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STABILITY UNDER DEFORMATIONS OF HERMITE-EINSTEIN ALMOST KÄHLER METRICS

by Mehdi LEJMI (*)

Abstract. — On a 4-dimensional compact symplectic manifold, we consider a smooth family of compatible almost-complex structures such that at time zero the induced metric is Hermite-Einstein almost-Kähler metric with zero or negative Hermitian scalar curvature. We prove, under certain hypothesis, the existence of a smooth family of compatible almost-complex structures, diffeomorphic at each time to the initial one, and inducing constant Hermitian scalar curvature metrics.

Résumé. — Sur une variété symplectique compacte de dimension 4, nous considérons une famille lisse de structures presque-complexes compatibles tel qu'en temps zéro, la métrique induite est presque-kählérienne de Hermite-Einstein avec une courbure scalaire hermitienne nulle ou négative. Nous prouvons, sous une certaine hypothèse, l'existence d'une famille lisse de structures presque-complexes, difféomorphe à chaque temps à la structure initiale et induisant une métrique à courbure scalaire hermitienne constante.

1. Introduction

On a $2n$-dimensional symplectic manifold (M,ω), an almost-complex structure J is ω-compatible if it induces a Riemannian metric g via the relation $g(\cdot,\cdot) = \omega(\cdot, J\cdot)$. The metric g is called then an almost-Kähler metric. When J is integrable, the induced metric is Kähler. Given an ω-compatible almost-complex structure J, there exists a canonical Hermitian connection with torsion ∇ [11, 20] on the tangent bundle, which preserves both ω and J. The curvature of the induced Hermitian connection on the anti-canonical bundle is of the form $\sqrt{-1}\rho^{\nabla}$, where ρ^{∇} is a closed real 2-form called the Hermitian Ricci form. The Hermitian scalar curvature s^{∇}

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is defined as the contraction of ρ^∇ by ω and coincides with the (usual) Riemannian scalar curvature when the metric is Kähler. An almost-Kähler metric is called Hermite-Einstein [16] (HEAK for short) if the Hermitian Ricci form ρ^∇ satisfies $\rho^\nabla = \frac{s^\nabla}{2n}\omega$ (in particular s^∇ is a constant). Note that the terminology 'Hermite-Einstein' here *does not imply the integrability* of the almost-complex structure.

On a compact symplectic manifold (M, ω), we consider the space AK_ω of all ω-compatible almost-complex structures. This is an infinite dimensional Fréchet space equipped with a formal Kähler structure described by Fujiki [5, 8]. Furthermore, there is a natural action of the Hamiltonian group $\mathrm{Ham}(M, \omega)$ on AK_ω and it turns out that this action is Hamiltonian [5, 8] with moment map identified with the Hermitian scalar curvature. A metric induced by a critical point of the square norm of the moment map $J \mapsto \int_M (s^\nabla)^2 \omega^n$ is called an extremal almost-Kähler metric [2, 16, 17]. Moreover, an almost-Kähler metric induced by J is extremal if and only if the symplectic gradient of its Hermitian scalar curvature is an infinitesimal isometry of J. Extremal almost-Kähler metrics are a generalization of Calabi extremal Kähler metrics [3]. Furthermore, almost-Kähler metrics with constant Hermitian scalar curvature are extremal.

In the Kähler setting, Fujiki–Schumacher [9] and Lebrun–Simanca [14] showed, in the abscence of holomorphic vector fields, that the existence of extremal Kähler metrics is an open condition. Moreover, Apostolov–Calderbank–Gauduchon–Friedman [1] proved the openess by fixing a maximal torus in the reduced automorphism group of the complex manifold (M, J). Furthermore, Rollin–Simanca–Tipler [23] showed with a certain hypothesis the stability of extremal Kähler metrics under complex deformations and hence generalized the results of [14, 13] (see also [24, 25]).

In the general almost-Kähler case, one expects, from the GIT standard picture [5, 22], the existence and uniqueness of extremal almost-Kähler metrics, up to the action of $\mathrm{Ham}(M, \omega)$, in every 'stable complexified' orbit of the action of $\mathrm{Ham}(M, \omega)$. The complexification of $\mathrm{Ham}(M, \omega)$ does not exist. However, one can complexify the action on the level of the Lie algebra and once we are restricted to the integrable ω-compatible almost-complex structures, a description of this complexified orbit is given when $H^1(M, \mathbb{R}) = 0$ [5]. It is identified with the space of Kähler forms in the cohomology class of ω.

In a previous paper [17], on a compact symplectic 4-manifold (M, ω), we considered a smooth path of ω-compatible almost-complex structures J_t invariant under a (fixed) maximal torus T in $\mathrm{Ham}(M, \omega)$ such that J_0

induces an extremal Kähler metric. In particular, J_0 *is integrable*. Furthermore, we supposed that $h^-_{J_t} = b^+ - 1$ for sufficiently small t, where $h^-_{J_t}$ is the dimension of g_t-harmonic J_t-anti-invariant 2-forms [7] (here g_t is the metric induced by J_t). Then, we showed, for a short time, the existence of smooth family of T-invariant ω-compatible almost-complex structures $\tilde{J}_t$ inducing extremal almost-Kähler metrics such that $\tilde{J}_0 = J_0$ and $\tilde{J}_t$ is diffeomorphic to J_t for each t. In the spirit of Lebrun–Simanca result [14], the proof consists mainly to deform the symplectic form by introducing a notion of almost-Kähler potential (defined only in dimension 4) and then using the Banach implicit function theorem for the Hermitian scalar curvature map. The hypothesis $h^-_{J_t} = b^+ - 1$ was necessarily to insure the continuity of the Hermitian scalar curvature map since a family of Green operators is involved in the definition of this almost-Kähler potential. A recent result of Tan–Wang–Zhang–Zhu [26] implies that one can drop the assumption $h^-_{J_t} = b^+ - 1$.

Now, if we suppose that J_0 *is not integrable*, it is not clear how to identify the kernel of the linearization of the Hermitian scalar curvature map with the Lie algebra of Hamiltonian Killing vector fields even in the simplest case namely when J_0 induces a HEAK metric. The idea in this paper is to define another suitable almost-Kähler potential for which it is possible to study the kernel of the derivative of the Hermitian scalar curvature map at least in the latter case. The almost-Kähler potential defined in this paper follows from a generalization of the dd^c-Lemma [4] to the almost-Kähler case. For instance, one can derive a Hodge decomposition of the Riemannian dual of a (real) holomorphic vector field on a compact almost-Kähler manifold. When J_0 induces a HEAK metric with zero or negative Hermitian scalar curvature, we obtain the following

Theorem 1.1. — *Let (M,ω) be a 4-dimensional compact symplectic manifold. Let J_t be any smooth family of ω-compatible almost-complex structures such that J_0 induces a HEAK metric with zero or negative Hermitian scalar curvature. Moreover, suppose that for a small t, $h^-_{J_t} = h^-_{J_0} = b^+ - 1$. Then, there exists a smooth family of ω-compatible almost-complex structures $\tilde{J}_t$, defined for small t, inducing almost-Kähler metrics with constant Hermitian scalar curvature such that $\tilde{J}_t$ is diffeomorphic to J_t for each t and $\tilde{J}_0 = J_0$.*

We note that, in the above theorem, the condition $h^-_{J_t} = h^-_{J_0} = b^+ - 1$ is not to ensure the continuity of the Hermitian scalar curvature map but to guarantee the J_t-invariance of the constructed symplectic forms. Moreover, by [26], one has only to suppose that $h^-_{J_0} = b^+ - 1$. The latter condition is satisfied in the cases mentionned in [17]. Moreover, by a result of Li and Tomassini [19], any homogeneous almost-Kähler structure (ω, J) on a 4-dimensional compact manifold $M = G/\Gamma$, where G is a simply-connected Lie group and $\Gamma \subset G$ a uniform discrete subgroup, verifies $h_J^- = b^+ - 1$. Namely, the Kodaira–Thurston manifold has non-integrable almost-Kähler metrics (ω, J) with vanishing Hermitian Ricci form satisfying the condition $h_J^- = b^+ - 1$ [6, 18, 28].

2. Preliminaries

Let (M, ω) be a symplectic manifold of dimension $2n$. An almost-complex structure J is ω-compatible if the induced 2-tensor field $g(\cdot,\cdot) := \omega(\cdot, J\cdot)$ is a Riemannian metric. Then, g is called an (ω-compatible) almost-Kähler metric. In the rest of the paper, we identify the induced metric g with the couple (ω, J). If, additionally, J is integrable, then (ω, J) is a Kähler metric.

The almost-complex structure J acts on the cotangent bundle T^*M by $J\alpha(X) = -\alpha(JX)$, where α is a 1-form and X a vector field on M. The action of J can be extend to any p-form ψ by $(J\psi)(X_1, \cdots, X_p) = (-1)^p \psi(JX_1, \cdots, JX_p)$. The bundle of 2-forms $\Lambda^2 M$ decomposes under the action of J as follows

$$\Lambda^2 M = \mathbb{R}\,.\,\omega \oplus \Lambda^{J,-}M \oplus \Lambda_0^{J,+}M, \tag{2.1}$$

where $\Lambda^{J,-}M$ is the subbundle of J-anti-invariant 2-forms and $\Lambda_0^{J,+}M$ is the subbundle of J-invariant 2-forms pointwise orthogonal to ω.

For an almost-Kähler metric (ω, J), the Hermitian connection ∇ on (TM, ω, J) is defined by

$$\nabla_X Y = D^g_X Y - \frac{1}{2} J\,(D^g_X J)\,Y,$$

where D^g is the Levi-Civita connection with respect to the induced metric g and X, Y are vector fields on M. Let R^∇ be the curvature of ∇. Then, the *Hermitian Ricci form* ρ^∇ is defined by

$$\rho^\nabla(X, Y) = -\operatorname{tr}(J \circ R^\nabla_{X,Y}),$$

where $R^\nabla_{X,Y}$ is viewed as an anti-Hermitian linear operator of (TM, ω, J). The form ρ^∇ is a de Rham representative of $2\pi c_1(TM, J)$ in $H^2(M, \mathbb{R})$, where $c_1(TM, J)$ is the first (real) Chern class. If we suppose that ω and $\tilde\omega$

are symplectic forms compatible with the same almost-complex structure J and satisfy $\tilde{\omega} = e^F\omega^n$ for some real-valued function F then

$$\tilde{\rho}^\nabla = -\frac{1}{2}dJdF + \rho^\nabla, \tag{2.2}$$

where $\tilde{\rho}^\nabla$ (resp. ρ^∇) is the Hermitian Ricci form of $(\tilde{\omega}, J)$ (resp. (ω, J)).

We define the *Hermitian scalar curvature* s^∇ of an almost-Kähler metric (ω, J) as the trace of ρ^∇ with respect to ω, *i.e.*,

$$s^\nabla\omega^n = 2n\left(\rho^\nabla \wedge \omega^{n-1}\right). \tag{2.3}$$

An almost Kähler metric (ω, J) is called *Hermite-Einstein* (HEAK for short) if

$$\rho^\nabla = \frac{s^\nabla}{2n}\omega.$$

In particular, s^∇ is a constant.

The Riemannian Hodge operator $*_g : \Lambda^pM \to \Lambda^{2n-p}M$ is defined to be the unique isomorphism such that $\psi_1 \wedge (*_g)\psi_2 = g(\psi_1, \psi_2)\frac{\omega^n}{n!}$, for any p-forms ψ_1 and ψ_2. Moreover, since the dimension of M is even, $(*_g)^2\psi = (-1)^p\psi$ on p-form ψ. In dimension 4, the bundle of 2-forms decomposes as

$$\Lambda^2M = \Lambda^+M \oplus \Lambda^+M,$$

where $\Lambda^\pm M$ corresponds to the eignevalue (± 1) under the action of the Riemannian Hodge operator $*_g$. This decomposition is related to the splitting 2.1 in the following way

$$\Lambda^+M = \mathbb{R}\,\omega \oplus \Lambda^{J,-}M \quad \text{and} \quad \Lambda^-M = \Lambda_0^{J,+}M.$$

2.1. Generalized dd^c-Lemma

In this section, we generalize the dd^c-Lemma [4] to the almost-Kähler case. For this purpose, we need to present some symplectic commutators.

Let (M, ω, J, g) be a compact almost-Kähler manifold of dimension $2n$. Let δ^g be the *codifferential* defined as the formal adjoint of the Levi-Civita connection D^g with respect to the almost-Kähler metric g when it is applied to sections of $\otimes^pT^*M$. In particular, it is the adjoint of the exterior derivative d when it is applied to p-forms and are related by $\delta^g = - *_g d *_g$ since the dimension of M is even. Denote by $\Delta^g = d\delta^g + \delta^g d$ the Laplacian and $\mathbb{G}$ the Green operator associated to Δ^g. The Riemannian Hodge operator $*_g$ commutes with Δ^g. It follows that $*_g$ commutes with $\mathbb{G}$.

Let $\delta^c = (-1)^pJ\delta^gJ$ be the *twisted codifferential* acting on p-forms. This is the *symplectic adjoint* of d. Define the *twisted differential* d^c by

$d^c = (-1)^p JdJ$ acting on p-forms and let $\Delta^c = d^c\delta^c + \delta^c d^c$ be the *twisted Laplacian* and $\mathbb{G}^c$ the Green operator associated to Δ^c. One can prove in elementary way that the codifferential δ^g and the exterior derivative d (resp. δ^c and d^c) commute with $\mathbb{G}$ (resp. $\mathbb{G}^c$).

We denote by Λ_ω the contraction by the symplectic form ω defined for a p-form ψ by $\Lambda_\omega(\psi) = \frac{1}{2}\sum_{i=1}^{2n}\psi(e_i, Je_i, \cdots)$, where $\{e_i\}$ is a local J-adapted orthonormal frame. The commutator of Λ_ω and d^c is given by [10, 21]

$$[\Lambda_\omega, d^c] = \delta^g. \tag{2.4}$$

It follows that $d^c\delta^g + \delta^g d^c = 0$. Furthermore, since Λ_ω commutes with J, the relation 2.4 implies [10]

$$[\Lambda_\omega, d] = -\delta^c. \tag{2.5}$$

Moreover, if L_ω is the adjoint of Λ_ω acting on a p-form ψ by $L_\omega\psi = \omega\wedge\psi$, then [10]

$$[L_\omega, \delta^g] = d^c. \tag{2.6}$$

Now, we are in position to derive a generalization of the the dd^c-Lemma.

Lemma 2.1. — *On a compact almost-Kähler manifold, let ψ be any J-invariant p-form satisfying $\psi = d\phi$ for some $(p-1)$ form ϕ. Then,*

$$\psi = d\mathbb{G}d^c\tilde{\psi} = \mathbb{G}dd^c\tilde{\psi},$$

for some $(p-2)$-form $\tilde{\psi}$.

Proof. — It follows from the Hodge decomposition with respect to Δ^g of ψ and since $d\psi = 0$ that

$$\psi = (\psi)_H + d\delta^g\mathbb{G}\psi = d\delta^g\mathbb{G}\psi. \tag{2.7}$$

Recall that $(\psi)_H$=0 because ψ is d-exact.

On the other hand, $d^c\psi = 0$ since ψ is J-invariant. So, it follows from the Hodge decomposition with respect to Δ^c that $\psi = (\psi)_{H^c} + d^c\delta^c\mathbb{G}^c\psi$ (here $(\psi)_{H^c}$ denotes the harmonic part with respect to Δ^c, in particular

$d^c(\psi)_{H^c} = \delta^c(\psi)_{H^c} = 0$). Plugging this in 2.7, we obtain

$$\begin{aligned}\psi &= d\delta^g\mathbb{G}\Big(\big((\psi)_{H^c}\big) + d^c\delta^c\mathbb{G}^c\psi\Big),\\ &= d\delta^g\mathbb{G}\big((\psi)_{H^c}\big) + d\delta^g\mathbb{G}(d^c\delta^c\mathbb{G}^c\psi),\\ &= d\mathbb{G}\delta^g\big((\psi)_{H^c}\big) + d\mathbb{G}\delta^g(d^c\delta^c\mathbb{G}^c\psi),\\ &= d\mathbb{G}[\Lambda_\omega, d^c]\big((\psi)_{H^c}\big) - d\mathbb{G}d^c(\delta^g\delta^c\mathbb{G}^c\psi),\\ &= -d\mathbb{G}d^c\Lambda_\omega((\psi)_{H^c}) - d\mathbb{G}d^c(\delta^g\delta^c\mathbb{G}^c\psi),\\ &= d\mathbb{G}d^c\Big(-\Lambda_\omega\big((\psi)_{H^c}\big) - \delta^g\delta^c\mathbb{G}^c\psi\Big).\end{aligned}$$

Here, we used the equality 2.4 and the fact that $d^c\delta^g + \delta^g d^c = 0$. The Lemma follows. □

In the Kähler case, remark that $\Delta = \Delta^c$ so $\mathbb{G} = \mathbb{G}^c$. Hence, $d^c\mathbb{G} = \mathbb{G}d^c$. Then, $\psi = d\mathbb{G}d^c\tilde{\psi} = dd^c(\mathbb{G}\tilde{\psi})$.

Proposition 2.2. — *On a compact almost-Kähler manifold, let ψ_1, ψ_2 be any two real J-invariant closed 2-forms and suppose that ψ_1, ψ_2 determine the same de Rham cohomology class. Then, there exists a real function f, uniquely defined up to an additive constant, such that*

$$\psi_1 - \psi_2 = d\mathbb{G}d^c f = \mathbb{G}dd^c f.$$

Proof. — This is a direct application of Lemma 2.1 for $\psi = \psi_1 - \psi_2$. If $\mathbb{G}dd^c f = 0$ then $dd^c f$ is harmonic. Since M is compact, $dd^c f = 0$. By the equality 2.4, it follows that $\Delta^g f = \delta^g df = [\Lambda_\omega, d^c]df = \Lambda_\omega d^c df = 0$ (because $d^c df = -Jdd^c f$). So f is constant as M is compact. □

As a consequence of Lemma 2.1, we obtain a Hodge decomposition of the Riemannian dual of a (real) holomorphic vector field on a compact almost-Kähler manifold (M, ω, J, g). Recall that a (real) vector field X is called holomorphic if it is an infinitesimal isometry of J *i.e.*, $\mathfrak{L}_X J = 0$, where $\mathfrak{L}$ is the Lie derivative.

Corollary 2.3. — *Let X be a holomorphic vector field on a compact almost-Kähler manifold and $\xi = X^{\flat_g}$ the dual of X with respect to the metric g. Then, we have*

$$\xi = (\xi)_{H^c} + d^c u - J\mathbb{G}d^c v, \tag{2.8}$$

where u, v are real functions, uniquely defined up to an additive constant. Here $(\xi)_{H^c}$ denotes the harmonic part with respect to Δ^c.

Remark that in the Kähler case, $(\xi)_{H^c} = \xi_H$ and $-J\mathbb{G}d^c v = d(\mathbb{G}v)$.

Proof. — Since X is holomorphic, a direct computation shows that $\mathfrak{L}_X\omega$ is a J-invariant 2-form. Hence, $dJ\xi$ is J-invariant. By Lemma 2.1, $dJ\xi = d\mathbb{G}d^c v$, for a function v uniquely defined up to a constant. The Hodge decomposition with respect to Δ^c of ξ is given by $\xi = (\xi)_{H^c} + d^c u + \delta^c\phi$ for some real function u and 2-form ϕ. Then

$$dJ\xi = -d\delta^g J\phi = d\mathbb{G}d^c v. \tag{2.9}$$

Moreover, using equality 2.6, we have $d\mathbb{G}d^c v = -d\mathbb{G}\delta^g(v\omega) = -d\delta^g\mathbb{G}(v\omega)$. So, from 2.9 we have $-d\delta^g J\phi = -d\delta^g\mathbb{G}(v\omega)$, thus $-\delta^g J\phi = -\delta^g\mathbb{G}(v\omega) = \mathbb{G}d^c v$. The Corollary follows. □

Now, given any function f, it is natural to wonder whether $d\mathbb{G}d^c f$ is J-invariant.

Proposition 2.4. — *In dimension $2n = 4$, for any smooth function f,*

$$(d\mathbb{G}d^c f)^{J,-} = \frac{1}{2}(f_0\omega)_H - \frac{1}{4}g\left((f_0\omega)_H, \omega\right)\omega.$$

In particular, if $h_J^- = b^+ - 1$, then $d\mathbb{G}d^c f$ is J-invariant (here f_0 is the orthogonal projection of f onto the complement of the constants).

Proof. — Using the equality 2.6 and the fact that the Hodge operator $*_g$ commutes with $\mathbb{G}$, we have

$$\begin{aligned}
(d\mathbb{G}d^c f)^{J,-} &= (-d\mathbb{G}\delta^g(f\omega))^{J,-}\\
&= \frac{1}{2}(I + *_g)(-\mathbb{G}d\delta^g(f\omega)) - \frac{1}{4}g\left((I + *_g)(-\mathbb{G}d\delta^g(f\omega)), \omega\right)\omega\\
&= -\frac{1}{2}\mathbb{G}\Delta^g(f\omega) + \frac{1}{4}g\left(\mathbb{G}\Delta^g(f\omega), \omega\right)\omega\\
&= -\frac{1}{2}f\omega + \frac{1}{2}(f\omega)_H + \frac{1}{2}f\omega - \frac{1}{4}g\left((f\omega)_H, \omega\right)\omega\\
&= \frac{1}{2}(f\omega)_H - \frac{1}{4}g\left((f\omega)_H, \omega\right)\omega\\
&= \frac{1}{2}(f_0\omega)_H - \frac{1}{4}g\left((f_0\omega)_H, \omega\right)\omega.
\end{aligned}$$

Here, we use the convention $g(\omega, \omega) = 2$. In case when $h_J^- = b^+ - 1$, we have $(f_0\omega)_H = 0$. Indeed, under the latter assumption, for any g-harmonic 2-form ψ (with respect to Δ^g), the pairing $g(\omega, \psi)$ is a constant function. Thus, given a function f, we obtain $< (f_0\omega)_H, \psi >_{L_2} = \int_M f_0 g(\omega, \psi)\frac{\omega^2}{2!} = 0$. Hence, $(f_0\omega)_H = 0$ and therefore $d\mathbb{G}d^c f$ is J-invariant. □

Thus, in dimension 4, when $h_J^- = b^+ - 1$, the symplectic form $\omega + d\mathbb{G}d^c f$ is J-invariant for any function f and so f is called *almost-Kähler potential* when $(\omega + d\mathbb{G}d^c f, J)$ induces a Riemannian metric. Again remark that in the Kähler case, $d\mathbb{G}d^c f = dd^c\mathbb{G}f$, hence $\mathbb{G}f$ coincides with the usual Kähler potential.

The 'potential' $\mathbb{G}f$ coincides with the potential defined by Weinkove [29] in the following way: Given a symplectic form $\tilde{\omega}$ compatible with J and cohomologous to ω, then $\tilde{\omega} - \omega = d\mathbb{G}d^c f$, for some function f. Now, let $\phi = \mathbb{G}f$ then a direct computation shows that

$$d\mathbb{G}d^c f = d\mathbb{G}d^c\Delta^g\phi = dd^c\phi - 2d\delta^g\mathbb{G}D^g_{(d\phi)^{\sharp_g}}\omega.$$

The function ϕ corresponds to ϕ_0 in the terminology of [29].

3. Proof of Theorem 1.1

Let (M,ω) be a 4-dimensional compact and connected symplectic manifold. Suppose that J_0 is an ω-compatible almost-complex structure which induces a HEAK metric with zero or negative Hermitian scalar curvature *i.e.*, the Hermitian Ricci form ρ^∇ of (ω, J_0) satisfies $\rho^\nabla = \frac{s^\nabla}{4}\omega$ with $s^\nabla \leqslant 0$. Moreover, suppose $h^-_{J_0} = b^+ - 1$, where $h^-_{J_0}$ is the dimension of g_0-harmonic J_0-anti-invariant 2-forms [7]. Let J_t be a smooth family of ω-compatible almost-complex structures in AK_ω such that $h^-_{J_t} = h^-_{J_0} = b^+ - 1$ for a small t. Denote by $g_t(\cdot,\cdot) = \omega(\cdot, J_t\cdot)$ the induced metric.

We consider the following almost-Kähler deformations

$$\omega_{t,f} = \omega + d\mathbb{G}_t J_t df,$$

where $\mathbb{G}_t$ is the Green operator associated to the Laplacian operator Δ^{g_t} with respect to the metric g_t and $f \in C^\infty_0(M,\mathbb{R})$ a smooth function with zero mean value.

By Proposition 2.4, the assumption $h^-_{J_t} = b^+ - 1$ implies that $\omega_{t,f}$ is J_t-invariant. Then, we define the map:

$$\begin{array}{rccc}\Phi: & \mathbb{R}\times C^\infty_0(M,\mathbb{R}) & \longrightarrow & C^\infty_0(M,\mathbb{R}) \\ & (t,f) & \longmapsto & \mathring{s}^{\nabla_{t,f}},\end{array}$$

where $\mathring{s}^{\nabla_{t,f}}$ is the zero integral part of the Hermitian scalar curvature $s^{\nabla_{t,f}}$ of the almost-Kähler metric $(\omega_{t,f}, J_t)$. We have $\Phi(t,f) = 0$ if and only if $(\omega_{t,f}, J_t)$ is an almost-Kähler metric with constant Hermitian scalar curvature. In particular, $\Phi(0,0) = 0$.

Let $W^{p,k}$ be the completion of $C^\infty_0(M,\mathbb{R})$ with respect to the Sobolev norm $\|\cdot\|_{p,k}$ involving derivatives up to order k. Denote by $\Phi^{(p,k)} : \mathbb{R}\times W^{p,k+2} \longrightarrow W^{p,k}$ the extension of Φ to the Sobolev completion of $C^\infty_0(M,\mathbb{R})$. The map $\Phi^{(p,k)}$ is well defined when $pk > 2n$. The kernel of the Laplacian

Δ^{g_t} are g_t-harmonic p-forms and thus the dimension of the kernel of Δ^{g_t} is independent of t. Hence, we deduce from [12, Theorem 7.6] that $\mathbb{G}_t$ is a C^1 map. Thus, the map $\Phi^{(p,k)}$ is clearly a C^1 map.

Using the formula 2.2 and definition of the Hermitan scalar curvature, we have the following

Proposition 3.1. — *Let (M,ω,J,g) be a 4-dimensional compact almost-Kähler manifold. Denote by $\mathbb{G}$ the Green operator associated to the Laplacian Δ^g. Suppose that $d\mathbb{G}d^c f$ is J-invariant for any function f. Then, for any almost-Kähler variation $\dot{\omega} = d\mathbb{G}d^c\dot{f}$ of the symplectic form ω ($\dot{f}$ with zero integral), the variation of the volume form, of the Hermitian Ricci form and the Hermitan scalar curvature are given by*

$$\dot{(\omega^2)} = (\delta^g J\mathbb{G}d^c\dot{f})\ \omega^2 = -\dot{f}\omega^2, \tag{3.1}$$

$$\dot{\rho^\nabla} = \frac{1}{2}dd^c\dot{f}, \tag{3.2}$$

$$\dot{s^\nabla} = -\Delta^g\dot{f} - 2g(\rho^\nabla, d\mathbb{G}d^c\dot{f}). \tag{3.3}$$

Remark that, in the Kähler case, if we substitute $\dot{f}$ by $\Delta^g\dot{\phi}$, then the above variations coincide with the variation $\dot{\omega} = dd^c\dot{\phi}$ of the Kähler form ω in the (fixed) Kähler class [15].

Proof. — Let $\omega_t = \omega + td\mathbb{G}d^c\dot{f}$. Then, using the relation 2.5, $\dot{(\omega^2)} = \frac{d}{dt}(\omega_t)^2|_{t=0} = g(d\mathbb{G}d^c\dot{f},\omega)\ \omega^2 = (\delta^g J\mathbb{G}d^c\dot{f})\ \omega^2$. Moreover, using the fact that $d\delta^c + \delta^c d = 0$ and the J-invariance of $d\mathbb{G}d^c\dot{f}$, we have

$$\begin{aligned} d\delta^g J\mathbb{G}d^c\dot{f} &= -d\delta^c\mathbb{G}d^c\dot{f} \\ &= \delta^c d\mathbb{G}d^c\dot{f} \\ &= J\delta^g d\mathbb{G}d^c\dot{f} \\ &= J\Delta^g\mathbb{G}d^c\dot{f} \\ &= Jd^c\dot{f} = -d\dot{f}. \end{aligned}$$

Since $\dot{f}$ has zero integral, we obtain the second equality in 3.1. The variation of the Hermitian Ricci form 3.2 follows from 2.2 while the expression 3.3 is a consequence of 2.3. □

Since (ω, J_0) is HEAK and $g(d\mathbb{G}d^c\dot{f},\omega) = -\dot{f}$, it follows from Proposition 3.1 that the partial derivative $\frac{\partial\Phi}{\partial f}|_{(0,0)}$ is given by

$$\frac{\partial\Phi}{\partial f}|_{(0,0)}(\dot{f}) = -\Delta^{g_0}\dot{f} + \frac{s^\nabla}{2}\dot{f}, \tag{3.4}$$

where s^∇ is the Hermitian scalar curvature (ω, J_0). Clearly, $\frac{\partial\Phi}{\partial f}|_{(0,0)}$ is a self-adjoint elliptic linear operator. Furthermore, it is an isomorphism of

$C_0^\infty(M,\mathbb{R})$. Indeed, suppose that $-\Delta^{g_0}\dot f + \frac{s^\nabla}{2}\dot f = 0$ for a function $\dot f$ (with zero integral), then $\Delta^{g_0}\dot f = \frac{s^\nabla}{2}\dot f$. By hypothesis, $s^\nabla \leqslant 0$. As M is compact, $\dot f \equiv 0$. The natural extension of $\frac{\partial\Phi}{\partial f}|_{(0,0)}$ to $W^{p,k+2}$ is again an isomorphism from $W^{p,k+2}$ to $W^{p,k}$. It follows from the implicit function theorem for Banach spaces that there exists $\epsilon,\delta>0$ such that for $|t|<\epsilon$, there exists f_t satisfying $\|f_t\|_{p,k}<\delta$ such that $\Psi^{(p,k)}(t,f_t)=0$. Hence, for each $|t|<\epsilon$, (ω_{t,f_t},J_t) is an almost-Kähler metric with constant Hermitian scalar curvature of regularity $W^{p,k+2}$. It follows from the bootstraping argument used in [17] that (ω_{t,f_t},J_t) are a family of smooth almost-Kähler metrics with constant Hermitian scalar curvature. Theorem 1.1 follows from the Moser Lemma.

Example 3.2. — Theorem 1.1 may be applied to the Kodaira–Thurston manifold given by $S^1\times(\mathrm{Nil}^3/\Gamma)$ where

$$\mathrm{Nil}^3 = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},\ x,y,z\in\mathbb{R} \right\},$$

and Γ is the subgroup of Nil^3 consisting of elements with integral entries. The 1-forms $dt, dx, dy, dz-xdy$ are invariant under the action of Γ (here t is the S^1 coordinate).

By a result of Li and Tomassini [19], any homogeneous almost-Kähler structure (ω,J) on $S^1\times(\mathrm{Nil}^3/\Gamma)$ has $h_J^- = b^+-1 = 1$. Namely, the following symplectic form

$$\omega = dx\wedge dt + dy\wedge(dz-xdy)$$

and the non-integrable ω-compatible almost-complex structure

$$Jdx = dt, \qquad Jdy = (dz-xdy)$$

verifies $h_J^- = b^+-1$. Moreover, the Hermitian Ricci form ρ^∇ of (ω,J) is zero [6, 18, 28].

Remark 3.3. — When the Hermitian scalar curvature is positive one can prove the following: Suppose that T is a maximal torus in $\mathrm{Ham}(M,\omega)$. Let J_t be any smooth family of ω-compatible T-invariant almost-complex structure such that J_0 induces a HEAK metric with positive Hermitian scalar curvature and close enough in C^∞-topology to an integrable T-invariant ω-compatible almost-complex structure J. Moreover, suppose that for a small t, $h_{J_t}^- = h_{J_0}^- = b^+-1$. Then, there exists a smooth family of ω-compatible T-invariant almost-complex structures $\tilde J_t$ inducing almost-Kähler metrics with constant Hermitian scalar curvature such that $\tilde{J}_t$ is diffeomorphic to J_t and $\tilde{J}_0 = J_0$.

Observe that our hypothesis here implies that (M, J) is a Fano complex surface. Since $b^+ = 1$, the condition $h^-_{J_t} = b^+ - 1 = 0$ is automatically satisfied for any family J_t. By Tian result [27], there exists a Kähler-Einstein metric except in the first Hirzebruch surface and its blown up at one point (actually these two surfaces are toric). In the latter two cases, the Futaki invariant of the anti-canonical class being non-zero implies that there is no toric HEAK metric [16].

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BIBLIOGRAPHY

[1] V. Apostolov, D. M. J. Calderbank, P. Gauduchon & C. W. Tønnesen-Friedman, "Extremal Kähler metrics on projective bundles over a curve", *Adv. Math.* **227** (2011), no. 6, p. 2385-2424.

[2] V. Apostolov & T. Drăghici, "The curvature and the integrability of almost-Kähler manifolds: a survey", in *Symplectic and contact topology: interactions and perspectives (Toronto, ON/Montreal, QC, 2001)*, Fields Inst. Commun., vol. 35, Amer. Math. Soc., Providence, RI, 2003, p. 25-53.

[3] E. Calabi, "Extremal Kähler metrics", in *Seminar on Differential Geometry*, Ann. of Math. Stud., vol. 102, Princeton Univ. Press, Princeton, N.J., 1982, p. 259-290.

[4] P. Deligne, P. Griffiths, J. Morgan & D. Sullivan, "Real homotopy theory of Kähler manifolds", *Invent. Math.* **29** (1975), no. 3, p. 245-274.

[5] S. K. Donaldson, "Remarks on gauge theory, complex geometry and 4-manifold topology", in *Fields Medallists' lectures*, World Sci. Ser. 20th Century Math., vol. 5, World Sci. Publ., River Edge, NJ, 1997, p. 384-403.

[6] T. Drăghici, "Lecture notes and private communications".

[7] T. Drăghici, T.-J. Li & W. Zhang, "Symplectic forms and cohomology decomposition of almost complex four-manifolds", *Int. Math. Res. Not. IMRN* (2010), no. 1, p. 1-17.

[8] A. Fujiki, "Moduli space of polarized algebraic manifolds and Kähler metrics [translation of Sûgaku **42** (1990), no. 3, 231–243; MR1073369 (92b:32032)]", *Sugaku Expositions* **5** (1992), no. 2, p. 173-191, Sugaku Expositions.

[9] A. Fujiki & G. Schumacher, "The moduli space of extremal compact Kähler manifolds and generalized Weil-Petersson metrics", *Publ. Res. Inst. Math. Sci.* **26** (1990), no. 1, p. 101-183.

[10] P. Gauduchon, "Calabi's extremal Kähler metrics: An elementary introduction", In preparation.

[11] ———, "Hermitian connections and Dirac operators", *Boll. Un. Mat. Ital. B (7)* **11** (1997), no. 2, suppl., p. 257-288.

[12] K. Kodaira, *Complex manifolds and deformation of complex structures*, english ed., Classics in Mathematics, Springer-Verlag, Berlin, 2005, Translated from the 1981 Japanese original by Kazuo Akao, x+465 pages.

[13] C. LeBrun & S. R. Simanca, "On the Kähler classes of extremal metrics", in *Geometry and global analysis (Sendai, 1993)*, Tohoku Univ., Sendai, 1993, p. 255-271.

[14] ———, "Extremal Kähler metrics and complex deformation theory", *Geom. Funct. Anal.* **4** (1994), no. 3, p. 298-336.

[15] ———, "On Kähler surfaces of constant positive scalar curvature", *J. Geom. Anal.* **5** (1995), no. 1, p. 115-127.

[16] M. Lejmi, "Extremal almost-Kähler metrics", *Internat. J. Math.* (2010), no. 12, p. 1639-1662.

[17] ———, "Stability under deformations of extremal almost-Kähler metrics in dimension 4", *Math. Res. Lett.* **17** (2010), no. 4, p. 601-612.

[18] T.-J. Li, "Symplectic Calabi-Yau surfaces", in *Handbook of geometric analysis, No. 3*, Adv. Lect. Math. (ALM), vol. 14, Int. Press, Somerville, MA, 2010, p. 231-356.

[19] T.-J. Li & A. Tomassini, "Almost Kähler structures on four dimensional unimodular Lie algebras", *J. Geom. Phys.* **62** (2012), no. 7, p. 1714-1731.

[20] P. Libermann, "Sur les connexions hermitiennes", *C. R. Acad. Sci. Paris* **239** (1954), p. 1579-1581.

[21] S. A. Merkulov, "Formality of canonical symplectic complexes and Frobenius manifolds", *Internat. Math. Res. Notices* (1998), no. 14, p. 727-733.

[22] D. Mumford, J. Fogarty & F. Kirwan, *Geometric invariant theory*, third ed., Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], vol. 34, Springer-Verlag, Berlin, 1994, xiv+292 pages.

[23] Y. Rollin, S. R. Simanca & C. Tipler, "Deformation of extremal metrics, complex manifolds and the relative Futaki invariant", *Math. Z.* **273** (2013), no. 1-2, p. 547-568.

[24] Y. Rollin & C. Tipler, "Deformations of extremal toric manifolds", preprint 2013, math.DG/1201.4137.

[25] G. Székelyhidi, "The Kähler-Ricci flow and K-polystability", *Amer. J. Math.* **132** (2010), no. 4, p. 1077-1090.

[26] Q. Tan, H. Wang, Y. Zhang & P. Zhu, "Symplectic cohomology and the stability of J-anti-invariant cohomology", preprint 2013, math.DG/1307.1513.

[27] G. Tian, "K-stability and Kähler-Einstein metrics", preprint 2013, math.DG/1211.4669.

[28] L. Vezzoni, "A note on canonical Ricci forms on 2-step nilmanifolds", *Proc. Amer. Math. Soc.* **141** (2013), no. 1, p. 325-333.

[29] B. Weinkove, "The Calabi-Yau equation on almost-Kähler four-manifolds", *J. Differential Geom.* **76** (2007), no. 2, p. 317-349.

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