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PSEUDO-REAL PRINCIPAL HIGGS BUNDLES ON COMPACT KÄHLER MANIFOLDS

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ABSTRACT. — Let X be a compact connected Kähler manifold equipped with an anti-holomorphic involution which is compatible with the Kähler structure. Let G be a connected complex reductive affine algebraic group equipped with a real form σ_G. We define pseudo-real principal G-bundles on X. These are generalizations of real algebraic principal G-bundles over a real algebraic variety. Next we define stable, semistable and polystable pseudo-real principal G-bundles. Their relationships with the usual stable, semistable and polystable principal G-bundles are investigated. We then prove that the following Donaldson-Uhlenbeck-Yau type correspondence holds: a pseudo-real principal G-bundle admits a compatible Einstein-Hermitian connection if and only if it is polystable. A bijection between the following two sets is established:

(1) The isomorphism classes of polystable pseudo-real principal G-bundles such that all the rational characteristic classes of positive degree of the underlying topological principal G-bundle vanish.
(2) The equivalence classes of twisted representations of the extended fundamental group of X in a σ_G-invariant maximal compact subgroup of G. (The twisted representations are defined using the central element in the definition of a pseudo-real principal G-bundle.)

All these results are also generalized to the pseudo-real Higgs G-bundle.

Keywords: Pseudo-real bundle, real form, Einstein-Hermitian connection, Higgs bundle, polystability.
Math. classification: 14P99, 53C07, 32Q15.

Résumé. — Soit X une variété kählerienne compacte et connexe, équipée d'une involution antiholomorphe compatible avec la structure Kählerienne. Soit G un groupe algébrique affine complexe, connexe et muni d'une forme réelle σ_G. Nous définissons des G-fibrés principaux holomorphes pseudo-réels sur X, ce qui généralise la notion de G-fibré principal réel sur une variété réelle. Nous introduisons ensuite les notions de G-fibré principal pseudo-réel stable, semi-stable et polystable. La relation de ces concepts avec les notions usuelles de G-fibré principal stable, semi-stable et polystable est discutée. Nous démontrons ensuite qu'il existe une correspondance de type Donaldson-Uhlenbeck-Yau : un G-fibré principal holomorphe pseudo-réel admet une connection Hermite-Einstein compatible si et seulement s'il est polystable. Nous établissons ensuite une bijection entre les deux ensembles suivants :

(1) Les classes d'isomorphisme de G-fibrés principaux holomorphes pseudo-réels sur X, dont toutes les classes caractéristiques rationnelles du G-fibré topologique sous-jacent s'annulent.
(2) Les classes d'équivalence de représentations tordues du groupe fondamental étendu de X dans un sous-groupe maximal compact σ_G-invariant de G. (Les représentations tordues sont définies en utilisant l'élément central qui entre dans la définition d'un G-fibré principal pseudo-réel.)

Tous ces résultats sont ensuite généralisés au cas du G-fibré de Higgs pseudo-réel.

1. Introduction

Let G be a connected reductive affine algebraic group defined over $\mathbb{C}$. Let

$$\sigma_G \colon G \longrightarrow G$$

be a real form on G. Fix a maximal compact subgroup $K_G \subset G$ such that $\sigma_G(K_G) = K_G$. Also, fix an element c in the center of K_G such that $\sigma_G(c) = c$. Let (X, ω) be a compact connected Kähler manifold equipped with an anti-holomorphic involution σ_X such that $\sigma_X^* \omega = -\omega$.

Using c, we define pseudo-real principal G-bundles on X (see Definition 2.1). We define stable, semistable and polystable pseudo-real principal G-bundles on X. These are related to the usual semistable and polystable principal G-bundles in the following way:

Proposition 1.1. — *A pseudo-real principal G-bundle (E_G, ρ) on X is semistable (respectively, polystable) if and only if the underlying holomorphic principal G-bundle E_G is semistable (respectively, polystable).*

Proposition 1.1 is proved in Lemma 2.5, Lemma 3.3 and Corollary 3.11.

Theorem 1.2. — *Let (E_G, ρ) be a pseudo-real principal G-bundle on X. The following two statements are equivalent:*

(1) *(E_G, ρ) is polystable.*

(2) *The holomorphic principal G-bundle E_G has an Einstein-Hermitian reduction of structure group $E_{K_G} \subset E_G$ to the maximal compact subgroup K_G such that $\rho(E_{K_G}) = E_{K_G}$.*

Theorem 1.2 is proved in Corollary 3.9 and Proposition 3.10.

Fix a point $x_0 \in X$ such that $\sigma_X(x_0) \neq x_0$. Let $\Gamma(X, x_0)$ be the homotopy classes of paths originating from x_0 that end in either x_0 or $\sigma_X(x_0)$. It is a group that fits in a short exact sequence

$$e \longrightarrow \pi_1(X, x_0) \longrightarrow \Gamma(X, x_0) \xrightarrow{\eta} \mathbb{Z}/2\mathbb{Z} \longrightarrow e.$$

Let $\widetilde{K} = K_G \rtimes (\mathbb{Z}/2\mathbb{Z})$ be the semi-direct product constructed using the involution σ_G of K_G. Let $\mathrm{Map}'(\Gamma(X, x_0), \widetilde{K})$ be the space of all maps $\delta\colon \Gamma(X, x_0) \to \widetilde{K}$ such that $\delta^{-1}(K_G) = \pi_1(X, x_0)$. We will write $\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$. Let $\mathrm{Hom}_c(\Gamma(X, x_0), \widetilde{K})$ be the space of all maps $\delta \in \mathrm{Map}'(\Gamma(X, x_0), \widetilde{K})$ such that

- the restriction of δ to $\pi_1(X, x_0)$ is a homomorphism of groups,
- $\delta(g'g) = c\delta(g')\delta(g)$, if $\eta(g) = 1 = \eta(g')$, where η is the above homomorphism, and
- $\delta(g'g) = \delta(g')\delta(g)$ if $\eta(g) \cdot \eta(g') = 0$.

A more intrinsic definition of $\mathrm{Hom}_c(\Gamma(X, x_0), \widetilde{K})$ is given in Remark 4.1.

Two elements $\delta', \delta' \in \mathrm{Hom}_c(\Gamma(X, x_0), \widetilde{K})$ are called *equivalent* if there is an element $g \in K_G$ such that $\delta'(z) = g^{-1}\delta(z)g$ for all $z \in \Gamma(X, x_0)$.

We prove the following (see Theorem 4.6):

Theorem 1.3. — *There is a natural bijective correspondence between the equivalence classes of elements of* $\mathrm{Hom}_c(\Gamma(X, x_0), \widetilde{K})$*, and the isomorphism classes of polystable pseudo-real principal G-bundles (E_G, ρ) satisfying the following two conditions:*

- $\int_X c_2(\mathrm{ad}(E_G)) \wedge \omega^{\dim_{\mathbb{C}}(X)-2} = 0$*, and*
- *for any character χ of G, the line bundle over X associated to E_G for χ is of degree zero.*

It may be mentioned that a polystable principal G-bundle E_G satisfies the above two numerical conditions if and only if all the rational characteristic classes of E_G of positive degree vanish.

In Section 5, we extends the above results to the more general context of pseudo-real principal G-bundle on X equipped with a Higgs field compatible with the pseudo-real structure. We prove the following (see Proposition 5.5 and Proposition 5.6):

Proposition 1.4. — *Let (E_G, ρ, θ) be a pseudo-real principal Higgs G-bundle. Then the principal Higgs G-bundle (E_G, θ) admits an Einstein-Hermitian structure $E_{K_G} \subset E_G$ with $\rho(E_{K_G}) = E_{K_G}$ if and only if (E_G, ρ, θ) is polystable.*

The definition of an Einstein-Hermitian structure on a principal Higgs G-bundle is recalled in Definition 5.4.

Let $\widetilde{G} := G \rtimes (\mathbb{Z}/2\mathbb{Z})$ be the semi-direct product constructed using σ_G. Define $\mathrm{Hom}_c(\Gamma(x_0), \widetilde{G})$ as before by replacing $\widetilde{K}$ with $\widetilde{G}$. See Section 5 for the equivalence classes of completely reducible elements of $\mathrm{Hom}_c(\Gamma(x_0), \widetilde{G})$.

Proposition 1.5. — *There is a natural bijective correspondence between the equivalence classes of completely reducible elements of $\mathrm{Hom}_c(\Gamma(x_0), \widetilde{G})$, and the isomorphism classes of polystable pseudo-real principal Higgs G-bundles (E_G, ρ, θ) satisfying the following conditions:*

- $\int_X c_2(\mathrm{ad}(E_G)) \wedge \omega^{\dim_{\mathbb{C}}(X)-2} = 0$, *and*
- *for any character χ of G, the line bundle over X associated to E_G for χ is of degree zero.*

When X is a compact Riemann surface, some of the above results were obtained in [6].

Without loss of any generality, we may assume that the element c is of order two (see Remark 3.6 for an explanation). The real principal G-bundles are very closely related to principal bundles on a variety defined over $\mathbb{R}$. This is elaborated in Remark 2.2.

A comment on the definition of (semi)stability is in order. As explained in [6, Section 2.3], when the base field is $\mathbb{R}$ (more generally, when it is not algebraically closed), the definition in [3], and not the one in [15], is the right one. Therefore, we have to follow the definition of [3] here.

2. Pseudo-real principal bundles

Let X be a compact connected Kähler manifold. The real tangent bundle of X will be denoted by $T^{\mathbb{R}}X$. The almost complex structure on X, which is a smooth section of $\mathrm{End}(T^{\mathbb{R}}X) = (T^{\mathbb{R}}X) \otimes (T^{\mathbb{R}}X)^*$, will be denoted by J. Let

$$\sigma_X : X \longrightarrow X$$

be a diffeomorphism such that

$$d\sigma_X \circ J = -J \circ d\sigma_X, \tag{2.1}$$

where

$$d\sigma_X : T^{\mathbb{R}}X \longrightarrow \sigma_X^* T^{\mathbb{R}}X \tag{2.2}$$

is the differential of σ_X.

Let ω be a Kähler form on X. The inner product on $T^{\mathbb{R}}X$ corresponding to ω will be denoted by $\widetilde{\omega}$. The Kähler form ω is said to be *compatible* with σ_X if $d\sigma_X$ preserves $\widetilde{\omega}$. Using (2.1) it is straightforward to check that ω is compatible with σ_X if and only if $\sigma_X^*\omega = -\omega$.

The Kähler manifold X admits a Kähler form compatible with σ_X. To see this, take any Kähler form ω on X, and define $\widetilde{\omega}$ as above. Let $\widetilde{\widetilde{\omega}}$ be the Riemannian metric on X defined by

$$\widetilde{\widetilde{\omega}}(v,w) := \widetilde{\omega}(v,w) + \widetilde{\omega}(d\sigma_X(v), d\sigma_X(w))$$

($d\sigma_X$ is defined in (2.2)). Since $d\sigma_X \circ J = -J \circ d\sigma_X$, and J is orthogonal with respect to $\widetilde{\omega}$, it follows that $\widetilde{\widetilde{\omega}}$ also defines a Kähler structure on X. In fact, the Kähler form for $\widetilde{\widetilde{\omega}}$ is $\omega - \sigma_X^*\omega$, hence the Kähler form is closed. This Kähler structure defined by $\widetilde{\widetilde{\omega}}$ is clearly compatible with σ_X.

Fix a Kähler form ω on X compatible with σ_X. For a torsionfree coherent analytic sheaf F on X, define

$$\mathrm{degree}(F) := \int_X c_1(F)\omega^{\dim_{\mathbb{C}}(X)-1} \in \mathbb{R}. \tag{2.3}$$

Let G be a connected reductive affine algebraic group defined over $\mathbb{C}$. We fix a real form σ_G of G. This means that

$$\sigma_G : G \longrightarrow G$$

is an anti-holomorphic isomorphism of order two. The Lie algebra of G will be denoted by $\mathfrak{g}$. The center of G will be denoted by Z_G. Let

$$Z_{\mathbb{R}} := Z_G \cap G^{\sigma_G}$$

be the group of fixed points in Z_G for the involution σ_G.

Let E_G be a holomorphic principal G-bundle over X. By $\overline{E}_G$ we denote the C^∞ principal G-bundle over X obtained by extending the structure group of E_G using the homomorphism σ_G:

$$\overline{E}_G = E_G \times^{\sigma_G} G.$$

In other words, $\overline{E}_G$ is the quotient of $E_G \times G$ where two points (z_1, g_1) and (z_1, g_1) are identified if there is an element $g \in G$ such that $z_2 = z_1 g$ and $g_2 = \sigma_G(g)^{-1}g_1$. The total space of $\overline{E}_G$ is canonically identified with the total space of E_G; this identification $\overline{E}_G \to E_G$ sends the equivalence class of (z, g) to $z\sigma_G(g)$ (see [6, p. 960, Remark 2.1]). The pullback $\sigma_X^*\overline{E}_G$ is a holomorphic principal G-bundle over X, although $\overline{E}_G$ is not equipped with a holomorphic structure. The holomorphic structure on $\sigma_X^*\overline{E}_G$ is uniquely determined by the following condition: a section of $\sigma_X^*\overline{E}_G$ defined over an open subset $U \subset X$ is holomorphic if and only if the corresponding section of E_G over $\sigma_X(U)$ is holomorphic.

Definition 2.1. — *A* pseudo-real *principal G-bundle on X is a pair of the form (E_G, ρ), where $E_G \to X$ is a holomorphic principal G-bundle, and*

$$\rho\colon E_G \longrightarrow \sigma_X^*\overline{E}_G$$

is a holomorphic isomorphism of principal G-bundles satisfying the condition that there is an element $c \in Z_{\mathbb{R}}$ such that the composition

$$E_G \xrightarrow{\rho} \sigma_X^*\overline{E}_G \xrightarrow{\sigma_X^*\overline{\rho}} \sigma_X^*\overline{\sigma_X^*\overline{E}}_G = \sigma_X^*\sigma_X^*\overline{\overline{E}}_G = E_G$$

coincides with the automorphism of E_G defined by $z \longmapsto zc$.

If (E_G, ρ) is a pseudo-real principal G-bundle such that $c = e$, then it is called a real principal G-bundle.

Using the C^∞ canonical identification between E_G and $\overline{E}_G$, the isomorphism ρ in Definition 2.1 produces an anti-holomorphic diffeomorphism of the total space of E_G over the involution σ_X. This diffeomorphism of E_G will also be denoted by ρ. Clearly, we have

$$\rho(zg) = \rho(z)\sigma_G(g) \tag{2.4}$$

for all $z \in E_G$ and $g \in G$. Also, $\rho^2(z) = zc$, where c is the element in Definition 2.1.

An *isomorphism* between two pseudo-real principal G-bundles (E_G, ρ) and (F_G, δ) is a holomorphic isomorphism of principal G-bundles

$$\mu\colon E_G \longrightarrow F_G$$

such that the following diagram commutes:

$$\begin{array}{ccc} E_G & \xrightarrow{\rho} & \sigma_X^*\overline{E}_G \\ \big\downarrow \mu & & \big\downarrow \sigma_X^*\overline{\mu} \\ F_G & \xrightarrow{\delta} & \sigma_X^*\overline{F}_G \end{array}$$

where $\sigma^*\overline{\mu}$ is the holomorphic isomorphism of principal G-bundles given by μ. The map $\sigma^*\overline{\mu}$ coincides with μ using the above mentioned identification of the total spaces of E_G and F_G with those of $\sigma^*\overline{E}_G$ and $\sigma^*\overline{F}_G$ respectively.

Let

$$\mathrm{Ad}(E_G) := E_G \times^G G \longrightarrow X$$

be the holomorphic fiber bundle associated to E_G for the adjoint action of G on itself. So $\mathrm{Ad}(E_G)$ is the quotient of $E_G \times G$ where two points (z_1, g_1)

and (z_2, g_2) are identified if there is an element $g \in G$ such that $z_2 = z_1 g$ and $g_2 = g^{-1} g_1 g$. Therefore, the fibers of $\mathrm{Ad}(E_G)$ are groups identified with G up to inner automorphisms. The fiber of $\mathrm{Ad}(E_G)$ over any point $x \in X$ is identified with the space of all automorphisms of the fiber $(E_G)_x$ that commute with the action of G on $(E_G)_x$. This identification is constructed as follows: the action of any $(z_1, g_1) \in (E_G)_x \times G$ on $(E_G)_x$ is $z_1 g \longmapsto z_1 g_1 g$. This action clearly descends to an action of the group $\mathrm{Ad}(E_G)_x$.

Let

$$\mathrm{Ad}(\overline{E}_G) := \overline{E}_G \times^G G \longrightarrow X$$

be the C^∞ fiber bundle associated to $\overline{E}_G$ for the adjoint action of G on itself. The homomorphism σ_G produces a C^∞ isomorphism of fiber bundles

$$\alpha_E \colon \mathrm{Ad}(E_G) \longrightarrow \mathrm{Ad}(\overline{E}_G)$$

whose restriction to each fiber is an isomorphism of groups. More precisely, α_E sends the equivalence class of $(z, g) \in E_G \times G$ to the equivalence class of $(z, \sigma_G(g)) \in \overline{E}_G \times G$ (recall that the fibers of E_G and $\overline{E}_G$ are naturally identified). The isomorphism ρ in Definition 2.1 produces an isomorphism

$$\rho'' \colon \mathrm{Ad}(E_G) \longrightarrow \mathrm{Ad}(\sigma_X^* \overline{E}_G) = \sigma_X^* \mathrm{Ad}(\overline{E}_G)$$

which is holomorphic. Let

$$(\sigma_X^* \alpha_E^{-1}) \circ \rho'' \colon \mathrm{Ad}(E_G) \longrightarrow \sigma_X^* \mathrm{Ad}(E_G)$$

be the composition. It defines a C^∞-isomorphism of fiber bundles

$$\rho' \colon \mathrm{Ad}(E_G) \longrightarrow \mathrm{Ad}(E_G) \tag{2.5}$$

over the map σ_X. This map ρ' is an anti-holomorphic involution, and it preserves the group-structure of the fibers of $\mathrm{Ad}(E_G)$. That ρ' is indeed an involution follows immediately from the fact that the adjoint action of $c \in Z_{\mathbb{R}}$ (see Definition 2.1) on G is trivial.

As before, the Lie algebra of G will be denoted by $\mathfrak{g}$. Let

$$\mathrm{ad}(E_G) := E_G \times^G \mathfrak{g} \longrightarrow X$$

be the holomorphic vector bundle associated to E_G for the adjoint action of G on $\mathfrak{g}$. It is the Lie algebra bundle corresponding to $\mathrm{Ad}(E_G)$. The anti-holomorphic involution ρ' in (2.5) produces an anti-holomorphic automorphism of order two of the vector bundle $\mathrm{ad}(E_G)$

$$\widetilde{\rho} \colon \mathrm{ad}(E_G) \longrightarrow \mathrm{ad}(E_G) \tag{2.6}$$

over σ_X. To describe $\widetilde{\rho}$ explicitly, recall that $\mathrm{ad}(E_G)$ is the quotient of $E_G \times \mathfrak{g}$ where two points (z_1, v_1) and (z_2, v_2) of $E_G \times \mathfrak{g}$ are identified if there is an; element $g \in G$ such that $z_2 = z_1 g$ and $v_2 = \mathrm{Ad}(g)(v_1)$ the automorphism

$\mathrm{Ad}(g)$ of $\mathfrak{g}$ is the differential at identity of the automorphism of G defined by $g' \longmapsto g^{-1}g'g$. Let

$$d\sigma_G \colon \mathfrak{g} \longrightarrow \mathfrak{g} \tag{2.7}$$

be the differential, at identity, of σ_G. The anti-holomorphic automorphism of $E_G \times \mathfrak{g}$ defined by $\rho \times d\sigma_G$ descends to an anti-holomorphic automorphism of the quotient $\mathrm{ad}(E_G)$. This automorphism of $\mathrm{ad}(E_G)$ will be denoted by $\widetilde{\rho}$ (this notation is justified below). Since the adjoint action of Z_G on $\mathfrak{g}$ is trivial, it follows that $\widetilde{\rho}$ is of order two. This map $\widetilde{\rho}$ preserves the Lie algebra structure of the fibers of $\mathrm{ad}(E_G)$. The homomorphism in (2.6) coincides with $\widetilde{\rho}$.

For a holomorphic vector bundle V on X, by $\overline{V}$ we will denote the C^∞ vector bundle whose underlying real vector bundle is identified with that of V, while multiplication by a complex number λ on $\overline{V}$ coincides with the multiplication by $\overline{\lambda}$ on V. If E_{GL} is the principal $\mathrm{GL}(r, \mathbb{C})$-bundle associated to V, where $r = \mathrm{rank}(V)$, then $\overline{V}$ corresponds to $\overline{E}_{\mathrm{GL}}$; here $\overline{E}_{\mathrm{GL}}$ is constructed using the anti-holomorphic involution of $\mathrm{GL}(r, \mathbb{C})$ defined by $A \longmapsto \overline{A}$. The pullback $\sigma_X^*\overline{V}$ has a natural holomorphic structure. This holomorphic structure is uniquely determined by the condition that a section of $\sigma_X^*\overline{V}$ defined over an open subset $U \subset X$ is holomorphic if and only if the corresponding section of V over $\sigma_X(U)$ is holomorphic.

Note that $\widetilde{\rho}$ in (2.6) coincides with the holomorphic isomorphism

$$\mathrm{ad}(E_G) \longrightarrow \mathrm{ad}(\sigma^*\overline{E}_G) = \sigma_X^*\overline{\mathrm{ad}(E_G)}$$

given by ρ in Definition 2.1 after we use the above conjugate linear identification of $\mathrm{ad}(E_G)$ with $\overline{\mathrm{ad}(E_G)}$ together with the natural identification between the total spaces of $\sigma_X^* \,\mathrm{ad}(E_G)$ and $\mathrm{ad}(E_G)$.

Remark 2.2. — Let X' be a geometrically irreducible smooth projective variety defined over the field $\mathbb{R}$ of real numbers. Let $X := X' \times_{\mathbb{R}} \mathbb{C}$ be the base change of it to $\mathbb{C}$. The Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$ produces an anti-holomorphic involution of X. The pair (G, σ_G) together define an algebraic group defined over the field $\mathbb{R}$. This group defined over $\mathbb{R}$ will be denoted by G'. The real principal G-bundles on X are precisely the algebraic principal G'-bundles over X'.

2.1. Stability and semistability

A complex linear subspace $S \subset \mathrm{ad}(E_G)_x$ is called a *parabolic subalgebra* if S is the Lie algebra of a parabolic subgroup of $\mathrm{Ad}(E_G)_x$. We recall that a connected Zariski closed subgroup P of $\mathrm{Ad}(E_G)_x$ is parabolic if $\mathrm{Ad}(E_G)_x/P$ is compact. A holomorphic subbundle $F \subset \mathrm{ad}(E_G)|_U$ defined over an open subset $U \subset X$ is called a *parabolic subalgebra bundle* if for each point $x \in U$, the fiber F_x is a parabolic subalgebra of $\mathrm{ad}(E_G)_x$.

Definition 2.3. — *A pseudo-real principal G-bundle (E_G, ρ) over X is called* semistable *(respectively,* stable*) if for every pair of the form $(U, \mathfrak{p})$, where*

- *$\iota_U : U \hookrightarrow X$ is a dense open subset with $\sigma_X(U) = U$ such that the complement $X \setminus U$ is a closed complex analytic subset of X of (complex) codimension at least two, and*
- *$\mathfrak{p} \subsetneq \mathrm{ad}(E_G)|_U$ is a parabolic subalgebra bundle over U such that $\widetilde{\rho}(\mathfrak{p}) = \mathfrak{p}$ (see (2.6) for $\widetilde{\rho}$), and the direct image $\iota_{U*}\mathfrak{p}$ is a coherent analytic sheaf (see Remark 2.4 below),*

we have

$$\mathrm{degree}(\iota_{U*}\mathfrak{p}) \leqslant 0 \ \textit{(respectively,} \ \mathrm{degree}(\iota_{U*}\mathfrak{p}) < 0)$$

(degree is defined in (2.3)).

Remark 2.4. — Let $\iota_U : U \hookrightarrow X$ is a dense open subset such that the complement $X \setminus U$ is a closed complex analytic subset of X of complex codimension at least two, and let V be a holomorphic vector bundle on U. If X is a complex projective manifold, then the direct image $\iota_{U*}V$ is a coherent analytic sheaf.

Lemma 2.5. — *A pseudo-real principal G-bundle (E_G, ρ) over X is semistable if and only if the vector bundle $\mathrm{ad}(E_G)$ is semistable.*

A pseudo-real principal G-bundle (E_G, ρ) is semistable if and only if the principal G-bundle E_G is semistable.

Proof. — If $\mathrm{ad}(E_G)$ is semistable, then clearly (E_G, ρ) is semistable.

To prove the converse, assume that $\mathrm{ad}(E_G)$ is not semistable. Let

$$V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset V_n = \mathrm{ad}(E_G)$$

be the Harder-Narasimhan filtration of $\mathrm{ad}(E_G)$. Then n is odd, and $V_{(n+1)/2}$ is a parabolic subalgebra bundle of $\mathrm{ad}(E_G)$ over a dense open subset $U \subset X$ such that the complement $X \setminus U \subset X$ is a complex analytic subset of complex codimension at least two (see [1, p. 216, Lemma 2.11]).

From the uniqueness of the Harder-Narasimhan filtration it follows immediately that

$$\widetilde{\rho}(V_{(n+1)/2}) = V_{(n+1)/2}$$

(see (2.6) for $\widetilde{\rho}$). Therefore, considering $V_{(n+1)/2} \subset \mathrm{ad}(E_G)$ we conclude that (E_G, ρ) is not semistable.

The vector bundle $\mathrm{ad}(E_G)$ is semistable if and only if the principal G-bundle E_G is semistable [1, p. 214, Proposition 2.10]. Therefore, the second statement of the lemma follows from the first statement. □

Lemma 2.6. — *Let (E_G, ρ) be a stable pseudo-real principal G-bundle over X. Then the vector bundle $\mathrm{ad}(E_G)$ is polystable. Also, the principal G-bundle E_G is polystable.*

Proof. — From the first part of Lemma 2.5 we know that $\mathrm{ad}(E_G)$ is semistable. A semistable sheaf V has a unique maximal polystable subsheaf F with

$$\mathrm{degree}(V)/\,\mathrm{rank}(V) = \mathrm{degree}(F)/\,\mathrm{rank}(F)$$

[13, page 23, Lemma 1.5.5]. This F is called the *socle* of V. Assume that $\mathrm{ad}(E_G)$ is not polystable. Then there is a unique filtration

$$0 = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_{n-1} \subset F_n = \mathrm{ad}(E_G) \tag{2.8}$$

such that for each $i \in [1, n]$, the quotient F_i/F_{i-1} is the socle of $\mathrm{ad}(E_G)/F_{i-1}$. Then n is odd, and $F_{(n+1)/2}$ is a parabolic subalgebra bundle of $\mathrm{ad}(E_G)$ over a dense open subset $U \subset X$ such that the complement $X \setminus U \subset X$ is a complex analytic subset of codimension at least two (see [1, p. 218]).

From the uniqueness of the filtration in (2.8) it follows immediately that $\widetilde{\rho}(F_{(n+1)/2}) = F_{(n+1)/2}$. Therefore, the subsheaf $F_{(n+1)/2} \subset \mathrm{ad}(E_G)$ shows that (E_G, ρ) is not stable. In view of this contradiction, we conclude that $\mathrm{ad}(E_G)$ is polystable.

The second statement of the lemma follows from the first statement and [1, p. 224, Corollary 3.8]. □

3. Polystable pseudo-real principal bundles and Einstein-Hermitian connections

Let (E_G, ρ) be a pseudo-real principal G-bundle. Let

$$\mathfrak{p} \subset \mathrm{ad}(E_G)$$

be a parabolic subalgebra bundle such that $\widetilde{\rho}(\mathfrak{p}) = \mathfrak{p}$, where $\widetilde{\rho}$ is the involution in (2.6). Let

$$R_u(\mathfrak{p}) \subset \mathfrak{p}$$

be the holomorphic subbundle over X whose fiber over any point $x \in X$ is the nilpotent radical of the parabolic subalgebra $\mathfrak{p}_x$. Therefore, the quotient $\mathfrak{p}/R_u(\mathfrak{p})$ is a bundle of reductive Lie algebras. Note that $\widetilde{\rho}(R_u(\mathfrak{p})) = R_u(\mathfrak{p})$.

A *Levi subalgebra bundle* of $\mathfrak{p}$ is a holomorphic subbundle

$$\ell(\mathfrak{p}) \subset \mathfrak{p}$$

such that for each $x \in X$, the fiber $\ell(\mathfrak{p})_x$ is a Lie subalgebra of $\mathfrak{p}_x$ with the composition

$$\ell(\mathfrak{p}) \hookrightarrow \mathfrak{p} \longrightarrow \mathfrak{p}/R_u(\mathfrak{p})$$

being an isomorphism, where $\mathfrak{p} \to \mathfrak{p}/R_u(\mathfrak{p})$ is the quotient map.

Let $\ell(\mathfrak{p}) \subset \mathfrak{p}$ be a Levi subalgebra bundle such that $\widetilde{\rho}(\ell(\mathfrak{p})) = \ell(\mathfrak{p})$. Since the fibers of $\ell(\mathfrak{p})$ are reductive subalgebras, we may extend the notion of (semi)stability to $\ell(\mathfrak{p})$ as follows.

Definition 3.1. — *A Levi subalgebra bundle $\ell(\mathfrak{p}) \subset \mathfrak{p}$ with $\widetilde{\rho}(\ell(\mathfrak{p})) = \ell(\mathfrak{p})$ is called* semistable *(respectively,* stable*) if for every pair of the form $(U, \mathfrak{q})$, where*

- *$\iota_U : U \hookrightarrow X$ is a dense open subset with $\sigma_X(U) = U$ such that the complement $X \setminus U$ is a closed complex analytic subset of X of complex codimension at least two, and*
- *$\mathfrak{q} \subsetneq \ell(\mathfrak{p})|_U$ is a parabolic subalgebra bundle over U such that $\widetilde{\rho}(\mathfrak{q}) = \mathfrak{q}$, and the direct image $\iota_{U*}\mathfrak{q}$ is a coherent analytic sheaf (see Remark 2.4),*

we have

$$\text{degree}(\iota_{U*}\mathfrak{q}) \leqslant 0 \ \textit{(respectively,}\ \text{degree}(\iota_{U*}\mathfrak{q}) < 0).$$

Definition 3.2. — *A semistable pseudo-real principal G-bundle (E_G, ρ) over X is called* polystable *if either (E_G, ρ) is stable, or there is a proper parabolic subalgebra bundle $\mathfrak{p} \subsetneq \mathrm{ad}(E_G)$, and a Levi subalgebra bundle $\ell(\mathfrak{p}) \subset \mathfrak{p}$, such that the following conditions hold:*

1. *$\widetilde{\rho}(\mathfrak{p}) = \mathfrak{p}$ and $\widetilde{\rho}(\ell(\mathfrak{p})) = \ell(\mathfrak{p})$, and*
2. *$\ell(\mathfrak{p})$ is stable (see Definition 3.1).*

In Definition 3.2, we start with a semistable pseudo-real principal bundle to rule out the analogs of direct sum of stable vector bundles of different slopes.

Lemma 3.3. — *Let (E_G, ρ) be a polystable pseudo-real principal G-bundle on X. Then the adjoint vector bundle* $\mathrm{ad}(E_G)$ *is polystable. Also, the principal G-bundle E_G is polystable.*

Proof. — If (E_G, ρ) is stable, then it follows by Lemma 2.6. So we assume that (E_G, ρ) is not stable. From the first part of Lemma 2.5 it follows that $\mathrm{ad}(E_G)$ is is semistable. Assume that $\mathrm{ad}(E_G)$ is not polystable. Let

$$F_1 \subset \mathrm{ad}(E_G)$$

be the socle (see (2.8)).

Recalling Definition 3.2, we observe that the vector bundle $\ell(\mathfrak{p})$ in Definition 3.2 is polystable with a proof identical to that of Lemma 2.6 (this is due to condition (2) in Definition 3.2). Therefore, we have

$$\ell(\mathfrak{p}) \subset F_1. \tag{3.1}$$

But $F_{(n-1)/2}$ in (2.8) is the nilpotent radical bundle of the parabolic subalgebra bundle $F_{(n+1)/2} \subset \mathrm{ad}(E_G)$. Therefore, all elements of $F_{(n-1)/2}$ are nilpotent. In particular, all elements of F_1 are nilpotent. On the other hand, $\ell(\mathfrak{p})$ is a Levi subalgebra bundle. So for each $x \in X$, the fiber $\ell(\mathfrak{p})_x$ is a reductive subalgebra of $\mathrm{ad}(E_G)_x$. Hence (3.1) is a contradiction. Therefore, we conclude that $\mathrm{ad}(E_G)$ is polystable.

The second statement of the lemma follows from the first statement and [1, p. 224, Corollary 3.8]. □

Consider the semi-direct product $G \rtimes (\mathbb{Z}/2\mathbb{Z})$ defined by the involution σ_G of G. So we have a short exact sequence of groups

$$e \longrightarrow G \longrightarrow G \rtimes (\mathbb{Z}/2\mathbb{Z}) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow e.$$

Take a maximal compact subgroup $\widetilde{K} \subset G \rtimes (\mathbb{Z}/2\mathbb{Z})$ containing the element $(e, 1) \in G \times (\mathbb{Z}/2\mathbb{Z})$ of order two. Define

$$K_G := \widetilde{K} \cap G \subset G. \tag{3.2}$$

It is a maximal compact subgroup of G which is preserved by σ_G.

By a *Hermitian structure* on a principal G-bundle E_G we will mean a C^∞ reduction of structure group of E_G to the subgroup K_G. If E_G is holomorphic, and $E_{K_G} \subset E_G$ is an Hermitian structure, then there is a unique connection ∇ on E_{K_G} such that the connection on E_G induced by ∇ has the property that the corresponding C^∞ splitting of the Atiyah exact sequence for E_G is $\mathbb{C}$-linear [2, pp. 191-192, Proposition 5]. This ∇ is called the *Chern connection* for the reduction E_{K_G}. The connection on E_G induced by ∇ is also called the Chern connection for the reduction E_{K_G}.

Let E_G be a holomorphic principal G-bundle, and let $E_{K_G} \subset E_G$ be an Hermitian structure on E_G. The corresponding Chern connection on E_G will be denoted by ∇. The curvature of ∇ will be denoted by $\mathcal{K}(\nabla)$. Let

$$\Lambda \colon \Omega^{p,q}_X \longrightarrow \Omega^{p-1,q-1}_X$$

be the adjoint of the exterior product with the Kähler form ω. The reduction E_{K_G} is said to be an *Einstein-Hermitian structure* on E_G if there is an element λ in the center of $\mathfrak{g}$ such that the section

$$\Lambda\mathcal{K}(\nabla) \in C^\infty(X, \mathrm{ad}(E_G))$$

coincides with the one given by λ (since the adjoint action of G on the center of $\mathfrak{g}$ is trivial, any element of it defines a section of $\mathrm{ad}(E_G)$).

A principal G-bundle E_G admits an Einstein-Hermitian structure if and only if E_G is polystable, and, moreover, the Einstein-Hermitian connection on a polystable principal G-bundle is unique [10], [19], [1, p. 208, Theorem 0.1], [16, p. 24, Theorem 1]. Therefore, Lemma 3.3 has the following corollary:

Corollary 3.4. — *Let (E_G, ρ) be a polystable pseudo-real principal G-bundle. Then E_G admits an Einstein-Hermitian structure.*

Assumption 3.5. — *Henceforth, we will always assume that $c \in Z_{\mathbb{R}}$ in Definition 2.1 lies in $Z_{\mathbb{R}} \bigcap K_G$. This assumption is justified in Remark 3.6.*

Remark 3.6. — Take any $\lambda \in Z_{\mathbb{R}}$. If we replace the isomorphism ρ in Definition 2.1 by the isomorphism ρ' defined by $z \longmapsto \rho(z)\lambda$, then the composition

$$E_G \xrightarrow{\rho'} \sigma_X^*\overline{E}_G \xrightarrow{\sigma_X^*\overline{\rho'}} \sigma_X^*\overline{\sigma_X^*\overline{E}}_G = \sigma_X^*\sigma_X^*\overline{\overline{E}}_G = E_G$$

differs from the composition in Definition 2.1 by multiplication with λ^2. We also note that the group of order two elements in $Z_{\mathbb{R}}$ surjects onto the quotient group $Z_{\mathbb{R}}/(Z_{\mathbb{R}})^2$. Consequently, without any loss of generality, the element $c \in \mathbb{Z}_{\mathbb{R}}$ in Definition 2.1 can be taken to be of order two (see also the end of Section 2.1 of [6]). But all elements of $Z_{\mathbb{R}}$ of order two lie in $Z_{\mathbb{R}} \bigcap K_G$. Hence Assumption 3.5 is not restrictive.

Let (V, h) be a holomorphic Hermitian vector bundle on a complex manifold M. Let h' be another Hermitian structure on V. Then there a unique C^∞ endomorphism A of V such that $A^{*_h} = A$, and

$$h'(v, w) = h(x)(\exp(A)(v), w),\ \forall\ x \in M \text{ and } v, w \in V_x,$$

where A^{*_h} is the adjoint of A with respect to h. Let ∇^h be the Chern connection on V for h.

Lemma 3.7. — *The Chern connection on V for h' coincides with ∇^h if and only if the above endomorphism A is flat with respect to ∇^h.*

Proof. — Let $\nabla^{h'}$ be the Chern connection on V for h'. Then

$$\nabla^{h'} - \nabla^h = \nabla^h(A)$$

(both sides are C^∞ one-forms with values in $\mathrm{ad}(E_G)$). □

Recall that ρ in Definition 2.1 produces an anti-holomorphic diffeomorphism of E_G which is also denoted by ρ (see (2.4)).

Proposition 3.8. — *Let (E_G, ρ) be a pseudo-real principal G-bundle such that the principal G-bundle E_G is polystable. Then E_G admits an Einstein-Hermitian structure*

$$E_{K_G} \subset E_G$$

such that $\rho(E_{K_G}) = E_{K_G}$.

Proof. — Let $E_{K_G} \subset E_G$ be a C^∞ reduction of structure group of the holomorphic principal G-bundle E_G to the subgroup K_G. Since $\sigma_G(K_G) = K_G$, from (2.4) it follows immediately that $\rho(E_{K_G}) \subset E_G$ is also a C^∞ reduction of structure group to K_G. Let ∇' be a connection on the principal G-bundle E_G; it is a $\mathfrak{g}$-valued one-form on the total space of E_G. Then $(d\sigma_G) \circ \rho^*\nabla'$ is also a connection on E_G, where $d\sigma_G$ is the homomorphism in (2.7) (recall that ρ is a self-map of the total space of E_G). If ∇' is the Chern connection for the Hermitian structure $E_{K_G} \subset E_G$, then it is straightforward to check that $(d\sigma_G) \circ \rho^*\nabla'$ is the Chern connection for the Hermitian structure $\rho(E_{K_G}) \subset E_G$.

The principal G-bundle E_G admits an Einstein-Hermitian structure, and the Einstein-Hermitian connection on E_G is unique (see Corollary 3.4). Let ∇ denote the Einstein-Hermitian connection on E_G. Since the Einstein-Hermitian connection ∇ is unique, it follows that ∇ is preserved by ρ, meaning $(d\sigma_G) \circ \rho^*\nabla = \nabla$. However, the Hermitian structure on E_G giving the Einstein-Hermitian connection is not unique in general.

Let

$$E_{K_G} \subset E_G$$

be an Hermitian structure on E_G giving the Einstein-Hermitian connection ∇. Define

$$E'_{K_G} = \rho(E_{K_G}) \subset E_G.$$

We noted above that E'_{K_G} is also a C^∞ reduction of structure group of E_G to K_G. Recall from above that the Chern connection on E_G for this Hermitian structure E'_{K_G} coincides with one given by ∇ using ρ. Since ∇ is preserved by ρ, the Chern connection on E_G for E'_{K_G} coincides with ∇.

Let $\mathcal{M}$ denote the space of all Hermitian structures on E_G that give the Einstein-Hermitian connection ∇. We note that every Hermitian structure in $\mathcal{M}$ is Einstein-Hermitian. If E_G is regularly stable (meaning E_G is stable and $\mathrm{Aut}(E_G) = Z_G$), then $\mathcal{M} = Z_G/(K_G \cap Z_G)$. Let

$$\rho_{\mathcal{M}} \colon \mathcal{M} \longrightarrow \mathcal{M} \tag{3.3}$$

be the map defined by $E_{K_G} \longmapsto \rho(E_{K_G})$ (constructed as above). Since the element c in Definition 2.1 lies in K_G (see Assumption 3.5), we conclude that $\rho_{\mathcal{M}}$ is an involution. The proposition is equivalent to the statement that $\rho_{\mathcal{M}}$ has a fixed point.

Fix a reduction

$$E^0_{K_G} \subset E_G$$

lying in $\mathcal{M}$. Fix an inner product $h_{\mathfrak{g}}$ on $\mathfrak{g}$ which is invariant under the adjoint action of K_G. We note that since K_G is compact, such an inner product exists. Using the reduction $E^0_{K_G}$, this $h_{\mathfrak{g}}$ produces an Hermitian structure on the adjoint vector bundle $\mathrm{ad}(E_G)$. To see this, note that $\mathrm{ad}(E_G)$ is identified with the vector bundle $E^0_{K_G} \times^{K_G} \mathfrak{g}$ associated to $E^0_{K_G}$ for the adjoint action of K_G on $\mathfrak{g}$. Therefore, $h_{\mathfrak{g}}$ induces an Hermitian structure on $E^0_{K_G} \times^{K_G} \mathfrak{g}$. So $\mathrm{ad}(E_G)$ gets an Hermitian structure using its identification with $E^0_{K_G} \times^{K_G} \mathfrak{g}$. This Hermitian structure on $\mathrm{ad}(E_G)$ will be denoted by $h_{\mathrm{ad}(E_G)}$.

Let

$$\mathcal{S} := \mathrm{ad}(E^0_{K_G})^{\perp} \subset \mathrm{ad}(E_G) \tag{3.4}$$

be the orthogonal complement of $\mathrm{ad}(E^0_{K_G})$ with respect to the Hermitian structure $h_{\mathrm{ad}(E_G)}$. This orthogonal complement is in fact independent of the choice of $h_{\mathfrak{g}}$. Given any Hermitian structure

$$E_{K_G} \subset E_G$$

on E_G, there is a unique C^∞ section $s \in C^\infty(X, \mathcal{S})$ such that

$$E_{K_G} = \exp(s)(E^0_{K_G})$$

(recall that $\mathrm{ad}(E_G)$ is the Lie algebra bundle associated to $\mathrm{Ad}(E_G)$). Conversely, for any

$$s \in C^\infty(X, \mathcal{S}),$$

the image $\exp(s)(E^0_{K_G}) \subset E_G$ is an Hermitian structure on E_G.

Let

$$s_0 \in C^\infty(X, \mathcal{S})$$

be the section such that $\exp(s_0)(E^0_{K_G}) = \rho_{\mathcal{M}}(E^0_{K_G})$, where $\rho_{\mathcal{M}}$ is constructed in (3.3).

Let ∇^{ad} be the connection on the vector bundle $\mathrm{ad}(E_G)$ induced by the Einstein-Hermitian connection ∇. From Lemma 3.7 it can be deduced that

s_0 is covariant constant (flat) with respect to ∇^{ad}. To prove this, take any faithful holomorphic representation $G \hookrightarrow \mathrm{GL}(W)$. Fix a maximal compact subgroup of $\mathrm{GL}(W)$ containing K_G. Consider the two Hermitian structures on the associated vector bundle $E_G \times^G W$ given by $E^0_{K_G}$ and $\rho_{\mathcal{M}}(E^0_{K_G})$. Since their Chern connections coincide, using Lemma 3.7 we deduce that s_0 is flat with respect to ∇^{ad}.

We will prove that that the Hermitian structure $\exp(s_0/2)(E^0_{K_G})$ on E_G is fixed by $\rho_{\mathcal{M}}$.

To prove that $\exp(s_0/2)(E^0_{K_G})$ lies in $\mathcal{M}$, note that $s_0/2$ is flat with respect to ∇^{ad} because s_0 is so. Therefore, using Lemma 3.7 we conclude that the Chern connection for the Hermitian structure $\exp(s_0/2)(E^0_{K_G})$ coincides with ∇ (as before, take a faithful holomorphic representation G and apply Lemma 3.7 to the associated vector bundle). Therefore,

$$\exp(s_0/2)(E^0_{K_G}) \in \mathcal{M}.$$

Take any point $x \in X$. Fix a point

$$z_0 \in (E^0_{K_G})_x.$$

Identify $(E^0_{K_G})_x$ and $(E_G)_x$ with K_G and G respectively by sending any element $z_0 g$ to g. The space of all reductions of the structure group of the principal G-bundle $(E_G)_x \to \{x\}$ to the subgroup K_G is identified with $(E_G)_x/K_G$. Hence using the above identification of $(E_G)_x$ with G, this space of reductions coincides with G/K_G.

Let $g_0 \in G$ be the unique element such that

$$\exp(s_0)(x)(z_0) = z_0 g_0. \tag{3.5}$$

For the element $g_0 K_G \in G/K_G$,

$$g_0 K_G = (\rho_{\mathcal{M}}(E^0_{K_G}))_x = \rho((E^0_{K_G})_{\sigma_X(x)}) \subset (E_G)_x$$

using the above identification between G/K_G and the space of all reductions of the principal G-bundle $(E_G)_x \to \{x\}$ to the subgroup K_G.

We note that using z_0, the fiber $\mathrm{ad}(E_G)$ is identified with the Lie algebra $\mathfrak{g}$. This identification sends any $v \in \mathfrak{g}$ to the equivalence class of (z_0, v) (recall that the total space of $\mathrm{ad}(E_G)$ is a quotient of $E_G \times \mathfrak{g}$). Let

$$v_0 \in \mathfrak{g}$$

be the element given by $s_0(x) \in \mathrm{ad}(E_G)_x$ using this identification. From (3.5) we have

$$\exp(v_0) = g_0. \tag{3.6}$$

Next we show that any reduction $E'_{K_G} \subset E_G$ lying in $\mathcal{M}$ is uniquely determined by its restriction $(E'_{K_G})_x \subset (E_G)_x$. To prove this, recall that the Chern connection on E_G for E'_{K_G} coincides with ∇. Hence we can reconstruct E'_{K_G} from $(E'_{K_G})_x$ by taking parallel translations of $(E'_{K_G})_x \subset (E_G)_x$ using ∇. Hence E'_{K_G} is uniquely determined by $(E'_{K_G})_x$.

Let

$$\mathcal{M}^x \subset G/K_G$$

be the image of the map $\mathcal{M} \to G/K_G$ that sends any $E'_{K_G} \subset E_G$ in $\mathcal{M}$ to the reduction $(E'_{K_G})_x \subset (E_G)_x$ (recall that the space of all reductions of the principal G-bundle $(E_G)_x \to \{x\}$ to the subgroup K_G is identified with G/K_G). Since any reduction $E'_{K_G} \subset E_G$ lying in $\mathcal{M}$ is uniquely determined by its restriction $(E'_{K_G})_x \subset (E_G)_x$, the map $\rho_{\mathcal{M}}$ in (3.3) produces a map

$$\widetilde{\rho}^x_{\mathcal{M}} \colon \mathcal{M}^x \longrightarrow \mathcal{M}^x. \tag{3.7}$$

Using (2.4) it follows that $\widetilde{\rho}^x_{\mathcal{M}}$ is the restriction of the map

$$f_{g_0} \colon G/K_G \longrightarrow G/K_G, \quad gK_G \longmapsto g_0\sigma_G(g)K_G, \tag{3.8}$$

where g_0 is the element of G in (3.5).

The direct sum of the Killing form on $[\mathfrak{g}, \mathfrak{g}]$ and an inner product on the center of $\mathfrak{g}$ is a nondegenerate G-invariant form on $\mathfrak{g}$. This form produces a Riemannian metric on G/K_G. The map f_{g_0} in (3.8) is an isometry with respect to this Riemannian metric. Given any two points of G/K_G, there is a unique geodesic passing through them.

The map $\widetilde{\rho}^x_{\mathcal{M}}$ in (3.7) interchanges the two points $(E^0_{K_G})_x$ and $(\rho_{\mathcal{M}}(E^0_{K_G}))_x$ of $\mathcal{M}^x$. Since $\widetilde{\rho}^x_{\mathcal{M}}$ is the restriction of the isometry f_{g_0}, the mid-point of the unique geodesic between the two points $(E^0_{K_G})_x$ and $(\rho_{\mathcal{M}}(E^0_{K_G}))_x$ is fixed by $\widetilde{\rho}^x_{\mathcal{M}}$, provided this mid-point lies in $\mathcal{M}^x$.

The earlier identification between G/K_G and the space of all reductions of the principal G-bundle $(E_G)_x \to \{x\}$ to K_G (given by z_0) sends the reduction $(E^0_{K_G})_x$ (respectively, $(\rho_{\mathcal{M}}(E^0_{K_G}))_x$) to eK_G (respectively, g_0K_G). The mid-point of the unique geodesic in G/K_G between eK_G and g_0K_G is $\exp(v_0/2)K_G$ (see (3.6)). Therefore, the mid-point of the unique geodesic between the two points $(E^0_{K_G})_x$ and $(\rho_{\mathcal{M}}(E^0_{K_G}))_x$ is $(\exp(s_0/2)(E^0_{K_G}))_x$.

We have shown above that $\exp(s_0/2)(E^0_{K_G})$ lies in $\mathcal{M}$. Consequently, for every point $x \in X$, the reduction

$$(\exp(s_0/2)(E^0_{K_G}))_x \subset (E_G)_x$$

coincides with $(\rho_{\mathcal{M}}(\exp(s_0/2)(E^0_{K_G})))_x \subset (E_G)_x$. Therefore, the Hermitian structure $\exp(s_0/2)(E^0_{K_G})$ on E_G is fixed by $\rho_{\mathcal{M}}$. □

Lemma 3.3 and Proposition 3.8 together give the following:

Corollary 3.9. — *Let (E_G, ρ) be a polystable pseudo-real principal G-bundle. Then E_G admits an Einstein-Hermitian structure $E_{K_G} \subset E_G$ such that $\rho(E_{K_G}) = E_{K_G}$.*

Proposition 3.10. — *Let (E_G, ρ) be a pseudo-real principal G-bundle admitting an Einstein-Hermitian structure $E_{K_G} \subset E_G$ such that $\rho(E_{K_G}) = E_{K_G}$. Then (E_G, ρ) is polystable.*

Proof. — As before, ∇^{ad} is the connection on $\mathrm{ad}(E_G)$ induced by the Einstein-Hermitian connection on E_G. This connection ∇^{ad} is clearly Einstein-Hermitian. Therefore, $\mathrm{ad}(E_G)$ is polystable, in particular, it is semistable. Hence the pseudo-real principal G-bundle (E_G, ρ) is semistable (see Lemma 2.5). If (E_G, ρ) is stable, then (E_G, ρ) is polystable. Therefore, assume that (E_G, ρ) is not stable.

Take a pair $(U, \mathfrak{p})$ as in Definition 2.3 such that

$$\mathrm{degree}(\iota_{U*}\mathfrak{p}) = 0.$$

Since $\mathrm{ad}(E_G)$ is polystable of degree zero, the subbundle $\mathfrak{p}$ of $\mathrm{ad}(E_G)|_U$ extends to a subbundle of $\mathrm{ad}(E_G)$ over X. To see this write, $\mathrm{ad}(E_G)$ as a direct sum of stable vector bundles. The statement is clear for a stable vector bundle; the statement for polystable case follows from this. This extended vector bundle will be denoted by $\mathfrak{p}'$. Clearly, $\mathfrak{p}'$ is a parabolic subalgebra bundle of $\mathrm{ad}(E_G)$. We also have $\widetilde{\rho}(\mathfrak{p}') = \mathfrak{p}'$, because $\widetilde{\rho}(\mathfrak{p}) = \mathfrak{p}$. Furthermore,

$$\mathrm{degree}(\mathfrak{p}') = \mathrm{degree}(\iota_{U*}\mathfrak{p}) = 0.$$

Let

$$\mathfrak{p} \subset \mathrm{ad}(E_G)$$

be a smallest parabolic subalgebra bundle over X such that

- $\widetilde{\rho}(\mathfrak{p}) = \mathfrak{p}$, and
- $\mathrm{degree}(\mathfrak{p}) = 0$.

It should be clarified that $\mathfrak{p}$ need not be unique.

We will show that the connection ∇^{ad} on $\mathrm{ad}(E_G)$ preserves the subbundle $\mathfrak{p}$.

The vector bundle $\mathrm{ad}(E_G)$ is polystable of degree zero. Since $\mathrm{degree}(\mathfrak{p}) = 0$, there is a holomorphic subbundle $W \subset \mathrm{ad}(E_G)$ such that the natural homomorphism

$$\mathfrak{p} \oplus W \longrightarrow \mathrm{ad}(E_G)$$

is an isomorphism. Hence both $\mathfrak{p}$ and W are of polystable of degree zero. Therefore, from the uniqueness of the Einstein-Hermitian connection it follows that the Einstein-Hermitian connection ∇^{ad} is the direct sum of the

Einstein-Hermitian connections on $\mathfrak{p}$ and W. In particular, the connection ∇^{ad} preserves the subbundle $\mathfrak{p}$.

The adjoint vector bundle $\mathrm{ad}(E_{K_G})$ is a totally real subbundle of $\mathrm{ad}(E_G)$, meaning $\mathrm{ad}(E_{K_G})\bigcap\sqrt{-1}\cdot\mathrm{ad}(E_{K_G})=0$. Since both the subbundles $\mathfrak{p}$ and $\mathrm{ad}(E_{K_G})$ are preserved by ∇^{ad}, it follows that $\mathfrak{p}\bigcap\mathrm{ad}(E_{K_G})$ is a real subbundle of $\mathrm{ad}(E_G)$ preserved by ∇^{ad}. Consider the complexified vector bundle

$$\mathcal{E} := (\mathfrak{p}\cap\mathrm{ad}(E_{K_G}))\otimes_{\mathbb{R}}\mathbb{C}.$$

Since $\mathrm{ad}(E_{K_G})$ is a totally real subbundle, this $\mathcal{E}$ is a complex subbundle of $\mathrm{ad}(E_{K_G})$. It is clearly preserved by ∇^{ad}. In particular, $\mathcal{E}$ is a holomorphic subbundle of $\mathfrak{p}$. This holomorphic subbundle $\mathcal{E}\subset\mathfrak{p}$ is a Levi subalgebra bundle of $\mathfrak{p}$.

The given condition that $\rho(E_{K_G})=E_{K_G}$ implies that $\widetilde{\rho}(\mathrm{ad}(E_{K_G}))=\mathrm{ad}(E_{K_G})$. Since we also have $\widetilde{\rho}(\mathfrak{p})=\mathfrak{p}$, it follows immediately that

$$\widetilde{\rho}(\mathcal{E})=\mathcal{E}.$$

From the minimality assumption on $\mathfrak{p}$ it can be deduced that the Levi subalgebra bundle $\mathcal{E}$ is stable. To see this, assume that $\mathfrak{q}\subset\mathcal{E}|_U$ is a parabolic subalgebra bundle violating the stability of the Levi subalgebra bundle $\mathcal{E}$. Then the direct sum $\mathfrak{q}\oplus R_n(\mathfrak{p})$, where $R_n(\mathfrak{p})\subset\mathfrak{p}|_U$ is the nilpotent radical, is properly contained in $\mathfrak{p}$, and it contradicts the minimality assumption on $\mathfrak{p}$. Hence we conclude that the Levi subalgebra bundle $\mathcal{E}$ is stable. Consequently, (E_G,ρ) is polystable. □

Proposition 3.8 and Proposition 3.10 together give the following:

Corollary 3.11. — *If (E_G,ρ) is a pseudo-real principal G-bundle such that the holomorphic principal G-bundle E_G is polystable. Then (E_G,ρ) is polystable.*

4. Representations of the extended fundamental group in a compact subgroup

Fix a point $x_0\in X$ such that $\sigma_X(x_0)\neq x_0$. Let

$$\Gamma(x_0)=\Gamma(X,x_0)$$

be the homotopy classes of paths $\gamma\colon[0,1]\to X$ such that $\gamma(0)=x_0$ and $\gamma(1)\in\{x_0,\sigma_X(x_0)\}$. Take two paths $\gamma_1,\gamma_2\in\Gamma(x_0)$. If $\gamma_2(1)=x_0$, then define $\gamma_2\cdot\gamma_1=\gamma_1\circ\gamma_2$, where "$\circ$" denotes composition of paths. If $\gamma_2(1)=\sigma_X(x_0)$, then define $\gamma_2\cdot\gamma_1=\sigma_X(\gamma_1)\circ\gamma_2$. These operations make $\Gamma(x_0)$ into a group (see [5]). The inverse of $\gamma\in\Gamma(x_0)$ with $\gamma(1)=\sigma_X(x_0)$ is represented by the path $t \longmapsto \sigma_X(\gamma(1-t))$. This group $\Gamma(x_0)$ fits in a short exact sequence of groups

$$e \longrightarrow \pi_1(X, x_0) \longrightarrow \Gamma(x_0) \xrightarrow{\eta} \mathbb{Z}/2\mathbb{Z} \longrightarrow e, \tag{4.1}$$

where $\eta(\gamma) = 0$ if $\gamma(1) = x_0$, and $\eta(\gamma) = 1$ if $\gamma(1) = \sigma_X(x_0)$. If there is a point $y \in X$ such that $\sigma_X(y) = y$, then (4.1) is a right-split (the exact sequence is isomorphic to a semi-direct product). To see this, fix a path γ_0 from x_0 to y. Then the composition $\gamma_1 := \sigma_X(\gamma_0)^{-1} \circ \gamma_0 \in \eta^{-1}(1)$ is of order two. So $1 \longmapsto \gamma_1$ is a right-splitting of (4.1).

Let K_G be the maximal compact subgroup of G defined earlier (see (3.2)). The group $\widetilde{K}$ in (3.2) is identified with the semi-direct product $K_G \rtimes (\mathbb{Z}/2\mathbb{Z})$ for the involution σ_G of K_G. In particular, the set $\widetilde{K}$ is identified with the set $K_G \times \{0, 1\}$.

Let $\mathrm{Map}'(\Gamma(x_0), \widetilde{K})$ be the space of all maps

$$\delta \colon \Gamma(x_0) \longrightarrow \widetilde{K}$$

such that the following diagram is commutative:

$$\begin{array}{ccccccccc} e & \longrightarrow & \pi_1(X, x_0) & \longrightarrow & \Gamma(x_0) & \xrightarrow{\eta} & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & e \\ & & \big\downarrow & & \big\downarrow \delta & & \| & & \\ e & \longrightarrow & K_G & \longrightarrow & \widetilde{K} & \xrightarrow{\eta'} & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & e \end{array} \tag{4.2}$$

We write $\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$. For any $c \in Z_{\mathbb{R}} \bigcap K_G$, let $\mathrm{Hom}_c(\Gamma(x_0), \widetilde{K})$ be the space of all maps

$$\delta \in \mathrm{Map}'(\Gamma(x_0), \widetilde{K})$$

such that

- the restriction of δ to $\pi_1(X, x_0)$ is a homomorphism of groups,
- $\delta(g'g) = c\delta(g')\delta(g)$, if $\eta(g) = 1 = \eta(g')$ (the homomorphism η is defined in (4.1)), and
- $\delta(g'g) = \delta(g')\delta(g)$ otherwise (meaning if $\eta(g) \cdot \eta(g') = 0$).

We note that if $c = e$, then $\mathrm{Hom}_c(\Gamma(x_0), \widetilde{K})$ is the space of all homomorphisms from $\Gamma(x_0)$ to $\widetilde{K}$ satisfying (4.2).

Remark 4.1. — We will give a more intrinsic definition of $\mathrm{Hom}_c(\Gamma(x_0), \widetilde{K})$. For that, we first recall that the semi-direct product $\widetilde{K} = K_G \rtimes (\mathbb{Z}/2\mathbb{Z})$ is constructed as follows: the underlying set is $K_G \times \{0, 1\}$, and the multiplication is given by

$$(g_1, e_1) \cdot (g_2, e_2) = (g_1(\sigma_G)^{e_1}(g_2), e_1 + e_2),$$

where $g_i \in K_G$ and $e_i \in \{0, 1\}$; note that $(\sigma_G)^{e_1}$ is either σ_G or the identity map of G depending on whether e_1 is 1 or 0. We now define a new group $\widetilde{\widetilde{K}}$.

The underlying set for $\widetilde{\widetilde{K}}$ is again $K_G \times \{0,1\}$, but the multiplication is now given by

$$(g_1, e_1) \cdot (g_2, e_2) = (g_1(\sigma_G)^{e_1}(g_2)c^{e_1 e_2}, e_1 + e_2); \tag{4.3}$$

as before, $c^{e_1 e_2}$ is either c or the identity element of K_G depending on whether $e_1 e_2$ is 1 or 0. The subset $\mathrm{Hom}_c(\Gamma(x_0), \widetilde{K})$ of $\mathrm{Map}'(\Gamma(x_0), \widetilde{K})$ consists of those elements that are homomorphisms from $\Gamma(x_0)$ to the group $\widetilde{\widetilde{K}}$ defined above.

Take any $\delta \in \mathrm{Hom}_c(\Gamma(x_0), \widetilde{K})$. We will construct from δ a polystable pseudo-real principal G-bundle on X.

Consider the restriction $\delta' := \delta|_{\pi_1(X,x_0)}$ (see (4.1)). It is a homomorphism from $\pi_1(X, x_0)$ to K_G. Therefore, δ' gives

- a principal K_G-bundle E_{K_G} equipped with a flat connection ∇^K, and
- a base point $z_0 \in (E_{K_G})_{x_0}$ over the base point x_0.

Let $E_G := E_{K_G} \times^{K_G} G \to X$ be the principal G-bundle obtained by extending the structure group of E_{K_G} using the inclusion of K_G in G. The flat connection ∇^K defines a holomorphic structure on E_G. This holomorphic principal G-bundle E_G is polystable because ∇^K is a flat Hermitian connection.

We will construct a diffeomorphism

$$\rho_{\sigma_X(x_0)} \colon (E_G)_{\sigma_X(x_0)} \longrightarrow (E_G)_{x_0} \tag{4.4}$$

between the fibers of E_G. For that, take any $\gamma \in \Gamma(x_0)$ such that $\eta(\gamma) = 1$ (see (4.1) for η). Let $g_\gamma \in K_G$ be the element such that the canonical identification of the set $\widetilde{K}$ with $K_G \rtimes \{0,1\}$ takes $\delta(\gamma)$ to $(g_\gamma, 1)$. Let

$$z'_0 \in (E_G)_{\sigma_X(x_0)}$$

be the element obtained by the parallel translation of the base point z_0 along γ for the connection ∇^K. The map $\rho_{\sigma_X(x_0)}$ in (4.4) is defined as follows:

$$\rho_{\sigma_X(x_0)}(z'_0 g) = z_0 \sigma_G(g_\gamma^{-1} g) \in (E_G)_{x_0}, \quad g \in G.$$

Lemma 4.2. — *The map $\rho_{\sigma_X(x_0)}$ defined above is independent of the choice of γ.*

Proof. — Take an element $\gamma_1 \in \pi_1(X, x_0)$, and replace γ by the element $\gamma_1\gamma \in \Gamma(x_0)$ represented by the path $\gamma \circ \gamma_1$. Let $g_{\gamma_1\gamma}$ be the element of K_G such that

$$\delta(\gamma_1\gamma) = (g_{\gamma_1\gamma}, 1).$$

Then $g_{\gamma_1\gamma} = \delta(\gamma_1)g_\gamma$. The element z_0' gets replaced by $z_0'\delta(\gamma_1)^{-1}$. Therefore, the map $\rho_{\sigma_X(x_0)}$ constructed as above using $\gamma_1\gamma$ in place of γ sends the point $z_0'\delta(\gamma_1)^{-1}$ to $z_0\sigma_G(g_\gamma^{-1})\sigma_G(\delta(\gamma_1))^{-1}$.

Consequently, the two maps $\rho_{\sigma_X(x_0)}$ constructed using γ and $\gamma_1\gamma$ respectively coincide on the point $z_0'\delta(\gamma_1)^{-1}$. On the other hand, both these maps satisfy the condition that

$$\rho_{\sigma_X(x_0)}(yh) = \rho_{\sigma_X(x_0)}(h)\sigma_G(h) \tag{4.5}$$

for all $y \in (E_G)_{\sigma_X(x_0)}$ and $h \in G$. These together imply the two maps coincide on the entire $(E_G)_{\sigma_X(x_0)}$. Therefore, the map $\rho_{\sigma_X(x_0)}$ is independent of the choice of γ. □

The map $\rho_{\sigma_X(x_0)}$ is clearly anti-holomorphic.

We will now show that $\rho_{\sigma_X(x_0)}$ is independent of the base point z_0.

Take any $g_0 \in K_G$. Define

$$\widetilde{\delta}\colon \Gamma(x_0) \longrightarrow \widetilde{K},\ z \longmapsto g_0^{-1}\delta(z)g_0$$

(recall that K_G is a subgroup of $\widetilde{K}$). Note that $\widetilde{\delta} \in \mathrm{Hom}_c(\Gamma(x_0), \widetilde{K})$. If we replace δ by $\widetilde{\delta}$, then the flat principal E_K-bundle (E_K, ∇^K) remains unchanged, but the base point z_0 gets replaced by z_0g_0.

Lemma 4.3. — *The map $\rho_{\sigma_X(x_0)}$ in (4.4) for δ coincides with the corresponding map for $\widetilde{\delta}$. In other words, $\rho_{\sigma_X(x_0)}$ does not change if δ is conjugated by an element of K_G.*

Proof. — Take the element $\gamma \in \Gamma(x_0)$ in the construction of the map in (4.4). Replace δ by $\widetilde{\delta}$. Then z_0 gets replaced by z_0g_0, and hence z_0' gets replaced by $z_0'g_0$. The element g_γ gets replaced by $g_0^{-1}g_\gamma\sigma_G(g_0)$. Therefore, the two maps constructed as in (4.4) for $\widetilde{\delta}$ and δ respectively coincide at the point $z_0'g_0$. Now from (4.5) we conclude that the two maps coincide on entire $(E_G)_{x_0}$. □

Take a point $x_1 \in X$. If $\sigma_X(x_1) \neq x_1$, then define $\Gamma(x_1) = \Gamma(X, x_1)$ as before by replacing x_0 with x_1. If $\sigma_X(x_1) = x_1$, then define $\Gamma(x_1)$ to be the semi-direct product

$$\Gamma(x_1) := \pi_1(X, x_1) \rtimes (\mathbb{Z}/2\mathbb{Z})$$

constructed using the involution of $\pi_1(X, x_1)$ given by σ_X.

Fix a path γ_0 in X from x_1 to x_0. Then we have an isomorphism $\pi_1(X, x_0) \to \pi_1(X, x_1)$ defined by $\gamma \longmapsto \gamma_0^{-1} \circ \gamma \circ \gamma_0$ (as before, "$\circ$" is composition of paths). This isomorphism extends to an isomorphism

$\Gamma(x_0) \to \Gamma(x_1)$ by sending any $\gamma \in \eta^{-1}(1)$ to $\sigma_X(\gamma_0^{-1}) \circ \gamma \circ \gamma_0$. The inverse of this isomorphism $\Gamma(x_0) \to \Gamma(x_1)$ produces a bijection

$$\beta\colon \operatorname{Hom}_c(\Gamma(x_0), \widetilde{K}) \to \operatorname{Hom}_c(\Gamma(x_1), \widetilde{K})$$

by composition of maps. The flat principal K-bundle corresponding to any

$$\delta \in \operatorname{Hom}_c(\Gamma(x_0), \widetilde{K})$$

is identified with the flat principal K-bundle corresponding to $\beta(\delta)$; the base point in the bundle changes by parallel translation along γ_0.

From Lemma 4.3 it can be deduced that the isomorphism

$$\rho_{\sigma_X(x_1)}\colon (E_G)_{\sigma_X(x_1)} \longrightarrow (E_G)_{x_1}$$

constructed as in (4.4) for $\beta(\delta)$ is independent of the choice of γ_0. Indeed, for two choices of γ_0, the corresponding isomorphisms $\Gamma(x_0) \to \Gamma(x_1)$ differ by an inner automorphism of $\Gamma(x_0)$ given by an element of $\pi_1(X, x_0)$. Therefore, for two choices of γ_0, the corresponding bijections from $\operatorname{Hom}_c(\Gamma(x_0), \widetilde{K})$ to $\operatorname{Hom}_c(\Gamma(x_1), \widetilde{K})$ differ by an inner automorphism of $\widetilde{K}$ by an element of K_G. By Lemma 4.3, an inner automorphism of $\widetilde{K}$ by an element of K_G does not affect the map in (4.4).

Therefore, we get a map

$$\rho_X\colon E_G \longrightarrow E_G$$

by running the base point x_1 over entire X. From the construction of ρ_X it follows immediately that

- $\rho_X(zg) = \rho_X(z)\sigma_G(g)$ for all $z \in E_G$ and $g \in G$, and
- ρ_X is anti-holomorphic.

Let

$$\rho\colon E_G \longrightarrow \sigma_X^* \overline{E}_G$$

be the map given by ρ_X and the natural identification of the total spaces of E_G and $\sigma_X^* \overline{E}_G$. From the above two properties of ρ_X it follows immediately that ρ is a holomorphic isomorphism of principal G-bundles.

Proposition 4.4. — *The pair (E_G, ρ) constructed above from $\delta \in \operatorname{Hom}_c(\Gamma(x_0), \widetilde{K})$ is a pseudo-real principal G-bundle such that the corresponding element in $Z_{\mathbb{R}}$ (see Definition 2.1) is c.*

Proof. — To prove the proposition it suffices to show that the composition

$$(E_G)_{\sigma_X(x_0)} \overset{\rho_{\sigma_X(x_0)}}{\longrightarrow} (E_G)_{x_0} \overset{\rho_{x_0}}{\longrightarrow} (E_G)_{\sigma_X(x_0)}$$

is multiplication by c.

Fix a path γ in X from x_0 to $\sigma_X(x_0)$. So $\gamma \in \eta^{-1}(1) \subset \Gamma(x_0)$. As before, z_0 is the base point in $(E_G)_{x_0}$. Let $z'_0 \in (E_G)_{\sigma_X(x_0)}$ be the point obtained by the parallel translation of z_0 along γ. We will identify $\Gamma(x_0)$ with $\Gamma(x_1)$ using the reverse path $\gamma'\colon [0,1] \to X$ from $\sigma_X(x_0)$ from x_0 defined by $\gamma'(t) = \gamma(1-t)$. Let

$$\delta' \in \mathrm{Hom}_c(\Gamma(x_1), \widetilde{K})$$

be the element given by δ using this isomorphism of $\Gamma(x_0)$ with $\Gamma(x_1)$. The base point in $(E_G)_{\sigma_X(x_0)}$ for δ' is z'_0.

We will use the path γ to construct $\rho_{\sigma_X(x_0)}$, and we will use the path $\sigma_X(\gamma)$ to construct ρ_{x_0}. Although these maps are independent of the choice of path (see Lemma 4.2), we need to fix paths for explicit computations.

As before, $g_\gamma \in K_G$ is such that the canonical identification of $\Gamma(x_0)$ with $K_G \rtimes \{0,1\}$ takes $\delta(\gamma)$ to $(g_\gamma, 1)$.

We have

$$\rho_{\sigma_X(x_0)}(z'_0) = z_0 \sigma_G(g_\gamma)^{-1}. \tag{4.6}$$

The parallel translation along the path $\sigma_X(\gamma)$ takes z'_0 to $z_0\delta(\gamma\gamma)^{-1}$ (the element $\gamma\gamma \in \pi_1(X, x_0)$ is given by the composition $\sigma_X(\gamma) \circ \gamma$). Therefore,

$$\rho_{x_0}(z_0\delta(\gamma\gamma)^{-1}) = z'_0 \sigma_G(g_\gamma^{-1});$$

this uses the fact that the above isomorphism between $\Gamma(x_0)$ and $\Gamma(x_1)$ takes $\gamma \in \eta^{-1}(1) \subset \Gamma(x_0)$ to the homotopy class of $\sigma_X(\gamma)$. Therefore, substituting $\sigma_X(x_0)$ in place of x_0 in the identity (4.5), we get

$$\rho_{x_0}(z_0\sigma_G(g_\gamma)^{-1}) = z'_0\sigma_G(g_\gamma^{-1})\sigma_G(\delta(\gamma\gamma)\sigma_G(g_\gamma)^{-1}). \tag{4.7}$$

But $\delta(\gamma\gamma) = \delta(\gamma)^2 c = g_\gamma \sigma_G(g_\gamma)c$. Hence

$$\sigma_G(g_\gamma^{-1})\sigma_G(\delta(\gamma\gamma)\sigma_G(g_\gamma)^{-1}) = \sigma_G(g_\gamma^{-1})\sigma_G(g_\gamma)g_\gamma(g_\gamma)^{-1}c = c.$$

Therefore, from (4.7) we have

$$\rho_{x_0}(z_0\sigma_G(g_\gamma)^{-1}) = z'_0 c.$$

Combining this with (4.6), we conclude that

$$\rho_{x_0} \circ \rho_{\sigma_X(x_0)}(z'_0) = z'_0 c. \tag{4.8}$$

From (4.5) it follows that $\rho_{x_0} \circ \rho_{\sigma_X(x_0)}$ commutes the action of G on $(E_G)_{\sigma_X(x_0)}$. Therefore, from (4.8) we conclude that $\rho_{x_0} \circ \rho_{\sigma_X(x_0)}$ coincides with multiplication by c. □

We noted earlier that the holomorphic principal G-bundle E_G is polystable. Therefore, from Corollary 3.11 it follows that the pseudo-real principal G-bundle (E_G, ρ) is polystable.

Since E_G admits a flat connection, it follows that all the rational characteristic classes of E_G of positive degree vanish.

We will now describe a reverse construction.

Let (E_G, ρ) be a polystable pseudo-real principal G-bundle such that the corresponding element in $Z_{\mathbb{R}}$ (see Definition 2.1) is $c \in Z_{\mathbb{R}} \bigcap K_G$.

Assume that the following two conditions hold:

- the second Chern class of $\mathrm{ad}(E_G)$ satisfies the condition
$$\int_X c_2(\mathrm{ad}(E_G)) \wedge \omega^{\dim_{\mathbb{C}}(X)-2} = 0,$$
and
- for any character χ of G, the line bundle over X associated to E_G for χ is of degree zero.

These two numerical conditions together imply that the Einstein-Hermitian connection on E_G is flat [14, p. 115, Lemma 4.12]. In [14], this is proved for vector bundles, but it extends to principal G-bundles by taking vector bundles associated to irreducible representations of G. Therefore, these numerical conditions imply that all the rational characteristic classes of E_G of positive degree vanish.

The Einstein-Hermitian connection on E_G will be denoted by ∇. Let
$$E_{K_G} \subset E_G$$
be an Hermitian structure that gives ∇ and satisfies the condition $\rho(E_{K_G}) = E_{K_G}$ (it exists by Proposition 3.8).

Fix a base point $z_0 \in (E_{K_G})_{x_0}$. Take any $\gamma \in \pi_1(X, x_0)$. Let $z_\gamma \in (E_{K_G})_{x_0}$ be the point obtained by the parallel translation of z_0 along γ for the connection ∇. Let
$$g_\gamma \in K_G$$
be the unique element such that $z_0 g_\gamma^{-1} = z_\gamma$.

Now take any $\gamma \in \eta^{-1}(1) \subset \Gamma(x_0)$. Let $y_\gamma \in (E_{K_G})_{\sigma_X(x_0)}$ be the point obtained by the parallel translation of z_0 along γ for the connection ∇. Let
$$h'_\gamma \in K_G$$
be the unique element such that $z_0 = \rho(y_\gamma)\sigma_G(h'_\gamma)$. Using the canonical set-theoretic identification of $(\eta')^{-1}(1)$ with G (see (4.2) for η'), the element h'_γ gives an element $h_\gamma \in (\eta')^{-1}(1)$. Let
$$\delta\colon \Gamma(x_0) \longrightarrow \widetilde{K} \tag{4.9}$$
be the map that sends any $\gamma \in \eta^{-1}(0)$ to g_γ constructed above and sends any $\gamma \in \eta^{-1}(1)$ to h_γ.

Proposition 4.5. — *The function* $\delta\colon \Gamma(x_0) \to \widetilde{K}$ *in* (4.9) *lies in* $\mathrm{Hom}_c(\Gamma(x_0), \widetilde{K})$.

Proof. — Clearly, $\delta^{-1}(K_G) = \pi_1(X, x_0)$. In other words, The diagram as in (4.2) is commutative. For any $\gamma, \gamma' \in \pi_1(X, x_0)$, it is easy to see that $\delta(\gamma\gamma') = \delta(\gamma)\delta(\gamma')$.

Now take $\gamma \in \pi_1(X, x_0)$ and $\gamma' \in \eta^{-1}(1)$. Let $g_{\gamma'}$ (respectively, $g_{\gamma\gamma'}$) be the element of K_G given by $\delta(\gamma')$ (respectively, $\delta(\gamma\gamma')$) using the set theoretic identification of $(\eta')^{-1}(1)$ with K_G (see (4.2) for η'). We need to show that

$$g_{\gamma\gamma'} = \delta(\gamma)g_{\gamma'}. \tag{4.10}$$

Let $z'_0 \in (E_{K_G})_{\sigma_X(x_0)}$ be the parallel translation of z_0 along γ'. Therefore, the parallel translation of z_0 along $\gamma\gamma'$ produces $z'_0\delta(\gamma)^{-1} \in (E_{K_G})_{\sigma_X(x_0)}$. Hence,

$$\rho(z'_0) = z_0\sigma_G(g_{\gamma'}^{-1}) \quad \text{and} \quad \rho(z'_0\delta(\gamma)^{-1}) = z_0\sigma_G(g_{\gamma\gamma'}^{-1}).$$

Since $\rho(yg) = \rho(y)\sigma_G(g)$, we conclude that

$$z_0\sigma_G(g_{\gamma\gamma'}^{-1}) = z_0\sigma_G(g_{\gamma'}^{-1})\sigma_G(\delta(\gamma)^{-1}) = z_0\sigma_G(g_{\gamma'}^{-1}\delta(\gamma)^{-1}).$$

Hence $g_{\gamma\gamma'}^{-1} = g_{\gamma'}^{-1}\delta(\gamma)^{-1}$. This implies (4.10).

Hence $g_{\gamma\gamma'} = \delta(\gamma)g_{\gamma'}$. This coincides with the corresponding identity in the definition of $\mathrm{Hom}_c(\Gamma(x_0), \widetilde{K})$.

Now take $\gamma \in \eta^{-1}(1)$ and $\gamma' \in \pi_1(X, x_0)$. Let g_γ (respectively, $g_{\gamma\gamma'}$) be the element of K_G given by $\delta(\gamma)$ (respectively, $\delta(\gamma\gamma')$) using the set theoretic identification of $(\eta')^{-1}(1)$ with K_G. We need to show that

$$g_{\gamma\gamma'} = g_\gamma\sigma_G(\delta(\gamma')). \tag{4.11}$$

Let $z'_0 \in (E_{K_G})_{\sigma_X(x_0)}$ be the parallel translation of z_0 along γ.

We will compute the parallel translation along the path $\sigma_X(\gamma') \circ \gamma$ which represents $\gamma\gamma' \in \Gamma(x_0)$.

Since ρ preserves the connection ∇, the image, under ρ, of the parallel translation along γ' is the parallel translation along the loop $\rho(\gamma')$.

Since z_0 is taken to $z_0\delta(\gamma')^{-1}$ by the parallel translation along γ', the parallel translation along $\rho(\gamma')$ takes $\rho(z_0)$ to $\rho(z_0)\sigma_G(\delta(\gamma')^{-1})$. We have

$$\rho(z'_0) = z_0\sigma_G(g_\gamma)^{-1}. \tag{4.12}$$

Hence $\rho(z_0\sigma_G(g_\gamma)^{-1}) = \rho \circ (z'_0) = z'_0c$. So,

$$\rho(z_0) = z'_0g_\gamma c. \tag{4.13}$$

Since the parallel translation along $\rho(\gamma')$ takes $z'_0g_\gamma c$ to

$$\rho(z_0)\sigma_G(\delta(\gamma')^{-1}) = z'_0g_\gamma\sigma_G(\rho(\gamma')^{-1})c,$$

we conclude that this parallel translation takes z_0' to $z_0' g_\gamma \sigma_G(\delta(\gamma')^{-1}) g_\gamma^{-1}$.

Consequently, the parallel translation along $\rho(\gamma') \circ \gamma$ takes z_0 to $z_0' g_\gamma \sigma_G(\delta(\gamma')^{-1}) g_\gamma^{-1}$. Hence

$$z_0 \sigma_G(g_{\gamma\gamma'})^{-1} = \rho(z_0' g_\gamma \sigma_G(\delta(\gamma')^{-1}) g_\gamma^{-1}).$$

Therefore, from (4.12),

$$\sigma_G(g_{\gamma\gamma'})^{-1} = \sigma_G(g_\gamma)^{-1} \sigma_G(g_\gamma \sigma_G(\delta(\gamma')^{-1}) g_\gamma^{-1}).$$

So we have

$$\sigma_G(g_{\gamma\gamma'})^{-1} = \sigma_G(\sigma_G(\delta(\gamma')^{-1}) g_\gamma^{-1}).$$

This implies (4.11).

Finally, take $\gamma, \gamma' \in \eta^{-1}(1)$. Let g_γ be as in the previous case. Let $g_{\gamma'}$ be the element of K_G given by $\delta(\gamma')$ using the set theoretic identification of $(\eta')^{-1}(1)$ with K_G. We need to show that

$$\delta(\gamma\gamma') = g_\gamma \sigma_G(g_{\gamma'}) c. \tag{4.14}$$

Define z_0' as before. From (4.12) it follows that

$$\rho(z_0' g_\gamma g_{\gamma'}^{-1}) = z_0 \sigma_G(g_{\gamma'}^{-1}).$$

Hence from the definition of $\delta(\gamma')$ we conclude that $z_0' g_\gamma g_{\gamma'}^{-1}$ is the parallel translation of z_0 along γ'.

Therefore, the parallel translation along $\sigma_X(\gamma')$ takes $\rho(z_0) = z_0' g_\gamma c$ (see (4.13)) to $\rho(z_0' g_\gamma g_{\gamma'}^{-1}) = z_0 \sigma_G(g_{\gamma'}^{-1})$ (see (4.12)). Hence the parallel translation along $\sigma_X(\gamma')$ takes z_0' to $z_0 \sigma_G(g_{\gamma'}^{-1})(g_\gamma)^{-1} c^{-1}$. Consequently, the parallel translation along the loop $\sigma_X(\gamma') \circ \gamma$, which represents $\gamma\gamma' \in \Gamma(x_0)$, takes z_0 to $z_0 \sigma_G(g_{\gamma'}^{-1})(g_\gamma)^{-1} c^{-1}$. Hence

$$\delta(\gamma\gamma')^{-1} = \sigma_G(g_{\gamma'}^{-1})(g_\gamma)^{-1} c^{-1}.$$

This implies (4.14). □

The above construction of an element of $\mathrm{Hom}_c(\Gamma(x_0), \widetilde{K})$ from a polystable pseudo-real principal G-bundle of vanishing characteristic classes of positive degrees is clearly the reverse of the earlier construction of a flat polystable pseudo-real principal G-bundle from an element of $\mathrm{Hom}_c(\Gamma(x_0), \widetilde{K})$.

Two elements $\delta', \delta' \in \mathrm{Hom}_c(\Gamma(x_0), \widetilde{K})$ are called *equivalent* if there is an element $g \in K_G$ such that $\delta'(z) = g^{-1}\delta(z) g$ for all $z \in \Gamma(x_0)$.

We have the following:

THEOREM 4.6. — *There is a natural bijective correspondence between the equivalence classes of elements of* $\mathrm{Hom}_c(\Gamma(x_0), \widetilde{K})$*, and the isomorphism classes of polystable pseudo-real principal* G*-bundles* (E_G, ρ) *satisfying the following conditions:*

- $\int_X c_2(\mathrm{ad}(E_G)) \wedge \omega^{\dim_{\mathbb{C}}(X)-2} = 0$,
- *for any character* χ *of* G*, the line bundle over* X *associated to* E_G *for* χ *is of degree zero, and*
- *the corresponding element in* $Z_{\mathbb{R}} \bigcap K_G$ *is* c *(see Definition 2.1).*

5. Pseudo-real Higgs G-bundles

Consider the differential $d\sigma_X$ in (2.2). Using the natural identification of the holomorphic tangent bundle TX with the real tangent bundle $T^{\mathbb{R}}X$, this $d\sigma_X$ produces a C^∞ involution of the total space of TX over the involution σ_X. Since $d\sigma_X \circ J = -J \circ d\sigma_X$, this involution of the total space of TX is anti-holomorphic. Let

$$\widehat{\sigma}\colon (TX)^* = \Omega^1_X \longrightarrow \Omega^1_X$$

be the anti-holomorphic involution given by the above involution of TX. Note that $\widehat{\sigma}$ is fiberwise conjugate linear.

Let (E_G, ρ) be a pseudo-real principal G-bundle on X. The involution $\widetilde{\rho}$ of $\mathrm{ad}(E_G)$ in (2.6) and the above involution $\widehat{\sigma}$ of Ω^1_X together produce an anti-holomorphic involution

$$\widetilde{\rho} \otimes \widehat{\sigma}\colon\ \mathrm{ad}(E_G) \otimes \Omega^1_X \longrightarrow \mathrm{ad}(E_G) \otimes \Omega^1_X. \tag{5.1}$$

A Higgs field on (E_G, ρ) is a holomorphic section

$$\theta \in H^0(X, \mathrm{ad}(E_G) \otimes \Omega^1_X)$$

such that

- $\widetilde{\rho} \otimes \widehat{\sigma}(\theta) = \theta$, where $\widetilde{\rho} \otimes \widehat{\sigma}(\theta)$ is defined in (5.1), and
- the holomorphic section $\theta \bigwedge \theta$ of $\mathrm{ad}(E_G) \otimes \Omega^2_X$ vanishes identically.

The above section $\theta \bigwedge \theta$ is defined using the Lie algebra structure of the fibers of $\mathrm{ad}(E_G)$ and the natural homomorphism $\Omega^1_X \otimes \Omega^1_X \to \Omega^2_X$.

A *pseudo-real principal Higgs* G*-bundle* is a pseudo-real principal G-bundle equipped with a Higgs field.

Definition 2.3 extends as follows:

DEFINITION 5.1. — *A pseudo-real principal Higgs* G*-bundle* (E_G, ρ, θ) *over* X *is called* semistable *(respectively,* stable*) if for every pair of the form* $(U, \mathfrak{p})$*, where*

- *$\iota_U : U \hookrightarrow X$ is a dense open subset with $\sigma_X(U) = U$ such that the complement $X \setminus U$ is a closed complex analytic subset of X of (complex) codimension at least two,*
- *$\mathfrak{p} \subsetneq \mathrm{ad}(E_G)|_U$ is a parabolic subalgebra bundle over U such that $\widetilde{\rho}(\mathfrak{p}) = \mathfrak{p}$, and $\iota_{U*}\mathfrak{p}$ is a coherent analytic sheaf (see Remark 2.4), and*
- *$\theta|_U \in H^0(U, \mathfrak{p} \otimes \Omega^1_U)$,*

we have $\mathrm{degree}(\iota_{U*}\mathfrak{p}) \leqslant 0$ *(respectively,* $\mathrm{degree}(\iota_{U*}\mathfrak{p}) < 0$*).*

Let $\mathfrak{p} \subset \mathrm{ad}(E_G)$ be a parabolic subalgebra bundle such that $\widetilde{\rho}(\mathfrak{p}) = \mathfrak{p}$ and $\theta \in H^0(X, \mathfrak{p} \otimes \Omega^1_X)$. Let $\ell(\mathfrak{p}) \subset \mathfrak{p}$ be a Levi subalgebra bundle such that $\widetilde{\rho}(\ell(\mathfrak{p})) = \ell(\mathfrak{p})$ and $\theta \in H^0(X, \ell(\mathfrak{p}) \otimes \Omega^1_X)$.

The pair $(\ell(\mathfrak{p}), \theta)$ is called *semistable* (respectively, *stable*) if for every pair of the form $(U, \mathfrak{q})$, where

- $\iota_U : U \hookrightarrow X$ is a dense open subset with $\sigma_X(U) = U$ such that the complement $X \setminus U$ is a closed complex analytic subset of X of (complex) codimension at least two,
- $\mathfrak{q} \subsetneq \ell(\mathfrak{p})|_U$ is a parabolic subalgebra bundle over U such that $\widetilde{\rho}(\mathfrak{q}) = \mathfrak{q}$, and the direct image $\iota_{U*}\mathfrak{q}$ is a coherent analytic sheaf, and
- $\theta|_U \in H^0(U, \mathfrak{q} \otimes \Omega^1_U)$,

we have

$$\mathrm{degree}(\iota_{U*}\mathfrak{q}) \leqslant 0 \text{ (respectively, } \mathrm{degree}(\iota_{U*}\mathfrak{q}) < 0).$$

Definition 5.2. — *A semistable pseudo-real principal Higgs G-bundle (E_G, ρ, θ) over X is called* polystable *if either (E_G, ρ) is stable, or there is a proper parabolic subalgebra bundle $\mathfrak{p} \subsetneq \mathrm{ad}(E_G)$, and a Levi subalgebra bundle $\ell(\mathfrak{p}) \subset \mathfrak{p}$, such that the following conditions hold:*

1. *$\widetilde{\rho}(\mathfrak{p}) = \mathfrak{p}$ and $\widetilde{\rho}(\ell(\mathfrak{p})) = \ell(\mathfrak{p})$,*
2. *$\theta \in H^0(X, \ell(\mathfrak{p}) \otimes \Omega^1_X)$, and*
3. *$(\ell(\mathfrak{p}), \theta)$ is stable (stability is defined above).*

Lemma 5.3. — *Let (E_G, ρ, θ) be a semistable pseudo-real principal Higgs G-bundle. Then the principal Higgs G-bundle (E_G, θ) is semistable.*

Let (E_G, ρ, θ) be a polystable pseudo-real principal Higgs G-bundle. Then (E_G, θ) is polystable.

Proof. — We begin by noting that the torsionfree part of the tensor product of two polystable (respectively, semistable) Higgs sheaves is again polystable (respectively, semistable); see [7, p. 553, Lemma 4.4] and [7, p. 553, Proposition 4.5]. Consequently, Proposition 2.10, Lemma 2.11 and Corollary 3.8 of [1] extends to Higgs G-bundles. In fact, as noted at then end of [1], once the polystability of the torsionfree part of the tensor product of polystable Higgs sheaves is established, the results of [1] extend to Higgs G-bundles. Therefore, the lemma follows exactly as Lemma 2.5 and Lemma 3.3 do. □

Let (E_G, θ) be a principal Higgs G-bundle on X. Let $E_{K_G} \subset E_G$ be a C^∞ reduction of structure group to the maximal compact subgroup K_G (see (3.2)). The Chern connection on E_G for E_{K_G} will be denoted by ∇, and the curvature of ∇ will be denoted by $\mathcal{K}(\nabla)$. Let θ^* be the adjoint of θ with respect to E_{K_G}. To describe θ^* explicitly, first note that we have a canonical C^∞ decomposition into a direct sum of real vector bundles

$$\mathrm{ad}(E_G) = \mathrm{ad}(E_{K_G}) \oplus \mathcal{S},$$

where $\mathcal{S}$ is defined in (3.4). If $\theta = \theta_1 + \theta_2$ with respect to this decomposition, then

$$\theta^* = -\overline{\theta_1} + \overline{\theta_2}, \tag{5.2}$$

where the conjugation is the usual conjugation of one-forms.

Definition 5.4. — *The Hermitian structure $E_{K_G} \subset E_G$ is said to be an* Einstein-Hermitian structure *if there is an element λ in the center of $\mathfrak{g}$ such that the section*

$$\Lambda(\mathcal{K}(\nabla) + [\theta, \theta^*]) \in C^\infty(X, \mathrm{ad}(E_G))$$

coincides with the one given by λ; here Λ as before is the adjoint of multiplication by the Kähler form. If $E_{K_G} \subset E_G$ is an Einstein-Hermitian structure, then the corresponding Chern connection ∇ is called an Einstein-Hermitian connection*.*

A principal Higgs G-bundle admits an Einstein-Hermitian structure if and only if it is polystable, and furthermore, the Einstein-Hermitian connection on a polystable principal Higgs G-bundle is unique [17], [11], [7, p. 554, Theorem 4.6].

If (E_G, θ) is a polystable principal Higgs G-bundle such that

(1) $\int_X c_2(\mathrm{ad}(E_G)) \wedge \omega^{\dim_{\mathbb{C}}(X)-2} = 0$, and
(2) for any character χ of G, the line bundle over X associated to E_G for χ is of degree zero.

Then all the rational characteristic classes of E_G of positive degree vanish [18, p. 20, Corollary 1.3].

Proposition 5.5. — *Let (E_G, ρ, θ) be a pseudo-real principal Higgs G-bundle. Then the principal Higgs G-bundle (E_G, θ) admits an Einstein-Hermitian structure $E_{K_G} \subset E_G$ with $\rho(E_{K_G}) = E_{K_G}$ if and only if (E_G, ρ, θ) is polystable.*

Proof. — The proof is similar to the proofs of Corollary 3.9 and Proposition 3.10. But the following observation is needed to make the proof of Proposition 3.8 work in the present situation (Corollary 3.9 is a consequence of Proposition 3.8).

Let (F_G, φ) be a polystable principal Higgs G-bundle on X. Let

$$F_{K_G} \subset F_G$$

be an Einstein-Hermitian structure on F_G. Let ∇^F be the corresponding Chern connection on F_G. The connection on $\mathrm{ad}(F_G)$ induced by ∇^F will be denoted by ∇^{ad}. As in (3.4), let

$$\mathcal{S} := \mathrm{ad}(F_{K_G})^{\perp} \subset \mathrm{ad}(F_G)$$

be the orthogonal complement with respect to an Hermitian structure on $\mathrm{ad}(F_G)$ induced by a K_G-invariant Hermitian form on $\mathfrak{g}$. There is a natural bijective correspondence between the Hermitian structures on F_G and the smooth sections of $\mathcal{S}$: the Hermitian structure corresponding to a section s is $\exp(s)(F_{K_G}) \subset F_G$.

An Hermitian structure $\exp(s)(F_{K_G}) \subset F_G$ is an Einstein-Hermitian structure for (F_G, φ) if and only if

- s is flat with respect to the connection ∇^{ad} on $\mathrm{ad}(F_G)$, and
- $[s, \varphi] = 0$ (it is the section of $\mathrm{ad}(F_G) \otimes \Omega^1_X$ given by the Lie bracket operation on the fibers of $\mathrm{ad}(F_G)$).

Therefore, if $\exp(s)(F_{K_G})$ is an Einstein-Hermitian structure for (F_G, φ), then the Hermitian structure

$$\exp(s/2)(F_{K_G}) \subset F_G$$

is also an Einstein-Hermitian structure for (F_G, φ).

The rest of the proof of Proposition 3.8 works as before once the above observation is incorporated. □

Let $\widetilde{G} := G \rtimes (\mathbb{Z}/2\mathbb{Z})$ be the semi-direct product defined by the involution σ_G. Consider $\Gamma(x_0)$ defined in Section 4. Let $\mathrm{Map}'(\Gamma(x_0), \widetilde{G})$ be the space of all maps

$$\delta \colon \Gamma(x_0) \longrightarrow \widetilde{G}$$

such that the following diagram is commutative:

$$\begin{array}{ccccccccc} e & \longrightarrow & \pi_1(X,x_0) & \longrightarrow & \Gamma(x_0) & \xrightarrow{\eta} & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & e \\ & & \downarrow & & \downarrow \delta & & \| & & \\ e & \longrightarrow & G & \longrightarrow & \widetilde{G} & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & e \end{array} \tag{5.3}$$

For an element $c \in Z_{\mathbb{R}} \bigcap K_G$, let $\mathrm{Hom}_c(\Gamma(x_0), \widetilde{G})$ be the space of all maps

$$\delta \in \mathrm{Map}'(\Gamma(x_0), \widetilde{G})$$

such that

- the restriction of δ to $\pi_1(X, x_0)$ is a homomorphism of groups,
- $\delta(g'g) = c\delta(g')\delta(g)$, if $\eta(g) = 1 = \eta(g')$ (the homomorphism η is defined in (4.1)), and
- $\delta(g'g) = \delta(g')\delta(g)$ otherwise (meaning if $\eta(g) \cdot \eta(g') = 0$).

If $c = e$, then $\mathrm{Hom}_c(\Gamma(x_0), \widetilde{G})$ is the space of all homomorphisms from $\Gamma(x_0)$ to $\widetilde{G}$ satisfying (5.3).

Imitating the construction of $\widetilde{\widetilde{K}}$ in Remark 4.1, we construct a group $\widetilde{\widetilde{G}}$ whose underlying set is $G \times \{0, 1\}$, and the group operation is given by

$$(g_1, e_1) \cdot (g_2, e_2) = (g_1(\sigma_G)^{e_1}(g_2)c^{e_1 e_2}, e_1 + e_2)$$

(see (4.3)). The subset $\mathrm{Hom}_c(\Gamma(x_0), \widetilde{G})$ of $\mathrm{Map}'(\Gamma(x_0), \widetilde{G})$ consists of those elements that are homomorphisms from $\Gamma(x_0)$ to the group $\widetilde{\widetilde{G}}$.

Two elements $\delta', \delta' \in \mathrm{Hom}_c(\Gamma(x_0), \widetilde{G})$ are called *equivalent* if there is an element $g \in G$ such that $\delta'(z) = g^{-1}\delta(z)g$ for all $z \in \Gamma(x_0)$.

Let H be a connected complex reductive affine algebraic group. A homomorphism

$$\gamma \colon \pi_1(X, x_0) \longrightarrow H$$

is called *irreducible* if the image $\gamma(\pi_1(X, x_0))$ is not contained in some proper parabolic subgroup of H. A homomorphism

$$\gamma \colon \pi_1(X, x_0) \longrightarrow G$$

is called *completely reducible* if there is a parabolic subgroup $P \subset G$ and a Levi factor $L(P)$ of P (see [12, p. 184], [8] for Levi factor) such that

- $\gamma(\pi_1(X, x_0)) \subset L(P)$, and
- the homomorphism $\gamma \colon \pi_1(X, x_0) \longrightarrow L(P)$ is irreducible.

A map

$$\delta \in \mathrm{Hom}_c(\Gamma(x_0), \widetilde{G})$$

is called *completely reducible* if the homomorphism $\delta|_{\pi_1(X,x_0)}$ is completely reducible. Note that if δ is is completely reducible, then all elements in $\mathrm{Hom}_c(\Gamma(x_0), \widetilde{G})$ equivalent to δ are also completely reducible.

Proposition 5.6. — *There is a natural bijective correspondence between the equivalence classes of completely reducible elements of* $\mathrm{Hom}_c(\Gamma(x_0), \widetilde{G})$*, and the isomorphism classes of polystable pseudo-real principal Higgs G-bundles (E_G, ρ, θ) satisfying the following conditions:*

- $\int_X c_2(\mathrm{ad}(E_G)) \wedge \omega^{\dim_{\mathbb{C}}(X)-2} = 0$,
- *for any character χ of G, the line bundle over X associated to E_G for χ is of degree zero, and*
- *the corresponding element in $Z_{\mathbb{R}} \bigcap K_G$ is c (see Definition 2.1).*

Proof. — The proof is similar to the proof of Theorem 4.6 after we incorporate Proposition 5.5. To explain this, take a polystable pseudo-real principal Higgs G-bundle (E_G, ρ, θ) such that

- $\int_X c_2(\mathrm{ad}(E_G)) \wedge \omega^{\dim_{\mathbb{C}}(X)-2} = 0$,
- for any character χ of G, the line bundle over X associated to E_G for χ is of degree zero, and
- the corresponding element in $Z_{\mathbb{R}} \bigcap K_G$ is c.

These conditions imply that all the rational characteristic classes of E_G of positive degree vanish [18, p. 20, Corollary 1.3]. From Proposition 5.5 we know that (E_G, ρ, θ) admits an Einstein-Hermitian structure $E_{K_G} \subset E_G$ such that $\rho(E_{K_G}) = E_{K_G}$. Let ∇^G be the corresponding Chern connection. Define θ^* as done in (5.2). Consider the connection

$$D := \nabla^G + \theta + \theta^*$$

on E_G. It is a flat connection because all the rational characteristic classes of E_G of positive degree vanish. The monodromy representation for D is completely reducible [18, p. 20, Corollary 1.3], [4, Theorem 1.1].

In Theorem 4.6, consider the construction of an element of $\mathrm{Hom}_c(\Gamma(x_0), \widetilde{K})$ from a polystable pseudo-real principal G-bundle F_G such that $\int_X c_2(\mathrm{ad}(F_G)) \wedge \omega^{\dim_{\mathbb{C}}(X)-2} = 0$ and for any character χ of G, the line bundle over X associated to F_G for χ is of degree zero (see Proposition 4.5). In this construction, replace the flat Einstein-Hermitian connection ∇ by the flat connection D constructed above. It yields a completely reducible element of $\mathrm{Hom}_c(\Gamma(x_0), \widetilde{G})$.

For the reverse direction, take a completely reducible element

$$\delta \in \mathrm{Hom}_c(\Gamma(x_0), \widetilde{G}).$$

Consider the homomorphism $\delta|_{\pi_1(X,x_0)}$. It gives a flat principal G-bundle (E_G, D) and a point $z_0 \in (E_G)_{x_0}$.

In Theorem 4.6, consider the construction of a pseudo-real principal G-bundle from an element of $\mathrm{Hom}_c(\Gamma(x_0), \widetilde{K})$ (see Proposition 4.4). In this construction, replace the flat Hermitian connection ∇ by the given flat connection D on E_G. It yields a pseudo-real structure

$$\rho\colon E_G \longrightarrow E_G \tag{5.4}$$

on the principal G-bundle E_G.

Since the monodromy representation for D is completely reducible, a theorem of Corlette says that E_G admits a harmonic reduction

$$E_{K_G} \subset E_G$$

(see [9, p. 368, Theorem 3.4], [18, p. 19, Theorem 1]). We will show that the harmonic reduction E_{K_G} can be so chosen that it satisfies the condition

$$\rho(E_{K_G}) = E_{K_G}, \tag{5.5}$$

where ρ is the pseudo-real structure obtained in (5.4).

To prove this, take a harmonic reduction $E_{K_G} \subset E_G$. As in (3.4), let

$$\mathcal{S} := \mathrm{ad}(E_{K_G})^{\perp} \subset \mathrm{ad}(E_G)$$

be the orthogonal complement with respect to an Hermitian structure on $\mathrm{ad}(E_G)$ induced by a K_G-invariant Hermitian form on $\mathfrak{g}$. We recall that every Hermitian structures on E_G is of the form $\exp(s)(E_{K_G})$, where s is a smooth sections of $\mathcal{S}$.

Let D^{ad} be the flat connection on $\mathrm{ad}(E_G)$ induced by the connection D on E_G. An Hermitian structure

$$\exp(s)(E_{K_G}) \subset E_G$$

is a harmonic reduction for (E_G, D) if and only if

$$D^{\mathrm{ad}}(s) = 0.$$

Therefore, if $\exp(s)(E_{K_G}) \subset E_G$ is a harmonic reduction for (E_G, D), then

$$\exp(s/2)(E_{K_G}) \subset E_G$$

is also a harmonic reduction for (E_G, D). Now the proof of Proposition 3.8 gives that there is a harmonic reduction E_{K_G} for (E_G, D) such that (5.5) holds.

Let (E'_G, θ) be the principal Higgs G-bundle corresponding to the triple (E, D, E_{K_G}), where E_{K_G} satisfies (5.5). So

$$D = \nabla + \theta + \theta^* = \nabla^{1,0} + \nabla^{0,1} + \theta + \theta^*,$$

such that the following three conditions hold:

(1) ∇ is a connection on E_G coming from a connection on E_{K_G}.

(2) $\nabla^{0,1} \circ \nabla^{0,1} = 0$, meaning $\nabla^{0,1}$ defines a holomorphic structure on the C^∞ principal G-bundle E_G. This holomorphic principal G-bundle $(E_G, \nabla^{0,1})$ is denoted by E'_G.

(3) θ is a Higgs field on the holomorphic principal G-bundle E'_G.

(See [18, p. 13].) The triple (E'_G, ρ, θ), where ρ is constructed in (5.4), is a polystable pseudo-real principal Higgs G-bundle. □

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