 6$ with one A_1-singularity and let $\overline{Y} \subset \mathbb{P}^4$ be its projective closure. A small resolution of $\overline{Y}$ is non-projective and a small resolution of Y is non-embeddable. The explicit example of [4] for the case of normal bundle $(-1,-1)$ is also of this form. The examples for $(0,-2)$ are similar. They start from an equation f_{2k} for a 3-fold A_{2k-1}-singularity, for which a small resolution with irreducible exceptional set is easily constructed. Let f_{2N} be a homogeneous polynomial of high degree $2N$ with isolated singularity at $0 \in \mathbb{C}^4$. Then $Y_k = \{f_{2k} + \varepsilon f_{2N} = 0\}$ is an affine hypersurface with non-embeddable small resolution. In [4] this is shown by explicit construction of a 3-chain with the exceptional set as boundary.

These examples suggest the following construction. Let $\{f = 0\} \subset \mathbb{C}^4$ be a hypersurface with terminal singularity at the origin, admitting a small resolution. Choose a general enough polynomial g of high enough degree. A small resolution X of $Y = \{f + \varepsilon g = 0\}$ should be a non-embeddable 1-convex manifold. Our main result states that this is indeed the case. The proof uses that X is non-embeddable if and only if the corresponding small resolution $\overline{X}$ of the projective closure $\overline{Y}$ is non-projective [21] (as g is general, the hyperplane section $\overline{Y}_\infty = \{g = 0\} \subset \mathbb{P}^3$ is smooth). This follows once the group of Weil divisors modulo algebraic equivalence has rank one [10]. We show the stronger result that the class group $\mathrm{Cl}(\overline{Y})$ is infinite cyclic: by the Grothendieck-Lefschetz theorem of [14] $\mathrm{Cl}(\overline{Y})$ injects into the class group of $\overline{Y}_\infty$, and by the classical Noether-Lefschetz theorem this smooth surface has Picard group $\mathbb{Z}$ for very general g (meaning for g outside a countable union of subvarieties in parameter space).

We also provide an explicit example of a non-embeddable X with irreducible exceptional curve with normal bundle $(1,-3)$. Such a curve blows down to a singularity with general hyperplane section of type D_4, E_6, E_7 or E_8 [8]. In the latter cases the formulas become very complicated, so we restrict ourselves to the simplest one (D_4). The strict transform of the general hyperplane section is a partial resolution of the D_4 singularity, and the total space is a 1-parameter smoothing. It can be obtained by pull-back from the versal deformation of the partial resolution, which is a simultaneous partial resolution of the versal deformation of the singularity, after a base change. We compute this base change and then construct the small modification, generalising Example (5.15) in Reid's pagoda paper [16]. We classify all 1-parameter smoothings, that is, all 3-dimensional singularities to which these blow down.

Our explicit example is a small resolution of

$$x^2 + (t+z)y^2 + (t-z)z^2 - (t^2 - z^2)t^{2k} + \varepsilon t^{2m} = 0.$$

This hypersurface is very singular at infinity, but the equation has the advantage of containing only a few terms. We explicitly show that (twice) the exceptional curve bounds a real 3-chain, and therefore the small resolution is not embeddable.

In the first section we recall the necessary definitions and known results about non-embeddable 1-convex manifolds. Then we show our main result on the existence of hypersurfaces with non-embeddable small resolution. In the second section we classify the cD_4-singularities which admit a small resolution, and construct this resolution explicitly. The final section is devoted to the specific example.

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1. Non-embeddable 1-convex manifolds

Definition 1.1. — *A complex space X is* 1-convex *(or* strongly pseudoconvex*) if there exists a proper surjective morphism $\pi\colon X \to Y$ onto a Stein space Y with $\pi_* \mathcal{O}_X = \mathcal{O}_Y$ and a finite subset $T \subset Y$ such that $X \smallsetminus \pi^{-1}(T) \to Y \smallsetminus T$ is biholomorphic. The* exceptional set *is $S = \pi^{-1}(T)$.*

Definition 1.2. — *A 1-convex space X is called* embeddable *if there exists a holomorphic embedding $X \to \mathbb{C}^M \times \mathbb{P}^N$ for some (M, N).*

A necessary and sufficient condition is given by the following result [17]:

Proposition 1.1. — *The 1-convex manifold X with exceptional set S is embeddable if and only if there exists a line bundle on X with $L_{|S}$ ample.*

We from now on only consider the case of one-dimensional exceptional sets S. There is the following topological criterion.

Theorem 1.2 ([1]). — *Let X be a 1-convex manifold with one-dimensional exceptional set S. Then X is Kähler if and only if S does not contain an effective curve C, whose class in $H_2(X, \mathbb{Z})$ vanishes. If moreover $H_2(X, \mathbb{Z})$ is finitely generated, then these conditions are equivalent to the fact that X is embeddable.*

Similar results are obtained in [18]. Non-embeddable 1-convex manifolds are very special.

THEOREM 1.3 ([6, 19]). — *If a* 1*-convex manifold* X *with one-dimensional exceptional set* S *is not embeddable, then* X *has dimension three and* S *has an irreducible component* C *with* $K_X \cdot C = 0$*.*

It follows that C is a rational curve, with normal bundle of type $\mathcal{O}_C(a) \oplus \mathcal{O}_C(b)$, satisfying $a + b = -2$. The only possibilities are $(-1, -1)$, $(0, -2)$ or $(1, -3)$ [11], see also [5, Lecture 16].

To describe the singularities of Y we look at the germ of X along S. Let $\pi\colon (X, S) \to (Y, p)$ be a small contraction (we call the map π a small contraction or small resolution depending on whether we view X or Y as the primary object) with X smooth and K_X π-trivial, i.e., $K_Y \cdot C = 0$ for every irreducible component of S. Then Y is Gorenstein terminal, so a cDV-singularity. This means that a hyperplane section through the singular point is a rational double point (a.k.a. DuVal singularity).

PROPOSITION 1.4 ([16]). — *If* (H, p) *is a generic hyperplane section of the cDV-singularity* (Y, p) *with small resolution* $\pi\colon (X, S) \to (Y, p)$*, then* $G := \pi^* H$ *is normal and the induced map* $f\colon (G, S) \to (Y, p)$ *is a partial resolution, dominated by the minimal resolution.*

In particular, if π contracts only one rational curve, the partial resolution is obtained by blowing down all exceptional curves on the minimal resolution $\widetilde{H}$ of H, except one. The type of H is determined by Kollár's length invariant [5, Lecture 16].

DEFINITION 1.3. — *The* length l *of the small contraction* $\pi\colon (X, C) \to (Y, p)$ *with irreducible exceptional curve* C *is*

$$l = \lg \mathcal{O}_X / \pi^* \mathfrak{m}_{Y,p}.$$

The length equals the multiplicity of the fundamental cycle of $\widetilde{H}$ at the strict transform of the exceptional curve C.

PROPOSITION 1.5 ([8]). — *The length of the small contraction* π *determines the type of the general hyperplane section* H *and the partial resolution* $G \to H$*.*

A simple proof is given by Kawamata [9]. For length $l = 1$ the general hyperplane section is of type A_1. This occurs for normal bundle of type $(-1, -1)$ or $(0, -2)$. For $(1, -3)$ the length lies between 2 and 6, with for $l = 2, 3, 4$ general hyperplane section D_4, E_6 and E_7. If $l = 5, 6$, then H has an E_8-singularity.

This result suggests how to construct examples of small contractions: start with a partial resolution of a rational double point with irreducible exceptional curve, and take a 1-parameter smoothing of it, such that the exceptional curve is isolated. As the singularity is rational, the deformation blows down to a deformation of the rational double point. Typically this construction leads to an affine hypersurface with embeddable small resolution. Examples of non-embeddable spaces with a singularity of length 1 were given by Colţoiu, Vo Van Tan and Bassanelli-Leoni [6, 18, 4], see also [7, 19].

For an affine threefold Y with small resolution $\pi\colon X \to Y$ embeddability of X is closely related to projectivity of the corresponding small resolution of the projective closure of Y. More precisely, we have the following result of [2], which was proved earlier in the special case of hypersurfaces in [21]. A similar result is proved in [18].

Theorem 1.6 ([2]). — *Let $\pi\colon X \to Y$ be a contraction of the 1-convex manifold X, with Y Stein and quasi-projective, of dimension at least 3. Let $(\overline{Y}, \overline{Y}_\infty)$ be the projective closure of Y and assume that* $\operatorname{Sing}(\overline{Y}) = \operatorname{Sing}(Y)$. *Let $(\overline{X}, \overline{X}_\infty)$ be the corresponding compactification, with the same divisor $\overline{X}_\infty = \overline{Y}_\infty$ at infinity. Suppose that the map $H_2(X,\mathbb{R}) \to H_2(\overline{X},\mathbb{R})$ is injective. Then X is embeddable (this is so if and only if X is Kähler) if and only if $\overline{X}$ is projective.*

The condition on the map $H_2(X,\mathbb{R}) \to H_2(\overline{X},\mathbb{R})$ is in particular satisfied if $\overline{Y}_\infty$ is a smooth projective hypersurface.

For small resolutions of threefolds in $\mathbb{P}^4$ we have the following result.

Theorem 1.7 ([10, Theorem 5.3.2]). — *Let $\pi\colon \overline{X} \to \overline{Y}$ be a small resolution of a projective threefold $\overline{Y}$ with at most terminal hypersurface singularities, such that the group of Weil divisors modulo algebraic equivalence has rank one. Then $\overline{X}$ is a Moishezon threefold which is nonprojective if π is not an isomorphism.*

Consider now any terminal hypersurface singularity, which admits a small resolution. Then there exists a projective hypersurface with this singularity as only singularity.

Lemma 1.8. — *Let the polynomial function $f\colon \mathbb{C}^4 \to \mathbb{C}$ define a hypersurface with isolated singularity at the origin. For generic homogeneous g of high enough degree the projective closure $\overline{Y} \subset \mathbb{P}^4$ of the affine hypersurface $V(f+\varepsilon g)$ has only one singularity, isomorphic to the singularity of $V(f)$ at the origin, and the hyperplane section at infinity $\overline{Y}_\infty$ is smooth.*

Proof. — As the singularity of f at the origin is finitely determined, the hypersurface $V(f+\varepsilon g)$ has an isomorphic singularity at the origin if the degree of g is at least the degree of determinacy. For generic g there are no other singularities and the hyperplane section at infinity $\overline{Y}_\infty = V(g) \subset \mathbb{P}^3$ is smooth. □

A polynomial is *very general*, if its parameter point lies outside a countable union of proper subvarieties in the space parametrising polynomials of given degree.

THEOREM 1.9. — *Let f and g be as in the lemma above, and let $\overline{Y} \subset \mathbb{P}^4$ be the projective closure of $V(f+\varepsilon g)$. If g is a very general polynomial, then the class group of $\overline{Y}$ satisfies* $\mathrm{Cl}(\overline{Y}) \cong \mathbb{Z}$.

Proof. — By the Grothendieck-Lefschetz theorem of Ravindra and Srinivas [14] the restriction homomorphism $\mathrm{Cl}(\overline{Y}) \to \mathrm{Cl}(\overline{Y}_\infty)$ is injective. The theorem as stated there gives only the conclusion for hyperplane sections in a Zariski dense open subset of sections, but as remarked by the same Authors in [15, p. 3378], it suffices that $\overline{Y}_\infty$ is smooth and does not pass through the singularity of $\overline{Y}$. By the classical Noether-Lefschetz theorem a very general smooth surface S of degree at least 4 in $\mathbb{P}^3$ satisfies $\mathrm{Cl}(S) = \mathrm{Pic}(S) \cong \mathbb{Z}$. An algebraic proof can be found in [15]. □

Combining the above results we obtain that every small contraction to a hypersurface singularity can occur on a non-embeddable 1-convex manifold.

COROLLARY 1.10. — *Suppose that the affine threefold $V(f) \subset \mathbb{C}^4$ has a terminal hypersurface singularity at the origin, which admits a non-trivial small resolution X_0. For very general homogeneous g of high enough degree the corresponding small resolution X of the affine hypersurface $V(f+\varepsilon g)$ is a non-embeddable* 1*-convex manifold.*

The above Corollary is a statement about affine 3-folds. A direct proof, without going to the projective closure, would be preferable. We have not been able to find it. The problem is that there exist affine hypersurfaces with terminal singularities, whose small resolution is embeddable: typically this is the case for the hypersurface $V(f)$, whose small resolution is given by explicit polynomial formulas. Adding the form g means specifying the hyperplane section at infinity, so one is naturally led to the projective closure.

2. Small resolutions for cD_4-singularities

In this section we construct a small resolution with irreducible exceptional set for certain cD_4-singularities. We view it as total space of a 1-parameter smoothing of a partial resolution of a D_4 surface singularity. As such it can be obtained by pull-back from the versal deformation of the partial resolution. We first describe Pinkham's construction of this versal deformation [13], see also [8]. We give explicit formulas. We then classify the occurring singularities.

The versal deformation $\mathcal{Y} \to S$ of a surface singularity Y of type A, D, E admits a simultaneous resolution after base change with the corresponding Weyl group W. We write $S = T/W$ and identify T with the vector space spanned by a root system of type A, D or E. The simultaneous resolution is the versal deformation $\widetilde{\mathcal{X}} \to T$ of the minimal resolution $\widetilde{X}$ of Y.

Now consider a partial resolution $\widehat{X} \to Y$ with irreducible exceptional set E_0; we denote strict transform of E_0 on the minimal resolution by the same name. It determines a one vertex subgraph Γ_0 of the resolution graph Γ. The connected components of the complement $\Gamma \setminus \Gamma_0$ are the graphs of the singularities on the partial resolution $\widehat{X}$; we can construct $\widehat{X}$ from the minimal resolution by blowing down the configurations of curves, given by $\Gamma \setminus \Gamma_0$. The versal deformation $\widehat{\mathcal{X}}$ of $\widehat{X}$ admits a simultaneous resolution after base change with the product W_0 of the Weyl groups corresponding to the connected components of $\Gamma \setminus \Gamma_0$; this simultaneous resolution is nothing else than $\widetilde{\mathcal{X}} \to T$. So the base space of $\widehat{\mathcal{X}}$ is T/W_0.

In the cases A and D it is rather easy to give the simultaneous partial resolution explicitly, but for E (and especially E_8) the formulas become too complicated (*cf.* [8]). We restrict ourselves in the following to the simplest case leading to normal bundle $(1,-3)$, the case D_4. Then the length of the small contraction (Definition 1.3) is equal to two.

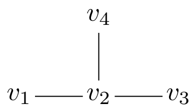
We start from the versal deformation $\mathcal{X} \to S = T/W$ of D_4, as in [20], see also [8], given by

$$x^2 + y^2 z - z^3 - t_2 z^2 - t_4 z - t_6 + 2 s_4 y = 0. \tag{2.1}$$

In $T \cong \mathbb{C}^4$, with negative definite inner product $\{e_i, e_j\} = -\delta_{ij}$, there is a root system with basis

$$\begin{aligned} v_1 &= e_1 - e_2 = (1,-1,0,0)\\ v_2 &= e_2 - e_3 = (0,1,-1,0)\\ v_3 &= e_3 - e_4 = (0,0,1,-1)\\ v_4 &= e_3 + e_4 = (0,0,1,1) \end{aligned}$$

and Dynkin diagram



The roots v_i correspond to the components of the exceptional divisor of the resolution of D_4. The partial resolution $\widetilde{X}$, which only pulls out the central curve, is obtained by blowing down the components corresponding to v_1, v_3 and v_4. The graph $\Gamma \setminus \Gamma_0$ has three components, all three of type A_1. Its Weyl group is $W_0 = W(A_1) \times W(A_1) \times W(A_1)$. To describe it explicitly, we take coordinates $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ on T. The reflection s_{v_4} acts as $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \mapsto (\alpha_1, \alpha_2, -\alpha_4, -\alpha_3)$, whereas for $i = 1, 2, 3$ the s_{v_i} are the transpositions $(i, i+1)$. The connection with the deformation (2.1) is that the coordinates on $S = T/W$ are the invariants $t_{2i} = \sigma_i(\alpha_1^2, \alpha_2^2, \alpha_3^2, \alpha_4^2)$ for $i = 1, 2, 3$ and $s_4 = \sigma_4(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, where the σ_i are the elementary symmetric functions.

Proposition 2.1. — *The versal deformation $\widehat{\mathscr{X}} \to T/W_0$ of the partial resolution $\widehat{X}$ is a simultaneous partial resolution (without base change) of the deformation*

$$F(x, y, z; \beta_1, \beta_2, \gamma_3, \beta_4) = x^2 - (z^2 + z\beta_1 + \beta_2^2)\gamma_3^2 + z(y + \beta_2)^2 - 2\beta_2(y + \beta_2)(z - \beta_4) - (z + \beta_1)(z - \beta_4)^2 \tag{2.2}$$

of the D_4 surface singularity Y.

Proof. — The invariants for W_0 are

$$\begin{aligned} \gamma_3 &= \alpha_1 + \alpha_2, \\ \beta_1 &= \alpha_3^2 + \alpha_4^2, \\ \beta_2 &= \alpha_3 \alpha_4, \\ \beta_4 &= \alpha_1 \alpha_2. \end{aligned}$$

We express the coordinates on S in these invariants:

$$\begin{aligned} t_2 &= \beta_1 + \gamma_3^2 - 2\beta_4, \\ t_4 &= \beta_2^2 + \beta_4^2 + \beta_1(\gamma_3^2 - 2\beta_4), \\ t_6 &= \beta_1 \beta_4^2 + \beta_2^2(\gamma_3^2 - 2\beta_4), \\ s_4 &= \beta_2 \beta_4. \end{aligned}$$

Inserting these values in the versal family (2.1) and rearranging gives the formula (2.2). According to Pinkham [13] a simultaneous partial resolution gives the desired versal deformation. □

Lemma 2.2. — *The (reduced) discriminant of the family* (2.2) *has five irreducible components, given by $4\beta_4 = \gamma_3^2$, $\beta_2 = 2\beta_1$, $\beta_2 = -2\beta_1$, $\gamma_3 = 0$ and*

$$(\beta_4^2 + \beta_1\beta_4 + \beta_2^2)^2 - \gamma_3^2(\beta_1\beta_2^2 + 4\beta_2^2\beta_4 + \beta_1\beta_4^2) + \beta_2^2\gamma_3^4 = 0. \tag{2.3}$$

Proof. — The discriminant is the image of the reflection hyperplanes $\alpha_i \pm \alpha_j = 0$ in T. The hyperplanes perpendicular to v_1, v_3 and v_4 are $\alpha_1 = \alpha_2$, $\alpha_3 = \alpha_4$ and $\alpha_3 = -\alpha_4$; they map to $4\beta_4 = \gamma_3^2$, $\beta_2 = 2\beta_1$ and $\beta_2 = -2\beta_1$. The fundamental cycle of the singularity corresponds to $v_1 + 2v_2 + v_3 + v_4 = (1,1,0,0)$ and determines the hyperplane $\alpha_1 + \alpha_2 = \gamma_3 = 0$. For $\gamma_3 = 0$ there is a singular point at $x = y + \beta_2 = z - \beta_4 = 0$.

The remaining hyperplanes give rise to the same irreducible component of the discriminant. To describe it we determine the corresponding component of the critical locus, which is the component of the singular locus of the total space, not contained in $\gamma_3 = 0$. A computation, which we suppress, shows that it is given by $x = 0$ and

$$\operatorname{Rank}\begin{pmatrix} z & \beta_2 & z-\beta_4 \\ \beta_2 & -z-\beta_1 & y+\beta_2 \\ z-\beta_4 & y+\beta_2 & -\gamma_3^2 \end{pmatrix} \leqslant 1.$$

This is indeed image of $\alpha_1 + \alpha_3 = 0$: we have $\gamma_3 = \alpha_1 + \alpha_2$, $\beta_1 = \alpha_1^2 + \alpha_4^2$, $\beta_2 = -\alpha_1\alpha_4$ and $\beta_4 = \alpha_1\alpha_2$, while the singular point lies at $x = 0$, $y = -\alpha_2\alpha_4$ and $z = -\alpha_1^2$. By eliminating the variables y and z we find the equation (2.3) for this component. □

To explicitly construct the simultaneous partial resolution we proceed as in Example (5.15) of [16]. We write the family (2.2) as

$$F = X^2 + (ac - b^2)T^2 + aY^2 - 2bYZ + cZ^2. \tag{2.4}$$

This generalises the family in [16], where $b = 0$. The coordinate change is given by $X = x$, $T = \gamma_3$, $Y = y + \beta_2$, $Z = z - \beta_4$, $a = z$, $b = \beta_2$ and $c = -z - \beta_1$. We consider the family (2.4) as quadric in X, Y, Z and T, with coefficients in $\mathbb{C}[a,b,c]$. It has two small resolutions. To give them explicitly we have to factorize $aY^2 - 2bYZ + cZ^2$. The idea is to put $a = -\alpha^2$ and write $X^2 - (\alpha Y + \frac{b}{\alpha}Z)^2 + (ac - b^2)(T^2 - (\frac{Z}{\alpha})^2) = 0$. Then one small resolution is obtained by blowing up the ideal $(X - (\alpha Y + \frac{b}{\alpha}Z), T - \frac{Z}{\alpha})$. We could as well set $c = -\gamma^2$; it leads to the same two small resolutions.

We eliminate α from the generators of the ideal by writing them as $(-1, \alpha)M$, where M is the matrix

$$M = \begin{pmatrix} Z & -aT & X - bT & -aY \\ T & Z & Y & X - bT \end{pmatrix}. \tag{2.5}$$

Lemma 2.3. — *The blow-up of the ideal generated by the minors of the 2×4 matrix M defines a small resolution $\widehat{\mathcal{X}}$ of the total space of the family* (2.4).

Proof. — The minors of the matrix M are not independent, and the ideal needs only four generators. The blow-up is the subset of $\mathbb{C}^3\times\mathbb{C}^4\times\mathbb{P}^3$, which is the closure of the graph of the rational map

$$(P\colon Q\colon R\colon S) = \big(Z^2+aT^2\colon (X-bT)^2+aY^2\colon \\ ZY-T(X-bT)\colon Z(X-bT)+aTY\big). \tag{2.6}$$

The relation between the minors gives $S^2-PQ+aR^2=0$. Furthermore one sees that $Q+cP-2bR$ is proportional to F, so vanishes on $F=0$. This allows us to eliminate $Q=2bR-cP$. We find

$$S^2+cP^2-2bPR+aR^2=0.$$

This formula shows that we have a small modification. The determinantal syzygies between the minors of the matrix give the following four equations, where Q is already eliminated:

$$\begin{aligned} YP-ZR-TS &= 0,\\ (X-bT)P+aTR-ZS &= 0,\\ -cTP+(X+bT)R-YS &= 0,\\ (cZ-bY)P+(aY-bZ)R+XS &= 0. \end{aligned}$$

These equations determine $(P\colon R\colon S)$ except when the rank of the coefficient matrix is at most one: this happens exactly at the singular points.

To check smoothness we look at affine charts. The exceptional curve over the origin is covered by the charts $P=1$ and $R=1$. In $R=1$ we can eliminate Z, X and a, leaving (Y,P,S,T,b,c) as coordinates:

$$\begin{aligned} Z &= YP-TS,\\ X+bT &= cTP+YS,\\ S^2+cP^2-2bP+a &= 0. \end{aligned}$$

This shows that the space $\widehat{\mathcal{X}}$ is smooth in this chart. Likewise we find in the chart $P=1$ that

$$\begin{aligned} Y &= ZR+TS,\\ X-bT &= -aTR+ZS,\\ S^2+c-2bR+aR^2 &= 0. \end{aligned}$$

□

To view $\widehat{\mathcal{X}}$ as simultaneous partial resolution of the deformation (2.2) of D_4 we have to go back to the original coordinates. The three singular points of the special fibre are visible in the chart $R = 1$, so we only look at this chart. We eliminate x and z via

$$\begin{aligned} x &= (y+\beta_2)S - (z+\beta_1)\gamma_3 P - \beta_2\gamma_3, \\ z &= (y+\beta_2)P - \gamma_3 S + \beta_4 \end{aligned} \tag{2.7}$$

and are left with one equation

$$S^2 - (y+\beta_2)P^3 - (\beta_1+\beta_4-\gamma_3 S)P^2 + (y-\beta_2)P + \beta_4 - \gamma_3 S = 0 \tag{2.8}$$

in the variables $(S, P, y; \beta_1, \beta_2, \gamma_3, \beta_4)$.

Over the non-smooth component (2.3) of the discriminant we have, using the parametrisation by the reflection hyperplane $\alpha_1 + \alpha_3 = 0$, that

$$\begin{aligned} x &= (\alpha_1+\alpha_2)(-\alpha_4 S + (\alpha_1+\alpha_2)P)(S+\alpha_4 P - \alpha_1), \\ z &= -\alpha_1^2 - (\alpha_1+\alpha_2)(S+\alpha_4 P - \alpha_1), \\ 0 &= (S+\alpha_4 P - \alpha_1)(S + (\alpha_1+\alpha_2)P^2 - \alpha_4 P - \alpha_2). \end{aligned}$$

So the curve $S + \alpha_4 P - \alpha_1 = 0$ is the exceptional curve (it extends to the $P = 1$ chart).

Over $\gamma_3 = 0$ we have $x = y + \beta_2 = z - \beta_4 = 0$ and the exceptional curve $S^2 - (\beta_1+\beta_4)P^2 - 2\beta_2 P + \beta_4 = 0$. Note that this curve is reducible if in addition $\beta_2^2 + \beta_1\beta_4 + \beta_4^2 = 0$, that is, over the intersection of the two components of the discriminant.

We now return to 3-dimensional cD_4 singularities. We use Arnol'd's notation for singularities, see [3].

Proposition 2.4. — *A 3-fold singularity with D_4 as general hyperplane section, which has a small resolution with irreducible exceptional curve, is of type $T_{3,3,2q+2}$, Q_{6q+5} or $Q_{\rho+1,\delta}$ with δ odd.*

Proof. — The singularity is a 1-parameter smoothing of the hyperplane section D_4, so can be obtained by pull-back from the versal family (2.2). We have to describe a curve in the base space, so now we take the β_i and γ_3 to be functions of a variable t. Having a small resolution with irreducible exceptional curve gives two conditions, that the curve does not lie in the discriminant, and that the total space of the 1-parameter deformation of the partial resolution of D_4 is smooth. The first condition translates into $\gamma_3(t) \not\equiv 0$ and a more complicated one for the other component. Smoothness of the total space can be checked in the $R = 1$ chart. We look at the equation (2.8) and its derivatives with respect to the variables y, S, P

and t. A possible singular point satisfies

$$\begin{aligned} (P^2-1)P &= 0, \\ 2S+\gamma_3(P^2-1) &= 0, \\ -3P^2(y+\beta_2)-2(\beta_1+\beta_4-\gamma_3 s)P+y-\beta_2 &= 0, \\ -\beta_1'P^2-\beta_2'(P^3+P)+(\gamma_3's-\beta_4')(P^2-1) &= 0. \end{aligned}$$

Here $\beta_i'(t)$ is the derivative of the power series $\beta_i(t)$. If $P=0$, $y=\beta_2$ and $2S=\gamma_3$, $S^2+\beta_4-\gamma_3 S=0$, so $4\beta_4=\gamma_3^2$. The condition of nonsingularity is then that $2\gamma_3\gamma_3'-4\beta_4'\neq 0$, at $t=0$. If $P=\pm 1$, $S=0$, $\beta_1\pm 2\beta_2=0$ and $y\pm\beta_4=0$ and we get the condition at $t=0$ that $\beta_1'\pm 2\beta_2'\neq 0$.

As $\gamma_3(0)=0$ the condition at $P=0$ becomes $\beta_4'\neq 0$, so by a coordinate change we may assume that $\beta_4(t)=t$. We now write $\beta_i(t)=t\bar\beta_i(t)=t(b_1+\dots)$ and $\gamma_3(t)=t\bar\gamma_3(t)=t(c_3+\dots)$. We put $F=x^2+G(y,z,t)$. The 3-jet of G is

$$j^3G=z(y+b_2t)^2-2b_2t(y+b_2t)(z-t)+(-z-b_1t)(z-t)^2.$$

This cubic defines a cubic curve in $\mathbb{P}^2$ with singular point in $(y\colon z\colon t)=(-b_2\colon 1\colon 1)$. We claim that it is irreducible. To show this we compute a parametrisation using the pencil $\lambda(y+b_2t)=\mu(z-t)$ of lines through the singular point. The curve is irreducible if and only if it has a rational parametrisation of degree 3. We find

$$(z-t)^2((\mu^2-\lambda^2)z-(2\mu b_2+b_1\lambda))\lambda t=0,$$

so the curve is reducible if and only if $\mu^2-\lambda^2$ and $(2\mu b_2+b_1\lambda)\lambda$ have a factor in common, but this only happens if $b_1\pm 2b_2=0$, which is excluded by non-singularity. The cubic has a cusp if $1+b_1+b_2^2=0$. Otherwise the curve is a nodal cubic, and the singularity is a cusp of type $T_{3,3,r}$. To determine the exact type of the singularity, also in the cuspidal cubic case, we blow up the origin. As said, we write $\beta_i=t\bar\beta_i$, $\gamma_3=t\bar\gamma_3$. We look at the appropriate chart, with coordinates (t,η,ζ), such that $(t,y,z)=(t,\eta t,\zeta t)$.

The strict transform of G is

$$(-\zeta^2-\zeta\bar\beta_1-\bar\beta_2^2)t\bar\gamma_3^2+\zeta(\eta+\bar\beta_2)^2-2\bar\beta_2(\eta+\bar\beta_2)(\zeta-1)+(-\zeta-\bar\beta_1)(\zeta-1)^2.$$

The singular point lies at $(t,\eta,\zeta)=(0,-b_2,1)$. We multiply the equation with the unit ζ and complete the square to obtain

$$(\zeta\eta+\bar\beta_2)^2-(\zeta^2+\zeta\bar\beta_1+\bar\beta_2^2)(\zeta t\bar\gamma_3^2+(\zeta-1)^2).$$

If $1+b_1+b_2^2\neq 0$, then $\zeta^2+\zeta\bar\beta_1+\bar\beta_2^2$ is a unit, and the singularity on the strict transform is an A_{2q-2} with $q=\operatorname{ord}_t\gamma_3=\operatorname{ord}_t\bar\gamma_3+1$ and the original singularity is of type $T_{3,3,2q+2}$.

Otherwise $\zeta^2 + \zeta\bar\beta_1 + \bar\beta_2^2$ is the equation of a curve, which is smooth in the point $(t,\zeta) = (0,1)$, as $b_1 = -2$ and $1 + b_1 + b_2^2 = 0$ gives $b_2^2 = 1$, contradicting the condition $b_1 \neq \pm 2b_2$ for nonsingularity. Let $\rho = \mathrm{ord}_t(1 + \bar\beta_1 + \bar\beta_2^2)$. The order of contact of the smooth branch with the cusp $\zeta t\bar\gamma_3^2 + (\zeta-1)^2$ is equal to $\min(2q-1, 2\rho)$. If the minimum is $2q-1$, then there is an E_{6q-5} and the original singularity is of type Q_{6q+5}. Otherwise we set $\delta = 2(q-\rho) - 1$; the singularity is of type $J_{\rho,\delta}$ with original singularity of type $Q_{\rho+1,\delta}$. □

Remark 2.5. — The original example of Laufer [11] of an exceptional curve with normal bundle of type $(1,-3)$ is $x^2 + y^3 + zt^2 + yz^{2q+1}$ of type Q_{6q+5}. One needs a coordinate transformation to bring this equation into our normal form. Note that the general hyperplane section does not give the standard quasi-homogeneous form for D_4.

3. A specific example

We now give an example of a non-embeddable 1-convex manifold. To have one we can compute with, we look for a simple formula with only a few terms.

In the versal family (2.2) we substitute $\beta_2 = 0$, $\beta_4 = t$, $\beta_1 = -2t$ and $\gamma_3 = it^k$. After the coordinate transformation $z \mapsto z + t$ we obtain the 3-fold singularity

$$f = x^2 + (t+z)y^2 + (t-z)z^2 - (t^2 - z^2)t^{2k}$$

of type $T_{3,3,2k+2}$. The small resolution of the previous section gives an embeddable 1-convex manifold. The given formula determines a curve in the base of the versal deformation (2.2), which intersects the discriminant in $t^4 - 2t^{2k+3} = 0$, so the hypersurface $\{f = 0\} \subset \mathbb{C}^4$ has singular points for $t^{2k-1} = \frac{1}{2}$. They are also resolved by the construction.

Now we perturb the function f by adding terms of high order. We take only one monomial, which makes the resulting hypersurface very singular at infinity. It is therefore not an example for Corollary 1.10. We show that a small resolution is non-embeddable by explicitly exhibiting a 3-chain with boundary on the exceptional curve.

Proposition 3.1. — *The affine hypersurface with equation*

$$h = x^2 + (t+z)y^2 + (t-z)z^2 - (t^2 - z^2)t^{2k} + \varepsilon t^{2m} = 0,$$

where $m > k+1$, has for almost all ε only one singular point, of type $T_{3,3,2k+2}$, isomorphic to that of f.

Proof. — We compute the singular locus $V(h,\partial_x h,\partial_y h,\partial_z h,\partial_t h)$:

$$\begin{aligned} h &= x^2+(t+z)y^2+(t-z)z^2-(t^2-z^2)t^{2k}+\varepsilon t^{2m},\\ \partial_x h &= 2x,\\ \partial_y h &= 2(t+z)y,\\ \partial_z h &= y^2+2tz-3z^2+2zt^{2k},\\ \partial_t h &= y^2+z^2-(2k+2)t^{2k+1}+2kz^2t^{2k-1}+2m\varepsilon t^{2m-1}. \end{aligned}$$

A singular point always satisfies $x=0$. If $z+t=0$, then $y^2=5t^2+2t^{2k+1}$ and $2t^3+\varepsilon t^{2m}=0$. This makes that $t\partial_t h=6t^3+2m\varepsilon t^{2m}=(6-4m)t^3$, so $t=z=y=x=0$. At the origin h has the same singularity as f (a cusp singularity has no moduli).

If $z+t\neq 0$, then $y=0$. If also $z=0$ holds, then $t^{2k+2}=\varepsilon t^{2m}$, and $t\partial_t h=2(m-k-1)t^{2k+2}\neq 0$, as $t\neq 0$.

If $z\neq 0$, then $3z=2t+2t^{2k}$. This shows that $t\neq 0$. Therefore

$$\begin{aligned} (t-2t^{2k})^2(4t+t^{2k})+27\varepsilon t^{2m} &= 0,\\ (t-2t^{2k})(4t-2t^{2k}-2k(5t+2t^{2k})t^{2k-1})+18m\varepsilon t^{2m-1} &= 0. \end{aligned}$$

The first equation shows that $t^{2k-1}=\frac{1}{2}$ gives no longer singular points for $\varepsilon\neq 0$ We eliminate ε, divide by $2t^2(t-2t^{2k})$ and find

$$(4m-6)+(15k+3-7m)t^{2k-1}+(6k-2m)t^{4k-2}=0.$$

This is a quadratic equation for t^{2k-1}. Only for finitely many values of ε there are singular points outside the origin. If $m\neq 3k$, then $3(m-3k)^2\varepsilon t^{2m-3}+(4m-10k-1)(2k-1)t^{2k-1}-2(2mk-m-2k^2-k+1)=0$. For $m=3k$ the equations simplify: $t^{2k-1}=2$, $\varepsilon t^{2m-3}=-2$, but on the other hand $t^{2m-3}=t^{6k-3}=8$, so singularities only exist for $\varepsilon=-1/4$. □

THEOREM 3.2. — *A small resolution of the affine hypersurface $\{h=0\}$, with h as in Proposition 3.1 and $\varepsilon>0$, is a non-embeddable 1-convex manifold, with rational irreducible exceptional curve with normal bundle of type $(1,-3)$.*

Proof. — The normal bundle on a small resolution is as stated, because the general hyperplane section through the singular point is of type D_4.

We prove that the manifold is not embeddable by showing that the exceptional curve C is rationally zero-homologous: $2C$ is a boundary.

We write $h(x,y,z,t)=x^2+\bar h_t(y,z)$ and $f(x,y,z,t)=x^2+\bar f_t(y,z)$, and consider $\bar h_t(y,z)$ and $\bar f_t(y,z)$ as a families of affine cubic curves. For all real $t>0$ the curve $\bar f_t(y,z)$ has three infinite branches and an oval with the origin in its interior, except for the t-value $t^{2k-1}=\frac{1}{2}$, when the

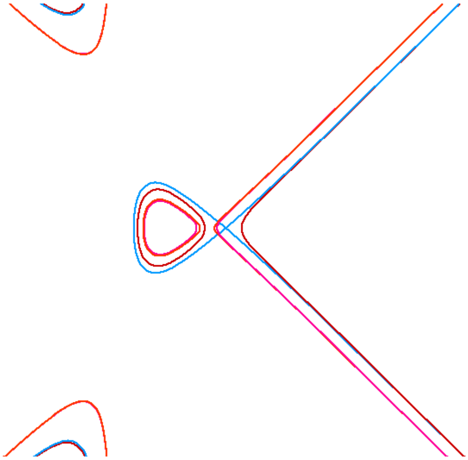


Figure 3.1. $\bar{h}_t$ *and* $\bar{f}_t$ *with* $k=2$, $m=6$ *and* $\varepsilon=1$ *for* $t=\frac{2}{3}, \sqrt[3]{\frac{1}{2}}$.

total space has a singular point. Then the intersection with the z-axis, given by $(t-z)(z^2-(t+z)t^{2k})=0$, has $z=t$ as double root. We obtain $\bar{h}_t(y,z)$ by adding the term εt^{2m} to $\bar{f}_t(y,z)$. As $\varepsilon>0$, there is no longer a double root. With increasing t the oval becomes smaller, and vanishes if $t^{2k+2}=\varepsilon t^{2m}$. This equation has only one real solution. For that t-value the curve $\bar{h}_t(y,z)$ has a singularity, as is easily seen from the computations in the proof of Proposition 3.1; this singularity is an isolated real point. We show the curves for $\varepsilon=1$, $k=2$ and $m=6$. The pictures are made with the Xalci web demo at `exacus.mpi-inf.mpg.de`. Figure 3.1 shows that for small t the curves $\bar{h}_t$ and $\bar{f}_t$ look almost the same, and that $\bar{h}_t$ does not have a double point. Figure 3.2 shows how the oval of $\bar{h}_t$ first grows and then vanishes. The family of surfaces $x^2+\bar{h}_t(y,z)$ is the family of double covers of the (y,z)-plane, branched along the curves $\bar{h}_t(y,z)$. For $0<t<\sqrt[2m-2k-2]{1/\varepsilon}$ there is a component of the real locus, which is a double covering of the interior of the oval, branched along the oval itself, while for $t=\sqrt[2m-2k-2]{1/\varepsilon}$ there is an isolated real point. The component is diffeomorphic to a 2-sphere. Together with the isolated point they form a smooth real 3-dimensional manifold M in the half-space $\{t>0\}$, which is compactified by the singular point at the origin.

On the small resolution the manifold M has boundary on the exceptional set. To compute it, we look at f. For small $t>0$ the value of z^2 on the oval is approximately at most t^{2k+1}, so $|z|\ll t$. We divide the equation by

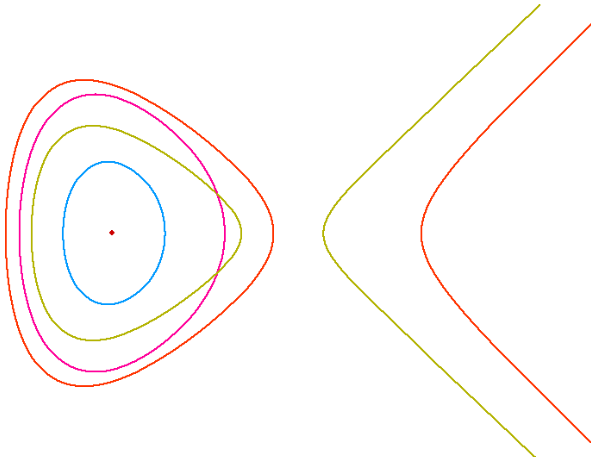


Figure 3.2. $\bar{h}_t$ *with* $k = 2$, $m = 6$ *and* $\varepsilon = 1$ *for* $t = \frac{2}{3}, \sqrt[3]{\frac{1}{2}}, \frac{19}{20}, \frac{99}{100}, 1.$

the unit $t^2 - z^2$ and use the coordinate transformation

$$\xi = \frac{x}{\sqrt{t^2 - z^2}}, \qquad \eta = \frac{y}{\sqrt{t - z}}, \qquad \zeta = \frac{z}{\sqrt{t + z}},$$

valid in a neighbourhood of the oval. Now

$$\frac{x^2}{t^2 - z^2} = \xi^2, \qquad \frac{y^2}{t - z} = \eta^2, \qquad \frac{z^2}{t + z} = \zeta^2,$$

so the transformation brings the 2-sphere in evidence:

$$\xi^2 + \eta^2 + \zeta^2 = t^{2k}.$$

We have to compute the limit for $t \to 0$ on the small resolution. We look at the chart $R = 1$. Rather than computing the inhomogeneous coordinates P and S from the homogeneous expressions P/R and S/R, we find P and S from the formulas (2.7) for z and x, which after our substitution and coordinate transformation $z \mapsto z + t$ become

$$x = yS - i(z - t)t^k P,$$
$$z = yP - it^k S.$$

This gives us

$$S = \frac{yx + iz(z - t)t^k}{y^2 - (t - z)t^{2k}}, \qquad P = \frac{yz + ixt^k}{y^2 - (t - z)t^{2k}}.$$

In the coordinates introduced above

$$S = \frac{\eta\xi - i\zeta t^k}{\eta^2 - t^{2k}}\sqrt{t+z}, \qquad P = \frac{\eta\zeta + i\xi t^k}{\eta^2 - t^{2k}}\sqrt{\frac{t+z}{t-z}}.$$

We do not express the square roots in the variable ζ, but observe that on our component of the real locus

$$\lim_{t\to 0}\sqrt{t+z} = 0, \qquad \lim_{t\to 0}\sqrt{\frac{t+z}{t-z}} = 1.$$

We parametrise the 2-sphere of radius t^k with the inverse of a stereographic projection: with $w = u + iv$ we put

$$\xi = \frac{2v}{w\bar{w}+1}t^k,$$
$$\eta = \frac{w\bar{w}-1}{w\bar{w}+1}t^k,$$
$$\zeta = \frac{2u}{w\bar{w}+1}t^k.$$

With these values we find

$$\lim_{t\to 0} S = 0, \qquad \lim_{t\to 0} P = \frac{2(w\bar{w}-1)u + 2i(w\bar{w}+1)v}{-4w\bar{w}} = \frac{1}{2}\left(\frac{1}{w} - w\right).$$

The map $P = (w^{-1} - w)/2$ is degree 2 map from $\mathbb{P}^1$ to $\mathbb{P}^1$, showing that the boundary of the real manifold M is $2C$. □

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