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INTEGRAL STRUCTURES ON p -ADIC FOURIER THEORY

by Kenichi BANNAI & Shinichi KOBAYASHI (*)

ABSTRACT. — In this article, we give an explicit construction of the p -adic Fourier transform by Schneider and Teitelbaum, which allows for the investigation of the integral property. As an application, we give a certain integral basis of the space of K -locally analytic functions on the ring of integers \mathcal{O}_K for any finite extension K of \mathbb{Q}_p , generalizing the basis constructed by Amice for locally analytic functions on \mathbb{Z}_p . We also use our result to prove congruences of Bernoulli-Hurwitz numbers at non-ordinary (i.e. supersingular) primes originally investigated by Katz and Chellali.

RÉSUMÉ. — Dans cet article, nous donnons une construction explicite de la transformation de Fourier p -adique de Schneider et Teitelbaum, qui nous permet d'étudier son intégralité. Comme application, pour toute extension finie K de \mathbb{Q}_p nous donnons une certaine base entière de l'espace de K -fonctions localement analytiques sur l'anneau des entiers \mathcal{O}_K , en généralisant la base construite par Amice pour les fonctions localement analytiques sur \mathbb{Z}_p . Nous utilisons également notre résultat pour démontrer certaines relations de congruence étudiées initialement par Katz et Chellali entre nombres de Bernoulli-Hurwitz aux places non-ordinaires (c'est-à-dire supersingulières).

1. Introduction

One important method in studying the congruences and p -adic properties of important invariants in number theory is the use of p -adic measures interpolating such values. Such theory was applied to obtain the Kummer congruence between special values of Riemann zeta function as well as the

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construction of the p -adic L -functions for elliptic curves with ordinary reduction at p . When dealing with the non-ordinary case, it is necessary to use the theory of p -adic analytic distributions, which is a generalization of the theory of p -adic measures. For such p -adic distributions on \mathbb{Z}_p , the Amice transform gives a one-to-one correspondence between \mathbb{C}_p -valued distributions on \mathbb{Z}_p and rigid analytic functions on the open unit disc. The general idea is to study the congruences and p -adic properties of the interpolated invariants through the p -adic property of the rigid analytic function corresponding to the p -adic distribution. However, contrary to the case of p -adic measures, the Amice transform is not well-behaved integrally for general p -adic distributions, hence it is necessary to investigate in detail the precise integral structure of this transform. Amice [1, §10] investigated the precise integral structure of the Amice transform.

Let \mathcal{O}_K be the ring of integers of a finite extension K of \mathbb{Q}_p . In [8, §4], Schneider and Teitelbaum constructed the p -adic Fourier transform, which is a one-to-one correspondence between \mathbb{C}_p -valued distributions on \mathcal{O}_K and rigid analytic functions on an open unit disc. The purpose of this article is to give an explicit and elementary construction of the p -adic Fourier transform of Schneider-Teitelbaum, which allows investigation of the precise integral structure of this correspondence. We then determine an integral structure on the ring of locally analytic functions on \mathcal{O}_K . The integrality of the p -adic Fourier transform for general K is even less well behaved than for the case of \mathbb{Q}_p ; even if the rigid analytic function corresponding to a p -adic distribution has bounded coefficients, the p -adic distribution may not necessarily be a p -adic measure. As an application of our result, we obtain the congruences originally proved by Katz [7, Theorem 3.11] and Chellali [4, Théorème 1.1] of Bernoulli-Hurwitz numbers, which are essentially special values of p -adic L -functions of CM elliptic curves at non-ordinary primes.

We now give the exact statements of our theorems. Let p be a rational prime and let $|\cdot|$ be the absolute value of \mathbb{C}_p such that $|p| = p^{-1}$. Let π be a uniformizer of \mathcal{O}_K , and let \mathbb{F}_q be the residue field of \mathcal{O}_K . We define $LA_N(\mathcal{O}_K, \mathbb{C}_p)$ to be the space of locally analytic functions on \mathcal{O}_K of order N which take values in \mathbb{C}_p . That is, $f(x) \in LA_N(\mathcal{O}_K, \mathbb{C}_p)$ if and only if $f(x)$ is defined as a convergent power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ on $a+\pi^N\mathcal{O}_K$ for any $a \in \mathcal{O}_K$. We let $\|f\|_{a,N} := \max_n \{|a_n\pi^{nN}|\}$. The space $LA_N(\mathcal{O}_K, \mathbb{C}_p)$ is a p -adic Banach space induced by the norm $\max_{a \in \mathcal{O}_K} \{\|f\|_{a,N}\}$ and we denote by $LA_N(\mathcal{O}_K, \mathbb{C}_p)_0$ the submodule of elements whose absolute values are less than or equal to 1. We let \mathcal{G} be a Lubin-Tate group of K corresponding to

π , and let $\varpi_p \in \mathbb{C}_p$ be a *p*-adic period of \mathcal{G} . We let

$$\bar{\rho}(k) = \max_{k \leq m} \{ |m! / \varpi_p^m| \}, \quad \underline{\rho}(k) = \min_{0 \leq m \leq k} \{ |m! / \varpi_p^m| \}.$$

See Proposition 3.1 for the properties of these numbers.

Let $\varphi(t)$ be a rigid analytic function on the open unit disc. In other words, $\varphi(t)$ is a power series of the form $\varphi(t) = \sum_{n=0}^\infty c_n t^n$ such that $|c_n| r_0^n \rightarrow 0$ for any $0 < r_0 < 1$. Let μ_φ be the distribution on \mathcal{O}_K corresponding to $\varphi(t)$ given by Schneider-Teitelbaum’s *p*-adic Fourier theory [8, Theorem 2.3]. Then we have the following:

THEOREM 1.1. — *Let $f \in LA_N(\mathcal{O}_K, \mathbb{C}_p)$. Then we have*

$$(1.1) \quad \left| \int_{a+\pi^N \mathcal{O}_K} f(x) d\mu_\varphi \right| \leq \bar{\rho}(0) \left| \frac{\pi}{q} \right|^N \|f\|_{a,N} \|\varphi\|_N$$

where

$$(1.2) \quad \|\varphi\|_N := \max_k \left\{ |c_k| \bar{\rho} \left(\left[\frac{k}{q^N} \right] \right) \right\}$$

and $[x]$ is the integral part of x .

The crucial difference from the case when $K = \mathbb{Q}_p$ is the fact that $|\pi/q| > 1$ when $K \neq \mathbb{Q}_p$. A finer version of the above is given as Theorem 4.3. Since $\bar{\rho} \left(\left[\frac{k}{q^N} \right] \right) \sim p^{-kr}$ where $r = 1/eq^N(q-1)$, the value $\|\varphi\|_N$ is approximated by

$$\|\varphi\|_{\overline{\mathbf{B}}(p^{-r})} = \max_{x \in \overline{\mathbf{B}}(p^{-r})} \{ |\varphi(x)| \},$$

where $\overline{\mathbf{B}}(p^{-r}) \subset \mathbb{C}_p$ is the closed disc of radius p^{-r} centered at the origin.

As an application of our main theorem, we obtain an estimate of the Fourier coefficients of Mahler like expansion of functions in $LA_N(\mathcal{O}_K, \mathbb{C}_p)$. Let $\lambda(t)$ be the formal logarithm of \mathcal{G} , and following [8], we define the polynomial $P_n(x)$ by

$$\exp(x\lambda(t)) = \sum_{n=0}^\infty P_n(x) t^n.$$

Note that when \mathcal{G} is the multiplicative formal group $\mathcal{G} = \widehat{\mathbb{G}}_m$, then $\lambda(t) = \log(1+t)$ and the above expansion is simply

$$(1+t)^x = \sum_{n=0}^\infty \binom{x}{n} t^n.$$

Hence the polynomial $P_n(x)$ is a generalization of the binomial polynomial

$$\binom{x}{n} = \frac{x(x-1) \cdots (x-n+1)}{n!}.$$

Then we have the following.

THEOREM 1.2 (Theorem 4.7). — *The series $\sum_{n=0}^{\infty} a_n P_n(x\varpi_p)$ converges to an element of $LA_N(\mathcal{O}_K, \mathbb{C}_p)_0$ for a_n satisfying*

$$|a_n| \leq \underline{\rho} \left(\left[\frac{n}{q^N} \right] \right), \quad \lim_{n \rightarrow 0} |a_n| / \underline{\rho} \left(\left[\frac{n}{q^N} \right] \right) = 0.$$

Conversely, if $f(x) \in LA_N(\mathcal{O}_K, \mathbb{C}_p)_0$, then it has an expansion

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x\varpi_p)$$

of the form

$$|a_n| \leq c \left| \frac{\pi}{q} \right|^N \bar{\rho} \left(\left[\frac{n}{q^N} \right] \right), \quad \lim_{n \rightarrow 0} |a_n| / \bar{\rho} \left(\left[\frac{n}{q^N} \right] \right) = 0,$$

where $c = 1$ if $e \leq p - 1$, and $c = \bar{\rho}(0)$ otherwise.

COROLLARY 1.3 (Corollary 4.8). — *Suppose*

$$e_{N,n} := \underline{\gamma} \left(\left[\frac{n}{q^N} \right] \right) P_n(x\varpi_p), \quad (n = 0, 1, \dots),$$

where $\underline{\gamma}(u)$ is an element in \mathbb{C}_p such that $\underline{\rho}(u) = |\underline{\gamma}(u)|$. If we denote by L_N the $\mathcal{O}_{\mathbb{C}_p}$ -module topologically generated by $e_{N,n}$, then

$$\bar{\rho}(0)^{-2} \left| \frac{q}{\pi} \right|^N LA_N(\mathcal{O}_K, \mathbb{C}_p)_0 \subset L_N \subset LA_N(\mathcal{O}_K, \mathbb{C}_p)_0.$$

In particular, $L_N \otimes \mathbb{Q}_p = LA_N(\mathcal{O}_K, \mathbb{C}_p)$. In other words, the functions $e_{N,n}$ form a p -adic Banach basis of $LA_N(\mathcal{O}_K, \mathbb{C}_p)$. Moreover, if $e \leq p - 1$, then

$$\left| \frac{q}{\pi} \right|^{N+1} LA_N(\mathcal{O}_K, \mathbb{C}_p)_0 \subset L_N \subset LA_N(\mathcal{O}_K, \mathbb{C}_p)_0.$$

This result for the case $\mathcal{O}_K = \mathbb{Z}_p$ gives the result of Amice [1, Théorème 3], namely that the functions

$$\left[\frac{n}{p^N} \right]! \binom{x}{n} \quad (n = 0, 1, \dots)$$

form a topological basis of $LA_N(\mathbb{Z}_p, \mathbb{C}_p)_0$ (actually, we can show that it is a basis of $LA_N(\mathbb{Z}_p, \mathbb{Q}_p)_0$).

As another application, in Theorem 5.8, we derive from our estimate of the integral the congruence of Bernoulli-Hurwitz numbers $BH(n)$ at supersingular primes established by Katz [7, Theorem 3.11] and Chellali [4, Théorème 1.1].

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2. Schneider-Teitelbaum’s *p*-adic Fourier theory.

Let K be a finite extension of \mathbb{Q}_p and $k = \mathbb{F}_q$ the residue field. Let e be the absolute ramification index of K . We fix a uniformizer π of K and let \mathcal{G} be a Lubin-Tate formal group of K associated to π . For a natural number N and an element a of \mathcal{O}_K , we define the space $A(a + \pi^N \mathcal{O}_K, \mathbb{C}_p)$ of K -analytic functions on $a + \pi^N \mathcal{O}_K$ by

$$A(a + \pi^N \mathcal{O}_K, \mathbb{C}_p) := \left\{ f : a + \pi^N \mathcal{O}_K \rightarrow \mathbb{C}_p \mid f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n, a_n \in \mathbb{C}_p, \pi^{nN} a_n \rightarrow 0 \right\}.$$

We equip the space $A(a + \pi^N \mathcal{O}_K, \mathbb{C}_p)$ with the norm

$$\|f\|_{a,N} := \max_n \{|\pi^{nN} a_n|\} = \max_{x \in a + \pi^N \mathcal{O}_{\mathbb{C}_p}} \{|f(x)|\}.$$

We also define the space $LA_N(\mathcal{O}_K, \mathbb{C}_p)$ of locally K -analytic functions on \mathcal{O}_K of order N by

$$LA_N(\mathcal{O}_K, \mathbb{C}_p) := \{f : \mathcal{O}_K \rightarrow \mathbb{C}_p \mid f|_{a + \pi^N \mathcal{O}_K} \in A(a + \pi^N \mathcal{O}_K, \mathbb{C}_p) \text{ for any } a \in \mathcal{O}_K\},$$

which is a p -adic Banach space by the norm $\max_a \{\|f\|_{a,N}\}$. We denote by $LA_N(\mathcal{O}_K, \mathbb{C}_p)_0$ the submodule of elements whose absolute values are less than or equal to 1. We put

$$LA(\mathcal{O}_K, \mathbb{C}_p) = \bigcup_N LA_N(\mathcal{O}_K, \mathbb{C}_p)$$

and equip it with the inductive limit topology. A continuous \mathbb{C}_p -linear function $LA(\mathcal{O}_K, \mathbb{C}_p) \rightarrow \mathbb{C}_p$ is called a \mathbb{C}_p -valued distribution on \mathcal{O}_K . We denote the space of \mathbb{C}_p -valued distributions on \mathcal{O}_K by $D(\mathcal{O}_K, \mathbb{C}_p)$, i.e.

$$D(\mathcal{O}_K, \mathbb{C}_p) = \varinjlim_N \text{Hom}_{\mathbb{C}_p}^{\text{cont}}(LA_N(\mathcal{O}_K, \mathbb{C}_p), \mathbb{C}_p).$$

We write an element of $D(\mathcal{O}_K, \mathbb{C}_p)$ symbolically as

$$\int d\mu : LA(\mathcal{O}_K, \mathbb{C}_p) \rightarrow \mathbb{C}_p, \quad f \mapsto \int f d\mu = \int_{\mathcal{O}_K} f(x) d\mu(x).$$

The space $D(\mathcal{O}_K, \mathbb{C}_p)$ has a product structure given by the convolution product. For a compact open set U of \mathcal{O}_K , we let

$$\int_U f(x) d\mu(x) := \int_{\mathcal{O}_K} f(x) \cdot 1_U(x) d\mu(x),$$

where 1_U is the characteristic function of U .

The structure of $D(\mathcal{O}_K, \mathbb{C}_p)$ is well-known for the case $K = \mathbb{Q}_p$ and described through the so-called Amice transform. We denote by R^{rig} the ring of rigid analytic functions on the open disc of radius 1, that is, the ring of power series of the form $\varphi(T) = \sum_{n=0}^{\infty} c_n T^n$ such that $|c_n| r_0^n \rightarrow 0$ for any $0 < r_0 < 1$. Then there exists an isomorphism of topological \mathbb{C}_p -algebras

$$(2.1) \quad D(\mathbb{Z}_p, \mathbb{C}_p) \cong R^{\text{rig}}, \quad \mu \mapsto \varphi$$

that is characterized by the equation

$$c_n = \int_{\mathbb{Z}_p} \binom{x}{n} d\mu(x)$$

or equivalently

$$\varphi(T) = \int_{\mathbb{Z}_p} (1 + T)^x d\mu(x).$$

For the Mahler expansion

$$f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n}$$

of $f \in LA(\mathbb{Z}_p, \mathbb{C}_p)$, Amice showed that $|a_n| r^n \rightarrow 0$ for some $r > 1$ and hence we can compute the integral as

$$(2.2) \quad \int_{\mathbb{Z}_p} f(x) d\mu = \sum_{n=0}^{\infty} a_n c_n.$$

Schneider-Teitelbaum [8, Theorem 2.3] constructed an isomorphism analogous to (2.1) for a general local field K .

Let ϖ_p be a p -adic period of \mathcal{G} . By Tate’s theory of p -divisible groups and Lubin-Tate theory, we have

$$\text{Hom}_{\mathcal{O}_{\mathbb{C}_p}}(\mathcal{G}, \widehat{\mathbb{G}}_m) \cong \text{Hom}_{\mathbb{Z}_p}(T_p \mathcal{G}, T_p \widehat{\mathbb{G}}_m) \cong \mathcal{O}_K.$$

(The last isomorphism is non-canonical.) Hence there exists a generator of the \mathcal{O}_K -module $\text{Hom}_{\mathcal{O}_{\mathbb{C}_p}}(\mathcal{G}, \widehat{\mathbb{G}}_m)$, which is written in the form of the integral power series $\exp(\varpi_p \lambda(t)) \in \mathcal{O}_{\mathbb{C}_p}[[t]]$ where $\lambda(t)$ is the logarithm of

\mathcal{G} . The element $\varpi_p \in \mathcal{O}_{\mathbb{C}_p}$ is determined uniquely up to an element of \mathcal{O}_K^\times . We fix such a ϖ_p and call it the *p*-adic period of \mathcal{G} (if the height of \mathcal{G} is equal to 1, then the *inverse* of ϖ_p is often called a *p*-adic period of \mathcal{G} . For example, see [9]). It is known that $|\varpi_p| = p^{-s}$, where $s = \frac{1}{p-1} - \frac{1}{e(q-1)}$ (see Appendix of [8] or an elementary proof in [3] when K/\mathbb{Q}_p is unramified). We define the polynomials $P_n(X) \in K[X]$ by the formal expansion

$$\exp(X\lambda(t)) = \sum_{n=0}^{\infty} P_n(X) t^n.$$

Note that in the case $\mathcal{G} = \widehat{\mathbb{G}}_m$, $\pi = p$ and $\lambda(t) = \log(1+t)$, the polynomial $P_n(X)$ is simply the binomial polynomial $\binom{X}{n}$. By construction, $P_n(x\varpi_p)$ is in $\mathcal{O}_{\mathbb{C}_p}$ if $x \in \mathcal{O}_K$.

THEOREM 2.1 (Schneider-Teitelbaum [8, §4]).

i) *The series*

$$\sum_{n=0}^{\infty} a_n P_n(x\varpi_p)$$

converges to an element of $LA(\mathcal{O}_K, \mathbb{C}_p)$ if $\overline{\lim}_n |a_n|^{\frac{1}{n}} < 1$. Conversely, any locally K -analytic function $f(x)$ on \mathcal{O}_K has a unique expansion

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x\varpi_p)$$

for some sequence $(a_n)_n$ in \mathbb{C}_p such that $\overline{\lim}_n |a_n|^{\frac{1}{n}} < 1$.

ii) *There exists an isomorphism of topological \mathbb{C}_p -algebras*

$$(2.3) \quad D(\mathcal{O}_K, \mathbb{C}_p) \cong R^{\text{rig}},$$

having the following characterization property: if $\varphi(T) = \sum_{n=0}^{\infty} c_n T^n$ corresponds to a distribution μ , then

$$c_n = \int_{\mathcal{O}_K} P_n(x\varpi_p) d\mu(x)$$

or equivalently

$$\varphi(t) = \int_{\mathcal{O}_K} \exp(x\varpi_p\lambda(t)) d\mu(x).$$

Schneider and Teitelbaum called the power series $\varphi(t)$ corresponding to μ the Fourier transform of μ and denoted it by $F_\mu(t)$.

3. Power sums

In this section, we give an estimate of the absolute value of the power sum

$$S_{N,n,k} := \partial_{\mathcal{G}}^n \sum_{t_N \in \mathcal{G}[\pi^N]} (t \oplus t_N)^k|_{t=0},$$

where $x \oplus y = \mathcal{G}(x, y)$, $\partial_{\mathcal{G}}$ is the differential operator $\lambda'(t)^{-1}(d/dt)$, and $\mathcal{G}[\pi^N]$ is the kernel of the multiplication $[\pi^N]$ of \mathcal{G} . This estimate is crucial for everything in this paper. We use Newton's method to compute this value.

We define $\bar{\rho}[l, n]$ and $\underline{\rho}[l, n]$ by

$$\bar{\rho}[l, n] = \max_{l \leq m \leq n} \{ |m!/\varpi_p^m| \}, \quad \underline{\rho}[l, n] = \min_{l \leq m \leq n} \{ |m!/\varpi_p^m| \}$$

for $l \leq n$. For $l > n$, we put $\bar{\rho}[l, n] = 0$ and $\underline{\rho}[l, n] = \infty$. Then $\bar{\rho}(k) = \bar{\rho}[k, \infty]$ and $\underline{\rho}(k) = \underline{\rho}[0, k]$ are the constants appearing in the introduction.

PROPOSITION 3.1.

- i) The values $\bar{\rho}(k)$ and $\underline{\rho}(k)$ are decreasing with k .
- ii) We have

$$\underline{\rho}(k) \leq \bar{\rho}(k), \quad \bar{\rho}(k) \leq \bar{\rho}(0)\underline{\rho}(k).$$

- iii) We have

$$\underline{\rho}(k_1 + \dots + k_n) \leq \underline{\rho}(k_1) \cdots \underline{\rho}(k_n).$$

- iv) We have

$$p^{\frac{1}{p-1} - \frac{k}{e(q-1)}} \leq \underline{\rho}(k) \leq 1.$$

Proof. — i) is clear. For ii), first we have $\bar{\rho}(k) \geq |k!/\varpi_p^k| \geq \underline{\rho}(k)$. Suppose $\bar{\rho}(k) = |k_1!/\varpi_p^{k_1}|$ and $\underline{\rho}(k) = |k_2!/\varpi_p^{k_2}|$. Then $k_1 \geq k \geq k_2$ and

$$\left| \frac{k_1!}{\varpi_p^{k_1}} / \frac{k_2!}{\varpi_p^{k_2}} \right| = \left| \binom{k_1}{k_2} \frac{(k_1 - k_2)!}{\varpi_p^{k_1 - k_2}} \right| \leq \bar{\rho}(0).$$

For iii), suppose that $\underline{\rho}(k_i) = |l_i!/\varpi_p^{l_i}|$ for $l_i \leq k_i$. Then the assertion for $\underline{\rho}$ follows from

$$\underline{\rho}(k_1 + \dots + k_n) \leq \left| \frac{(l_1 + \dots + l_n)!}{\varpi_p^{l_1 + \dots + l_n}} \right| \leq \left| \frac{(l_1 + \dots + l_n)!}{l_1! \cdots l_n!} \right| \left| \frac{l_1!}{\varpi_p^{l_1}} \right| \cdots \left| \frac{l_n!}{\varpi_p^{l_n}} \right|.$$

For iv), suppose that $\underline{\rho}(k) = |l!/\varpi_p^l|$ for $l \leq k$. Then

$$p^{\frac{1}{p-1} - \frac{k}{e(q-1)}} \leq p^{\frac{1}{p-1} - \frac{l}{e(q-1)}} \leq \left| \frac{l!}{\varpi_p^l} \right| = \underline{\rho}(k).$$

□

If $e \leq p - 1$, then we can determine $\bar{\rho}(k)$ and $\underline{\rho}(k)$ explicitly.

LEMMA 3.2. — *Let k be a non-negative integer and let q be a power of p .*

- i) *For any integer $0 \leq r < q$, we have $\binom{kq+r}{r} \equiv 1 \pmod p$.*
- ii) *We have $\binom{k}{q} \in [k/q]\mathbb{Z}_p$.*

Proof. — i) is clear. For ii), we write $k = aq + r$ with $0 \leq r < q$. We put $(1 + x)^q = 1 + x^q + pf(x)$ for some integral polynomial $f(x)$. Then

$$(1 + x)^k = (1 + x^q + pf(x))^a (1 + x)^r \equiv (1 + x^q)^a (1 + x)^r \pmod{ap\mathbb{Z}_p[x]}.$$

Hence the coefficient of x^q in the above is in $a\mathbb{Z}_p$. □

PROPOSITION 3.3. — *Let i, e and h be natural numbers. We put $q = p^h$. Then we have*

$$v_p(i!) \geq \frac{i}{p-1} - \frac{i}{e(q-1)} - h + \frac{1}{e} + \left[\frac{i}{q} \right] \left(\frac{1}{e} - \frac{1}{p-1} + \frac{1}{e(q-1)} \right) + v_p \left(\left[\frac{i}{q} \right]! \right).$$

In the above, equality holds if and only if $i \equiv -1 \pmod q$. In particular, if $e \leq p - 1$ or $i < q$, then we have

$$v_p(i!) \geq \frac{i}{p-1} - \frac{i}{e(q-1)} - h + \frac{1}{e}$$

and equality holds if and only if $i = q - 1$. In this case, we have $\bar{\rho}(0) = |\pi/q|$.

Proof. — First, we assume that $i < q$. We prove the inequality by induction on h . If $h = 1$, then $i < p$. Hence the left-hand side $v_p(i!)$ is equal to zero, and the right-hand side takes the maximum value when $i = p - 1$, which is also equal to zero. We assume that the inequality holds for natural numbers less than h . Since the right-hand side is strictly increasing for i , and $v_p(i!)$ strictly increases only when p divides i , we may assume that i is of the form $i = kp - 1$ for some natural number $k \leq p^{h-1}$. We have

$$v_p(i!) = v_p((kp)!) - v_p(kp) = k - 1 + v_p((k - 1)!).$$

On the other hand, we have

$$\begin{aligned} & \frac{i}{p-1} - \frac{i}{e(q-1)} - h + \frac{1}{e} \\ &= (k-1) + \frac{k-1}{p-1} - \frac{k-1}{e(p^{h-1}-1)} - (h-1) + \frac{1}{e} + \frac{k-1}{e(p^{h-1}-1)} - \frac{kp-1}{e(q-1)} \\ &\leq k-1 + v_p((k-1)!). \end{aligned}$$

In the last inequality, we used the inductive hypothesis and $k \leq p^{h-1}$. Hence we have the desired inequality, and equality holds only when $k = p^{h-1}$, i.e. when $i = q - 1$. For $i \geq q$, by Lemma 3.2 ii) and by induction, we have

$$\begin{aligned} v_p(i!) &\geq v_p((i - q)!) + v_p(q!) + v_p\left(\left[\frac{i}{q}\right]\right) \\ &\geq \frac{i}{p - 1} - \frac{i}{e(q - 1)} - h + \frac{1}{e} \\ &\quad + \left[\frac{i}{q}\right] \left(\frac{1}{e} - \frac{1}{p - 1} + \frac{1}{e(q - 1)}\right) + v_p\left(\left[\frac{i}{q}\right]!\right). \end{aligned}$$

From the above argument and by induction, to have the equality, i must be congruent to -1 modulo q . On the other hand, if $i \equiv -1 \pmod q$, then direct calculations give the equality. \square

PROPOSITION 3.4. — Suppose that $e \leq p - 1$, and that $e > 1$ or $h > 1$.

- i) We have $|n!/\varpi_p^n| > 1$ for $0 < n < q$.
- ii) For any non-negative integer n , $\rho(n) = |n_0!/\varpi_p^{n_0}|$ where $n_0 = [n/q]q$.
- iii) For $n \equiv -1 \pmod q$ and a natural number $i \neq q$, we have

$$\left| \frac{n!}{\varpi_p^n} \right| > \left| \frac{(n + q)!}{\varpi_p^{n+q}} \right| > \left| \frac{(n + i)!}{\varpi_p^{n+i}} \right|$$

In particular, for any non-negative integer n , we have $\bar{\rho}(n) = |n_1!/\varpi_p^{n_1}|$ where $n_1 = [n/q]q + q - 1$.

Proof. — We prove i) by induction on h of $q = p^h$. If $h = 1$, then $n!$ is a p -adic unit and the assertion is clear. Assume that $h > 1$. We write as $n = kp + r$ with $0 \leq r < p$. Then

$$\frac{n!}{\varpi_p^n} = \binom{n}{r} \frac{(kp)!}{\varpi_p^{kp}} \frac{r!}{\varpi_p^r}.$$

Hence by Lemma 3.2 i) and the induction on n , we may assume that $r = 0$ and $k \geq 1$. Then

$$v_p\left(\frac{(kp)!}{\varpi_p^{kp}}\right) = v_p((kp)!) - \frac{kp}{p - 1} + \frac{kp}{e(q - 1)} < v_p(k!) - \frac{k}{p - 1} + \frac{k}{e(p^{h-1} - 1)}.$$

By the inductive hypothesis for h , the right-hand side is negative or 0.

Next we prove ii). Suppose that $m < n_0$. Then

$$\left| \frac{n_0!}{\varpi_p^{n_0}} / \frac{m!}{\varpi_p^m} \right| = \left| \frac{n_0}{\varpi_p} \binom{n_0 - 1}{m} \frac{(n_0 - m - 1)!}{\varpi_p^{n_0 - m - 1}} \right| \leq \left| \frac{n_0}{\varpi_p} \right| \bar{\rho}(0) = \left| \frac{n_0 \pi}{q \varpi_p} \right| < 1.$$

Suppose that $n \geq m > n_0$. We write as $m = [n/q]q + r$ with $0 < r < q$. Then i) and Lemma 3.2 i) show that

$$\left| \frac{n_0!}{\varpi_p^{n_0}} / \frac{m!}{\varpi_p^m} \right| = \left| \binom{m}{r}^{-1} \frac{\varpi_p^r}{r!} \right| < 1.$$

Finally, we show iii). Let n be such that $n \equiv -1 \pmod q$. We have

$$\frac{(n+i)!}{\varpi_p^{n+i}} / \frac{(n+q)!}{\varpi_p^{n+q}} = \frac{(n+i)!}{(n+q)!} \varpi_p^{q-i} = u \frac{q}{\pi} \frac{(i-1)!}{\varpi_p^{i-1}} \frac{\pi \varpi_p^{q-1}}{q!},$$

where $u = \binom{n+q}{q-1}^{-1} \binom{n+i}{i-1}$ is a p -adic integer by Lemma 3.2 i). By Proposition 3.3, the p -adic (additive) valuation of the right-hand side is positive. Since $v_p(\pi/\varpi_p) > 0$, the p -adic (additive) valuation of

$$\frac{(n+q)!}{\varpi_p^{n+q}} / \frac{n!}{\varpi_p^n} = \binom{n+q}{q} \frac{q!}{\pi \varpi_p^{q-1}} \frac{\pi}{\varpi_p}$$

is positive. □

Next we investigate the absolute values of the coefficients of a power of the logarithm and the exponential map of the Lubin-Tate group. The case $k = 1$ in the proposition below is obtained in [10].

PROPOSITION 3.5. — We put $\partial = d/dt$. Then we have

$$\left| \frac{\varpi_p^k \partial^n \lambda(t)^k}{k!n!} \Big|_{t=0} \right| \leq \underline{\rho}[k, n]^{-1}, \quad |\partial^n \exp_{\mathcal{G}}^k(t)|_{t=0}| \leq |\varpi_p^n| \bar{\rho}[k, n].$$

Proof. — The case for $n < k$ or $k = 0$ is trivial. Suppose that $n \geq k \geq 1$. We first assume that the formal logarithm of \mathcal{G} is given by

$$\lambda(t) = \sum_{m=0}^{\infty} \frac{t^m}{\pi^m}.$$

Then it suffices to show inequalities

$$|\partial^n \lambda(t)^k|_{t=0}| \leq |k! \varpi_p^{n-k}|, \quad |\partial^n \exp_{\mathcal{G}}^k(t)|_{t=0}| \leq |k! \varpi_p^{n-k}|.$$

When $k = 1$, the inequality for $\lambda(t)$ is proven by direct calculations. We prove the general case by induction on k . We have

$$\begin{aligned} \partial^n \lambda(t)^k \Big|_{t=0} &= k \partial^{n-1} (\lambda(t)^{k-1} \lambda'(t)) \Big|_{t=0} \\ &= k \partial^{n-1} \sum_{m=0}^{\infty} \lambda(t)^{k-1} \frac{q^m t^{q^m-1}}{\pi^m} \Big|_{t=0} \\ &= \sum_{m=0}^{\infty} \binom{n-1}{q^m-1} \frac{q^m! k}{\pi^m} \partial^{n-q^m} \lambda(t)^{k-1} \Big|_{t=0}. \end{aligned}$$

Hence we have $|\partial^n \lambda(t)^k|_{t=0} \leq |k! \varpi_p^{n-k}|$.

We put $\exp_{\mathcal{G}}^k(t) = \sum_{n=k}^{\infty} a_n t^n$. We prove that $|n! a_n| \leq |k! \varpi_p^{n-k}|$ by induction on n . If $n = k$, then the assertion is true since $a_k = 1$. We assume that the assertion is true for integers less than n . Since $\exp_{\mathcal{G}}^k(\lambda(t)) = t^k$, we have

$$t^k = a_k \lambda(t)^k + a_{k+1} \lambda(t)^{k+1} + \dots + a_n \lambda(t)^n + \dots$$

By i) and the inductive hypothesis, we have

$$|a_m \partial^n \lambda(t)^m|_{t=0} \leq |k! \varpi_p^{n-k}|$$

for $m < n$. Since $\partial^n \lambda(t)^n|_{t=0} = n!$ and $\partial^n \lambda(t)^m|_{t=0} = 0$ for $n < m$, the assertion is also true for n .

Now we consider a general parameter s . Then the logarithm and the exponential for \mathcal{G} with parameter s are of the form $\lambda(\phi(s))$ and $\psi(\exp_{\mathcal{G}}(s))$ for some $\phi(s), \psi(s) \in s\mathcal{O}_K[[s]]^\times$. We put $\lambda(t)^k = \sum_{n=k}^{\infty} c_n^{(k)} t^n$ and $\lambda(\phi(s))^k = \sum d_n^{(k)} s^n$. Then we have shown that $|c_n^{(k)}| \leq |k! \varpi_p^{n-k} / n!|$. Since $d_n^{(k)}$ is a linear sum of $c_l^{(k)}$ ($k \leq l \leq n$) with integral coefficients, we have

$$\left| \frac{\varpi_p^k d_n^{(k)}}{k!} \right| \leq \max_{k \leq l \leq n} \left\{ \left| c_l^{(k)} \frac{\varpi_p^k}{k!} \right| \right\} \leq \max_{k \leq l \leq n} \left\{ \left| \frac{\varpi_p^l}{l!} \right| \right\} = \rho[k, n]^{-1}.$$

Hence we have the inequality for the logarithm. The inequality for the exponential is straightforward. □

LEMMA 3.6.

- i) Suppose that $f(t) \in \mathcal{O}_K[[t]]$ satisfies $f(t \oplus t_N) = f(t)$ for all $t_N \in \mathcal{G}[\pi^N]$. Then there exists a power series $g(t) \in \mathcal{O}_K[[t]]$ such that $f(t) = g([\pi^N]t)$.
- ii) There exists an integral power series $g_k(t) \in \mathcal{O}_K[[t]]$ such that

$$\pi^{-N} \sum_{t_N \in \mathcal{G}[\pi^N]} (t \oplus t_N)^k = g_k([\pi^N]t).$$

Proof. — See [5], Chapter III. □

We put

$$F(t, X) = \prod_{t_N \in \mathcal{G}[\pi^N]} (1 - (t \oplus t_N)X) = 1 + \alpha_1(t)X + \dots + \alpha_{q^N}(t)X^{q^N}.$$

For $\partial_X = \partial/\partial X$, we consider the power series

$$(3.1) \quad \frac{\pi^{-N} \partial_X F(t, X)}{F(t, X)} = - \sum_{k=0}^{\infty} \left(\pi^{-N} \sum_{t_N \in \mathcal{G}[\pi^N]} (t \oplus t_N)^{k+1} \right) X^k.$$

By Lemma 3.6 and the above formula, $\pi^{-N} \partial_X F(t, X) \in \mathcal{O}_K[[t]][X]$.

PROPOSITION 3.7. — *Let k, n be non-negative integers and N a natural number. Then we have*

$$(3.2) \quad \left| \pi^{-N} \sum_{t_N \in \mathcal{G}[\pi^N]} \partial_{\mathcal{G}}^n (t \oplus t_N)^k \Big|_{t=0} \right| \leq \left| \pi^{Nn+k_0(1-\frac{1}{q})} \varpi_p^n \right| \bar{\rho} \left(\left[\frac{k}{q^N} \right] \right) \bar{\rho}(0),$$

where $k_0 = \max\{[k/q^N] - n, 0\}$. We also have

$$(3.3) \quad \left| \pi^{-N} \sum_{t_N \in \mathcal{G}[\pi^N]} \partial_{\mathcal{G}}^n (t \oplus t_N)^k \Big|_{t=0} \right| \leq |\pi^{Nn} \varpi_p^n| \bar{\rho}[0, n].$$

Moreover, if $e \leq p - 1$, we have

$$(3.4) \quad \left| \pi^{-N} \sum_{t_N \in \mathcal{G}[\pi^N]} \partial_{\mathcal{G}}^n (t \oplus t_N)^k \Big|_{t=0} \right| \leq |\pi^{Nn} \varpi_p^n| \bar{\rho} \left(\left[\frac{k}{q^N} \right] \right).$$

Proof. — We put $G(t, X) = F(0, X) - F(t, X)$, then $G(0, X) = G(t, 0) = 0$. We have

$$\frac{1}{F(t, X)} = \frac{1}{F(0, X) - G(t, X)} = \sum_{l=0}^{\infty} \frac{G(t, X)^l}{F(0, X)^{l+1}} \in \mathcal{O}_K[[t, X]].$$

Since $G(0, X) = 0$ and $G(t, X)$ is invariant for the translation $t \mapsto t_N$, it is of the form

$$(3.5) \quad G(t, X) = ([\pi^N]t)H([\pi^N]t, X)$$

for some element H in $\mathcal{O}_K[[t]][X]$. Since $F(0, X) \equiv 1 \pmod{\pi}$, the power series $F(0, X)^{-l-1}$ is equal to

$$\sum_{m=0}^{\infty} \binom{-l-1}{m} (F(0, X) - 1)^m = \sum_{m=0}^{\infty} \binom{l+m}{m} \pi^m \left(\frac{1 - F(0, X)}{\pi} \right)^m.$$

Hence we have

$$(3.6) \quad \begin{aligned} \frac{\pi^{-N} \partial_X F(t, X)}{F(t, X)} &= \sum_{l=0}^{\infty} \pi^{-N} \partial_X F(t, X) \cdot G(t, X)^l \cdot F(0, X)^{-l-1} \\ &= \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \binom{l+m}{m} \pi^m (\pi^{-N} \partial_X F(t, X)) G(t, X)^l \left(\frac{1 - F(0, X)}{\pi} \right)^m. \end{aligned}$$

To show the assertion for $k + 1$, we look the coefficient of X^k in the last term of (3.6). We consider the coefficients of the terms X^a, X^b and X^c with $a + b + c = k$ of $\pi^{-N} \partial_X F(t, X), G(t, X)^l$ and $(1 - F(0, X))^m \pi^{-m}$ respectively. Since $\deg \partial_X F(t, X) = q^N - 1, \deg G(t, X) = q^N$ and $\deg(1 -$

$F(0, X) = q^N - 1$ as polynomials for X , we have $a \leq q^N - 1, b \leq lq^N$ and $c \leq m(q^N - 1)$. Then by (3.5), the product of these coefficients is an integral linear combination of the terms of the form

$$\binom{l+m}{m} \pi^m G_l([\pi^N]t)$$

where $G_l(t)$ is a power series in $t^l \mathcal{O}_K[[t]]$ and l, m satisfies

$$(3.7) \quad a + lq^N + m(q^N - 1) \geq a + b + c = k.$$

We estimate the absolute value of

$$(3.8) \quad \binom{l+m}{m} \pi^m \partial_{\mathcal{G}}^n G_l([\pi^N]t) |_{t=0}.$$

By Proposition 3.5, we have

$$|\partial_{\mathcal{G}}^n([\pi^N]t)^d |_{t=0}| = \left| \pi^{Nn} \frac{d^n}{dz^n} \exp_{\mathcal{G}}^d(z) |_{z=0} \right| \leq |\pi^{Nn} \varpi_p^n| \bar{\rho}[d, n].$$

Therefore, we have

$$|\partial_{\mathcal{G}}^n G_l([\pi^N]t) |_{t=0}| \leq |\pi^{Nn} \varpi_p^n| \bar{\rho}[l, n].$$

Hence we have (3.3). If $n < l$, then (3.8) is zero and there is nothing to prove. We assume that $n \geq l$. We let $l' \geq l$ be such that $\bar{\rho}(l) = |l'|/\varpi_p^{l'}$. Then

$$(3.9) \quad \left| \binom{l+m}{m} \pi^m \partial_{\mathcal{G}}^n G_l([\pi^N]t) |_{t=0} \right| \leq \left| \binom{l+m}{m} \pi^{m+Nn} \varpi_p^n \right| \bar{\rho}[l, n]$$

$$(3.10) \quad \leq \left| \pi^{Nn} \varpi_p^n \frac{(l+m)! (l'-l)! (l')}{\varpi_p^{l+m} \varpi_p^{l'-l} \binom{l'}{l}} \frac{\varpi_p^m \pi^m}{m!} \right|.$$

First we consider the case $a \leq q^N - 2$ or $m \neq 0$. Then by (3.7) we have

$$l+m \geq \left\lfloor \frac{k+1}{q^N} \right\rfloor.$$

In particular, $m \geq [(k+1)/q^N] - n$ and the value (3.10) is less than or equal to

$$|\pi^{Nn+k_0(1-\frac{1}{q-1})} \varpi_p^n| \bar{\rho} \left(\left\lfloor \frac{k+1}{q^N} \right\rfloor \right) \bar{\rho}(0)$$

where $k_0 = \max\{[(k+1)/q^N] - n, 0\}$. Hence in this case we have (3.2). Suppose that $e \leq p-1$. If $l' < l+m$, then $|\varpi_p^m| < |\varpi_p^{l'-l}|$ and hence the value (3.10) is less than $|\pi^{Nn} \varpi_p^n| \bar{\rho} \left(\left\lfloor \frac{k+1}{q^N} \right\rfloor \right)$. If $l' \geq l+m$, then

$$\bar{\rho}(l) = \left| \frac{l!}{\varpi_p^{l'}} \right| \leq \bar{\rho}(l+m) \leq \bar{\rho} \left(\left\lfloor \frac{k+1}{q^N} \right\rfloor \right).$$

Hence the value (3.9) is also less than or equal to $|\pi^{m+Nn}\varpi_p^n|\bar{\rho}\left(\left[\frac{k+1}{q^N}\right]\right)$. Hence in this case we have (3.4).

Finally we consider the case when $a = q^N - 1$ and $m = 0$. Then the coefficient of $\pi^{-N}\partial_X F(t, X)$ of degree a is $(q/\pi)^N\alpha_{q^N}(t)$, which is divisible by $[\pi^N]t$. Hence in this case the product of the coefficient of X^a in $\pi^{-N}\partial_X F(t, X)$, the coefficient of X^b in $G(t, X)^l$ and the coefficient of X^c in $(1 - F(0, X))^m\pi^{-m}$ is an integral linear combination of terms in the form $G_{l+1}([\pi^N]t)$ for some $G_{l+1}(t) \in t^{l+1}\mathcal{O}_K[[t]]$. In this case l satisfies $l + 1 \geq [(k + 1)/q^N]$. Therefore

$$|\partial_{\mathcal{G}}^n G_{l+1}([\pi^N]t)|_{t=0} \leq |\pi^{Nn}\varpi_p^n|\bar{\rho}[l + 1, n] \leq |\pi^{Nn}\varpi_p^n|\bar{\rho}\left(\left[\frac{k + 1}{q^N}\right]\right).$$

If $n < l + 1$, then (3.8) is zero and there is nothing to prove. We assume that $n \geq l + 1$. In particular, by (3.7) we have $n \geq [(k + 1)/q^N]$, and hence $k_0 = \max\{[(k + 1)/q^N] - n, 0\} = 0$. Therefore we have (3.2) and (3.4). \square

4. Integral structures on *p*-adic Fourier theory

In this section, we give an explicit construction of Schneider-Teitelbaum’s *p*-adic distribution associated to a rigid analytic function on the open unit disc.

Let $\varphi(t)$ be a rigid analytic function on the open unit disc. We will construct a distribution μ_φ on \mathcal{O}_K such that

$$\int_{\mathcal{O}_K} \exp(x\varpi_p\lambda(t))d\mu_\varphi = \varphi(t).$$

If we were able to first prove a Mahler like expansion for *K*-analytic functions as in the case of $K = \mathbb{Q}_p$, then it would be possible to define the integral by (2.2). However, as in [8], we will first define the integral then use this integral to prove the existence of the Mahler like expansion for *K*-analytic functions. Our construction of the integral is different from that of [8] in that we investigate directly the explicit power series corresponding to the moments of the integral, instead of formally reducing to the case of \mathbb{Z}_p .

We fix a Lubin-Tate formal group \mathcal{G} associated to π , and denote its addition by \oplus . For $a \in \mathcal{O}_K$ and a natural number N , we let

$$\int_{a+\pi^N\mathcal{O}_K} (x - a)^n d\mu_\varphi := \frac{1}{q^N\varpi_p^n} \left(\partial_{\mathcal{G}}^n \sum_{t_N \in \mathcal{G}[\pi^N]} \varphi_a(t \oplus t_N) \right) \Big|_{t=0}$$

where

$$\varphi_a(t) := \exp(-a\varpi_p\lambda(t))\varphi(t).$$

We put $\varphi(t) = \sum_{k=0}^\infty c_k t^k$ and $\varphi_a(t) = \sum_{k=0}^\infty c_k^{(a)} t^k$. Then by Proposition 3.7, we have

$$\begin{aligned} (4.1) \quad \left| \int_{a+\pi^N\mathcal{O}_K} (x-a)^n d\mu_\varphi \right| &\leq \bar{\rho}(0) \left| \frac{\pi}{q} \right|^N |\pi|^{Nn} \sup_k \{ |c_k^{(a)}| \bar{\rho} \left(\left[\frac{k}{q^N} \right] \right) \} \\ &\leq \bar{\rho}(0) \left| \frac{\pi}{q} \right|^N |\pi|^{Nn} \sup_k \{ |c_k| \bar{\rho} \left(\left[\frac{k}{q^N} \right] \right) \}. \end{aligned}$$

Here for the last estimate, we used the facts that $c_k^{(a)}$ is an integral linear combination of c_0, \dots, c_k and the function $\bar{\rho}(m)$ for m is decreasing.

We define the distribution μ_φ on $LA_N(\mathcal{O}_K, \mathbb{C}_p)$ as follows. For an element f of $LA_N(\mathcal{O}_K, \mathbb{C}_p)$, suppose f is of the form $\sum_{n=0}^\infty a_n(x-a)^n$ such that $a_n\pi^{nN} \rightarrow 0$ if $n \rightarrow \infty$ on $a + \pi^N\mathcal{O}_K$. Then we define the integral of f on $a + \pi^N\mathcal{O}_K$ by

$$(4.2) \quad \int_{a+\pi^N\mathcal{O}_K} f(x) d\mu_\varphi := \sum_{n=0}^\infty a_n \int_{a+\pi^N\mathcal{O}_K} (x-a)^n d\mu_\varphi.$$

We define

$$(4.3) \quad \int_{\mathcal{O}_K} f(x) d\mu_\varphi = \sum_{a \bmod \pi^N} \int_{a+\pi^N\mathcal{O}_K} f(x) d\mu_\varphi.$$

We have to show the well-definedness of the integral.

PROPOSITION 4.1.

- i) The integral (4.2) converges and does not depend on the choice of the representative of $a \bmod \pi^N$. The integral (4.3) does not depend on the choice of N . Hence μ_φ gives a well-defined element of $D(\mathcal{O}_K, \mathbb{C}_p)$.
- ii) For a polynomial $f(x)$, we have

$$\int_{\mathcal{O}_K} f(x) d\mu_\varphi = f(\varpi_p^{-1}\partial_{\mathcal{G}})\varphi(t)|_{t=0}.$$

Proof. — Since $\bar{\rho}([k/q^N]) \leq Ckp^{-\frac{k}{eqN(q-1)}}$ for some constant C which depends only on e, q and N , the value $\sup_k \{ |c_k| \bar{\rho} \left(\left[\frac{k}{q^N} \right] \right) \}$ is finite. Hence the convergence follows from (4.1). We show that the integral (4.2) depends only on the class of a modulo π^N . Since the integral is convergent, we may

assume that f is a monomial $(x - a)^n$. For a' such that $a' \equiv a \pmod{\pi^N}$, we put $b = a' - a$. Since

$$(x - a)^n|_{a'+\pi^N \mathcal{O}_K} = \sum_{l=0}^n \binom{n}{l} b^{n-l} (x - a')^l|_{a'+\pi^N \mathcal{O}_K},$$

it suffices to show that

$$\int_{a+\pi^N \mathcal{O}_K} (x - a)^n d\mu_\varphi = \sum_{l=0}^n \binom{n}{l} b^{n-l} \int_{a'+\pi^N \mathcal{O}_K} (x - a')^l d\mu_\varphi.$$

This follows from

$$\begin{aligned} \varpi_p^{-n} \partial_{\mathcal{G}}^n \varphi_a(t \oplus t_m) &= \varpi_p^{-n} \partial_{\mathcal{G}}^n (\exp(b\varpi_p \lambda(t)) \varphi_{a'}(t \oplus t_N)) \\ &= \exp(b\varpi_p \lambda(t)) \sum_{l=0}^n \binom{n}{l} b^{n-l} \varpi_p^{-l} \partial_{\mathcal{G}}^l (\varphi_{a'}(t \oplus t_m)). \end{aligned}$$

Now we show that the integral (4.3) does not depend on N . It is sufficient to show the distribution relation

$$(4.4) \quad \int_{a+\pi^N \mathcal{O}_K} f(x) d\mu_\varphi = \sum_{b \equiv a \pmod{\pi^N}} \int_{b+\pi^{N+1} \mathcal{O}_K} f(x) d\mu_\varphi$$

where the sum runs over a representative b of \mathcal{O}_K/π^{N+1} such that $b \equiv a \pmod{\pi^N}$. To show this, replacing φ by φ_a , we may assume that $a = 0$ and $f(x) = x^n$. Then

$$\begin{aligned} &q^{N+1} \varpi_p^n \sum_{b \equiv 0 \pmod{\pi^N}} \int_{b+\pi^{N+1} \mathcal{O}_K} x^n d\mu_\varphi \\ &= \sum_{b \equiv 0 \pmod{\pi^N}} \sum_{i=0}^k \binom{n}{k} b^{n-k} \left(\varpi_p^{n-k} \partial_{\mathcal{G}}^k \sum_{t_{N+1} \in \mathcal{G}[\pi^{N+1}]} \varphi_b(t \oplus t_{N+1}) \right) \Big|_{t=0} \\ &= \sum_{b \equiv 0 \pmod{\pi^N}} \left(\partial_{\mathcal{G}}^n \sum_{t_{N+1} \in \mathcal{G}[\pi^{N+1}]} \exp(b\varpi_p \lambda(t)) \varphi_b(t \oplus t_{N+1}) \right) \Big|_{t=0} \\ &= \sum_{t_{N+1} \in \mathcal{G}[\pi^{N+1}]} \left(\sum_{b \equiv 0 \pmod{\pi^N}} \exp(-b\varpi_p \lambda(t))|_{t=t_{N+1}} \right) \partial_{\mathcal{G}}^n \varphi(t \oplus t_{N+1}) \Big|_{t=0} \\ &= q \left(\partial_{\mathcal{G}}^n \sum_{t_N \in \mathcal{G}[\pi^N]} \varphi(t \oplus t_N) \right) \Big|_{t=0} = q^{N+1} \varpi_p^n \int_{\pi^N \mathcal{O}_K} x^n d\mu_\varphi. \end{aligned}$$

The above calculation is also true when $a = N = 0$, and hence we have

$$\varpi_p^n \sum_{b \in \mathcal{O}_K/\pi} \int_{b+\pi \mathcal{O}_K} x^n d\mu_\varphi = \partial_{\mathcal{G}}^n \varphi(t)|_{t=0}.$$

Assertion ii) follows from this equality. □

For $\varphi(t) = \sum_{k=0}^{\infty} c_k t^k \in R^{\text{rig}}$, we define $\|\varphi\|_N$ by

$$(4.5) \quad \|\varphi\|_N := \max_k \left\{ |c_k| \bar{\rho} \left(\left[\frac{k}{q^N} \right] \right) \right\}.$$

Since $\bar{\rho} \left(\left[\frac{k}{q^N} \right] \right) \sim p^{-kr}$ where $r = 1/eq^N(q-1)$, the value $\|\varphi\|_N$ is approximately,

$$\|\varphi\|_{\overline{\mathbf{B}}(p^{-r})} = \max_{x \in \overline{\mathbf{B}}(p^{-r})} \{ |\varphi(x)| \}$$

where $\overline{\mathbf{B}}(p^{-r}) \subset \mathbb{C}_p$ is the closed disc with radius p^{-r} at origin.

LEMMA 4.2. — *For an element $a \in \mathcal{O}_K$, let $\varphi_a(t) = \exp(-a\varpi_p \lambda(t))\varphi(t)$ as before. Then $\|\varphi_a\|_N = \|\varphi\|_N$.*

Proof. — It suffices to show that $\|\varphi_a\|_N \leq \|\varphi\|_N$. This follows from the same argument showing (4.1). □

Then Proposition 3.7 may be rewritten as follows, which is a precise version of Theorem 1.1 of the introduction.

THEOREM 4.3.

i) *Suppose that the function $f \in LA_N(\mathcal{O}_K, \mathbb{C}_p)$ is given by a polynomial of degree d on $a + \pi^N \mathcal{O}_K$ for $a \in \mathcal{O}_K$. For $\varphi_k(t) = t^k$, we have*

$$(4.6) \quad \left| \int_{a+\pi^N \mathcal{O}_K} f(x) d\mu_{\varphi_k} \right| \leq \bar{\rho}[0, d] \left| \frac{\pi}{q} \right|^N \|f\|_{a,N}.$$

We also have

$$(4.7) \quad \left| \int_{a+\pi^N \mathcal{O}_K} f(x) d\mu_{\varphi_k} \right| \leq \bar{\rho}(0) \left| \frac{\pi^{k_0(1-\frac{1}{q-1})+N}}{q^N} \right| \|f\|_{a,N} \bar{\rho} \left(\left[\frac{k}{q^N} \right] \right)$$

where $k_0 = \max\{[k/q^N] - d, 0\}$. Moreover, if $e \leq p - 1$, then we have

$$(4.8) \quad \left| \int_{a+\pi^N \mathcal{O}_K} f(x) d\mu_{\varphi_k} \right| \leq \left| \frac{\pi}{q} \right|^N \|f\|_{a,N} \bar{\rho} \left(\left[\frac{k}{q^N} \right] \right).$$

ii) *We have*

$$\left| \int_{a+\pi^N \mathcal{O}_K} f(x) d\mu_{\varphi} \right| \leq \bar{\rho}(0) \left| \frac{\pi}{q} \right|^N \|f\|_{a,N} \|\varphi\|_N.$$

Moreover, if $e \leq p - 1$, then

$$\left| \int_{a+\pi^N \mathcal{O}_K} f(x) d\mu_{\varphi} \right| \leq \left| \frac{\pi}{q} \right|^N \|f\|_{a,N} \|\varphi\|_N.$$

COROLLARY 4.4. — We have

$$(4.9) \quad \left| \int_{a+\pi^N \mathcal{O}_K} f(x) d\mu_\varphi \right| \leq p^{\frac{p}{p-1} + \frac{1}{e(q-1)}} \bar{\rho}(0) |\pi|^N \|f\|_{a,N} \|\varphi\|_{\mathbf{B}'(p^{-r})}$$

where $r = 1/eq^N(q-1)$ and

$$\|\varphi\|_{\mathbf{B}'(p^{-r})} := \max_k \{ |c_k| k p^{-kr} \}.$$

Moreover, if $e \leq p-1$, then

$$(4.10) \quad \left| \int_{a+\pi^N \mathcal{O}_K} f(x) d\mu_\varphi \right| \leq p^{\frac{p}{p-1} + \frac{1}{e(q-1)}} |\pi|^N \|f\|_{a,N} \|\varphi\|_{\mathbf{B}'(p^{-r})}.$$

Proof. — The formula follows from

$$\bar{\rho} \left(\left[\frac{k}{q^N} \right] \right) \leq kq^{-N} p^{\frac{p}{p-1} + \frac{1}{e(q-1)} - \frac{k}{eq^N(q-1)}}.$$

□

As before, we define polynomials P_n by

$$\exp(x\lambda(T)) = \sum_{n=0}^{\infty} P_n(x) T^n.$$

Then by formal computation, we have

$$P_k(\partial_{\mathcal{G}})\varphi(t)|_{t=0} = \frac{1}{k!} \partial^k \varphi(t)|_{t=0}$$

where $\partial = d/dt$ (for example, formula 6 of Lemma 4.2 of [8]). We let $\varphi_n(t) = t^n$ and μ_{φ_n} the distribution associated to $\varphi_n(t)$. Then by Proposition 4.1 ii) we have

$$\int_{\mathcal{O}_K} P_k(x\varpi_p) d\mu_{\varphi_n} = \sum_{n=0}^{\infty} P_k(\partial_{\mathcal{G}})\varphi_n(t)|_{t=0} = \begin{cases} 1 & (k = n) \\ 0 & (k \neq n). \end{cases}$$

Hence if $\varphi(t) = \sum_{k=0}^{\infty} c_k t^k$, then

$$\int_{\mathcal{O}_K} P_k(x\varpi_p) d\mu_\varphi = c_k.$$

Equivalently,

$$\varphi(t) = \int_{\mathcal{O}_K} \exp(x\varpi_p \lambda(t)) d\mu_\varphi.$$

PROPOSITION 4.5. — For $N \geq 1$, we have

$$\left| \frac{q}{\pi} \right|^N \bar{\rho} \left(\left[\frac{n}{q^N} \right] \right)^{-1} c^{-1} \leq \|P_n(x\varpi_p)\|_N \leq \underline{\rho} \left(\left[\frac{n}{q^N} \right] \right)^{-1}$$

where $c = 1$ if $e \leq p-1$ and $c = \bar{\rho}(0)$, otherwise.

Proof. — We have

$$1 = \left| \int_{\mathcal{O}_K} P_n(x\varpi_p) d\mu_{\varphi_n} \right| \leq \max_a \left\{ \left| \int_{a+\pi^N \mathcal{O}_K} P_n(x\varpi_p) d\mu_{\varphi_n} \right| \right\} \\ \leq \left| \frac{\pi}{q} \right|^N \|P_n(x\varpi_p)\|_N \bar{\rho} \left(\left[\frac{n}{q^N} \right] \right) \bar{\rho}(0).$$

Similarly, if $e \leq p - 1$, then by using (4.8), we obtain the lower estimate.

For the upper estimate, we put $P_n(x\pi^N \varpi_p) = \sum_{k=1}^n a_k^{(n)} x^k$ for $n \geq 1$. By the definition of P_n , the value $a_k^{(n)}$ is the coefficient of t^n of $\varpi_p^k \lambda([\pi^N]t)^k/k!$. Since $\underline{\rho}(k)$ is decreasing with k , we may assume that $\lambda(t) = \sum_{l=0}^\infty t^{q^l}/\pi^l$. Since $[\pi^N]t \equiv t^{q^N} \pmod{\pi}$, we have

$$\lambda([\pi^N]t) \equiv \lambda(t^{q^N}) + \pi t f(t)$$

for some $f(t) \in \mathcal{O}_{\mathbb{C}_p}[[t]]$. (cf. [6, Lemma 4].) Hence we have

$$\frac{\varpi_p^k \lambda([\pi^N]t)^k}{k!} = \sum_{i=0}^k t^i f(t)^i \frac{\varpi_p^i \pi^i \varpi_p^{k-i} \lambda(t^{q^N})^{k-i}}{i! (k-i)!}.$$

Therefore by Proposition 3.5 we have

$$|a_k^{(n)}| \leq \underline{\rho} \left(\left[\frac{n}{q^N} \right] \right)^{-1}.$$

Hence we have $\|P_n(x\varpi_p)\|_{0,N} \leq \underline{\rho} \left(\left[\frac{n}{q^N} \right] \right)^{-1}$. Then by the formula before Lemma 4.4 of [8], for $a \in \mathcal{O}_K$, we have

$$\|P_n(x\varpi_p)\|_{a,N} \leq \max_{0 \leq i \leq n} \|P_i(x\varpi_p)\|_{0,N} \leq \underline{\rho} \left(\left[\frac{n}{q^N} \right] \right)^{-1}.$$

□

Now we prove that our definition of the distribution coincides with that of Schneider-Teitelbaum. Namely, we will prove that the distribution has the characterization property (2.3).

THEOREM 4.6. — *Let μ_φ be the distribution associated to a rigid analytic function $\varphi(t)$ on the open unit disc. Then*

$$\varphi(t) = \int_{\mathcal{O}_K} \exp(x\varpi_p \lambda(t)) d\mu_\varphi.$$

Conversely, for every distribution μ , there exists a unique rigid analytic function φ such that $\mu = \mu_\varphi$. Then φ is the Fourier transform of μ , and we have $F_{\mu_\varphi} = \varphi$. In particular, we have an isomorphism of algebras,

$$D(\mathcal{O}_K, \mathbb{C}_p) \cong R^{\text{rig}}.$$

Proof. — We have already shown the first assertion. For a given μ , we put

$$c_k := \int_{\mathcal{O}_K} P_k(x\varpi_p) d\mu.$$

Since the distribution is a continuous linear operator on the *p*-adic Banach space $LA_N(\mathcal{O}_K, \mathbb{C}_p)$ for every natural number *N*, there exists a positive constant *C* depending only on μ and *N* such that

$$|c_k| = \left| \int_{\mathcal{O}_K} P_k(x\varpi_p) d\mu \right| \leq C \|P_k(x\varpi_p)\|_N \leq Cp^{-\frac{1}{p-1} + \frac{k}{eq^N(q-1)}}$$

where for the last inequality, we used Proposition 3.1 and Proposition 4.5. Hence for any $0 \leq r < 1$, if we choose sufficiently large *N*, we have $|c_k|r^k \rightarrow 0$ when $k \rightarrow \infty$. Hence $\varphi(t) = \sum_{k=0}^\infty c_k t^k$ is a rigid analytic function on the open unit disc. Then by construction

$$\varphi(t) = \int_{\mathcal{O}_K} \exp(x\varpi_p \lambda(t)) d\mu.$$

Since the function $(x - a)|_{a+\pi^N \mathcal{O}_K}$ is given by

$$\frac{1}{q^N \varpi_p^n} \partial_{\mathcal{G}}^n \left(\sum_{t_N \in \mathcal{G}[\pi^N]} \exp((x - a)\varpi_p \lambda(t))|_{t=t \oplus t_N} \right) |_{t=0},$$

we have

$$\begin{aligned} \int_{a+\pi^N \mathcal{O}_K} (x - a)^n d\mu &= \frac{1}{q^N \varpi_p^n} \partial_{\mathcal{G}}^n \sum_{t_N \in \mathcal{G}[\pi^N]} \varphi_a(t \oplus t_N)|_{t=0} \\ &= \int_{a+\pi^N \mathcal{O}_K} (x - a)^n d\mu_\varphi. \end{aligned}$$

Since $\pi^{-nN}(x - a)^n|_{a+\pi^N \mathcal{O}_K}$ for $a \in \mathcal{O}_K$ and $n = 0, 1, \dots$ are topological generators of $LA_n(\mathcal{O}_K, \mathbb{C}_p)$, we have

$$\int_{\mathcal{O}_K} f(x) d\mu = \int_{\mathcal{O}_K} f(x) d\mu_\varphi$$

for all $f \in LA_N(\mathcal{O}_K, \mathbb{C}_p)$. Hence $\mu = \mu_\varphi$. □

Now we prove Theorem 4.7.

THEOREM 4.7.

i) *The series $\sum_{n=0}^\infty a_n P_n(x\varpi_p)$ converges to an element of $LA_N(\mathcal{O}_K, \mathbb{C}_p)_0$ for a_n satisfying*

$$|a_n| \leq \rho \left(\left[\frac{n}{q^N} \right] \right), \quad \lim_{n \rightarrow 0} |a_n| / \rho \left(\left[\frac{n}{q^N} \right] \right) = 0.$$

ii) If $f(x) \in LA_N(\mathcal{O}_K, \mathbb{C}_p)_0$, then it has an expansion

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x\varpi_p)$$

of the form

$$|a_n| \leq c \left| \frac{\pi}{q} \right|^N \bar{\rho} \left(\left[\frac{n}{q^N} \right] \right), \quad \lim_{n \rightarrow 0} |a_n| / \bar{\rho} \left(\left[\frac{n}{q^N} \right] \right) = 0,$$

where $c = 1$ if $e \leq p - 1$, and $c = \bar{\rho}(0)$, otherwise.

Proof. — i) follows from Proposition 4.5. For ii), we proceed as in the proof of Theorem 4.7 of [8] except the estimate of the Mahler coefficients. We put

$$a_n := \int_{\mathcal{O}_K} f(x) d\mu_{\varphi_n}.$$

Then by Theorem 4.3, we have

$$|a_n| = \left| \int_{\mathcal{O}_K} f(x) d\mu_{\varphi_n} \right| \leq c \left| \frac{\pi}{q} \right|^N \bar{\rho} \left(\left[\frac{n}{q^N} \right] \right).$$

We next prove the limit in ii). We may assume that $f(x) = \sum_{i=0}^{\infty} c_i(x-a)^i$ on $a + \pi^N \mathcal{O}_K$ and $f(x) = 0$ outside of $a + \pi^N \mathcal{O}_K$. For a given $\epsilon > 0$, we can take N_0 so that

$$\left\| \sum_{i=N_0}^{\infty} c_i(x-a)^i \right\|_{a,N} < \epsilon.$$

Hence by (4.7), we have

$$\left| \int_{a+\pi^N \mathcal{O}_K} \sum_{i=N_0}^{\infty} c_i(x-a)^i d\mu_{\varphi_n} \right| \leq \epsilon C_1 \bar{\rho} \left(\left[\frac{n}{q^N} \right] \right)$$

where C_1 is a positive constant independent of n . On the other hand, also by 4.7, we have

$$\left| \int_{a+\pi^N \mathcal{O}_K} \sum_{i=0}^{N_0} c_i(x-a)^i d\mu_{\varphi_n} \right| \leq C_2 p^{-\frac{n_0}{e} (1 - \frac{1}{q-1})} \bar{\rho} \left(\left[\frac{n}{q^N} \right] \right)$$

where $n_0 = \max\{[n/q^N] - N_0, 0\}$ and C_2 is a positive constant independent of n . Hence we have

$$\left| \int_{a+\pi^N \mathcal{O}_K} f(x) d\mu_{\varphi_n} \right| \leq \epsilon C_1 \bar{\rho} \left(\left[\frac{n}{q^N} \right] \right)$$

for sufficiently large n . Hence we have $|a_n|/\bar{\rho} \left(\left[\frac{n}{q^N} \right] \right) \rightarrow 0$ when $n \rightarrow \infty$. Then by i), the series $\sum_{k=0}^{\infty} a_n P_n(x\varpi_p)$ converges to a function in $LA_N(\mathcal{O}_K, \mathbb{C}_p)$. We put

$$g(x) = f(x) - \sum_{k=0}^{\infty} a_n P_n(x\varpi_p).$$

Then we have $\int_{\mathcal{O}_K} g(x) d\mu_{\varphi_n} = 0$ for all n , and hence $\int_{\mathcal{O}_K} g(x) d\mu = 0$ for all distribution μ . Considering the Dirac distribution $\delta_a : h \mapsto h(a)$, we have $g(a) = 0$ for any a . Hence $f(x) = \sum_{n=0}^{\infty} a_n P_n(x\varpi_p)$. \square

COROLLARY 4.8. — *Suppose*

$$e_{N,n} = \underline{\gamma} \left(\left[\frac{n}{q^N} \right] \right) P_n(x\varpi_p), \quad (n = 0, 1, \dots),$$

where $\underline{\gamma}(u)$ is an element in \mathbb{C}_p satisfying $\underline{\rho}(u) = |\underline{\gamma}(u)|$. If L_N is the $\mathcal{O}_{\mathbb{C}_p}$ -module topologically generated by $e_{N,n}$, then

$$\bar{\rho}(0)^{-2} \left| \frac{q}{\pi} \right|^N LA_N(\mathcal{O}_K, \mathbb{C}_p)_0 \subset L_N \subset LA_N(\mathcal{O}_K, \mathbb{C}_p)_0.$$

In particular, the functions e_n form a topological basis of the p -adic Banach space $LA_N(\mathcal{O}_K, \mathbb{C}_p)$. Moreover, if $e \leq p - 1$, then

$$\left| \frac{q}{\pi} \right|^{N+1} LA_N(\mathcal{O}_K, \mathbb{C}_p)_0 \subset L_N \subset LA_N(\mathcal{O}_K, \mathbb{C}_p)_0.$$

In addition, if $\mathcal{O}_K = \mathbb{Z}_p$, we recover Amice’s result, namely that

$$\left[\frac{n}{p^N} \right]! \binom{x}{n}$$

for $n = 0, 1, \dots$ form a topological basis of $LA_N(\mathbb{Z}_p, \mathbb{C}_p)_0$.

5. Relations to Katz’s and Chellali’s results.

As an application, we reprove Katz’s and Chellali’s results ([4], [7]) by using our results.

First we recall results of Katz [7] and Chellali [4]. Let E be an elliptic curve with complex multiplication by the ring of integer $\mathcal{O}_{\mathcal{K}}$ of an imaginary quadratic field \mathcal{K} . For simplicity, we assume that E is defined over \mathcal{K} , and fix a Weierstrass model

$$y^2 = 4x^3 - g_2x - g_3, \quad g_2, g_3 \in \mathcal{O}_{\mathcal{K}}$$

of E/\mathcal{K} . Let p be an odd prime. We assume that p is inert in \mathcal{K} and does not divide the discriminant of the above Weierstrass model, or equivalently, E

has good *supersingular* reduction at p . Then the Bernoulli-Hurwitz number $BH(n)$ is defined by

$$\wp(z) = \frac{1}{z^2} + \sum_{n \geq 2} \frac{BH(n+2)}{n+2} \frac{z^n}{n!},$$

where $\wp(z)$ is the Weierstrass \wp -function for the model. Let ϵ be a root of unity in \mathcal{O}_K such that the multiplication by $-\epsilon p$ gives the Frobenius $(x, y) \mapsto (x^{p^2}, y^{p^2})$ of $E \pmod p$. Let γ be a unit in the Witt ring $W(\overline{\mathbb{F}}_p)$ such that

$$\gamma^{p^2-1} = -\epsilon^{-1} \frac{p^2!}{p^{p+1}(p^2-1)}.$$

For a fixed $b \in \mathcal{O}_K$ prime to p , we put

$$L(n) = \frac{(1-b^{n+2})(1-p^n)}{\gamma^n p^{\lfloor np/(p^2-1) \rfloor}} \frac{BH(n+2)}{n+2}.$$

THEOREM 5.1 (Katz [7]). — *The number $L(n)$ is integral. Let l and n be non-negative integers. Then*

$$L(n + p^l(p^2 - 1)) \equiv L(n) \pmod{p^l}.$$

Later, Chellali [4] refined the congruences as follows.

THEOREM 5.2 (Chellali [4]). — *Let l and n be non-negative integers. If $n \not\equiv 0 \pmod{p^2 - 1}$, we have*

$$L(n + p^l(p^2 - 1)) \equiv L(n) \pmod{p^{l+1}}.$$

If $n \equiv 0 \pmod{p^2 - 1}$ and $n \neq 0$, put $L'(n) = L(n)/n$, then

$$L'(n + p^l(p^2 - 1)) \equiv L'(n) \pmod{p^{l+1}}.$$

In the following, let K be the unramified quadratic extension of \mathbb{Q}_p and let \mathcal{G} be the Lubin-Tate group of height $h = 2$ associated to the uniformizer $\pi = -\epsilon p$. We assume that $[\pi]T = \pi T + T^q$ for $q = p^2$ is an endomorphism of \mathcal{G} . It is known that the formal group of E at p is isomorphic to \mathcal{G} .

PROPOSITION 5.3. — *Let φ be an integral power series and let μ_φ be the corresponding distribution associated to φ .*

i) *We have*

$$\left| \int_{\mathcal{O}_K^\times} x^n d\mu_\varphi \right| \leq p.$$

ii) *If $m \equiv n \pmod{p^l(q-1)}$, then*

$$\left| \int_{\mathcal{O}_K^\times} (x^m - x^n) d\mu_\varphi \right| \leq p^{-l + \frac{p}{q-1}}.$$

iii) If $(q - 1)|n$ and $m \equiv n \pmod{p^l(q - 1)}$, then

$$\left| \int_{\mathcal{O}_K^\times} \left(\frac{x^m - 1}{m} - \frac{x^n - 1}{n} \right) d\mu_\varphi \right| \leq p^{-l-1+\frac{2p}{q-1}}.$$

Proof. — We have

$$\int_{a+\pi\mathcal{O}_K} x^n d\mu_\varphi = a^n \int_{a+\pi\mathcal{O}_K} d\mu_\varphi + \sum_{k=1}^n \int_{a+\pi\mathcal{O}_K} \binom{n}{k} (x-a)^k a^{n-k} d\mu_\varphi.$$

Then by the estimate (4.6) the absolute value of the first integral is less than or equal to p . By the estimate (4.8), the absolute value of the second integral is also less than or equal to p since $\|(x-a)\|_{a,1}\bar{\rho}(0) = 1$. We put $m - n = k(q - 1)$. Then

$$\begin{aligned} x^m - x^n &= x^n \sum_{i=1}^k \binom{k}{i} (x^{q-1} - 1)^i \\ &= kx^n(x^{q-1} - 1) + x^n \sum_{i=2}^k k \binom{k-1}{i-1} \frac{(x^{q-1} - 1)^i}{i} \\ &= k \left(c_0 + c_1(x-a) + c_2 \frac{(x-a)^2}{2} + c_3 \frac{(x-a)^3}{3} + \dots \right) \end{aligned}$$

where c_i are integers satisfying $p|c_0$. Since $\|(x-a)^i/i\|_{a,1} \leq p^{-2}$ for $i \geq 2$, the assertion ii) follows from the estimates (4.6).

For an integer s , we have

$$\begin{aligned} \frac{(x^{q-1})^s - 1}{s} &= \sum_{i=1}^\infty \frac{(\log_p x^{q-1})^i}{i!} s^{i-1} \\ &= \sum_{i=1}^\infty \sum_{n=i}^\infty c_{i,n} \frac{(x^{q-1} - 1)^n}{n!} \\ &= \sum_{i=1}^\infty \sum_{j+k \geq i}^\infty c_{i,j,k} \frac{\pi^k}{k!} \frac{(x-a)^j}{j!} s^{i-1} \end{aligned}$$

for some integers $c_{i,n}$ and $c_{i,j,k}$. If we write $m = s_1(q - 1)$ and $n = s_2(q - 1)$, then

$$\frac{(x^{q-1})^{s_1} - 1}{s_1} - \frac{(x^{q-1})^{s_2} - 1}{s_2} = \sum_{i \geq 2, j+k \geq i}^\infty c_{i,j,k} \frac{\pi^k}{k!} \frac{(x-a)^j}{j!} (s_1^{i-1} - s_2^{i-1})$$

By the estimate (4.6), the integral of $\frac{\pi^k}{k!} \frac{(x-a)^j}{j!}$ is divisible by $p^{1-\frac{2p}{q-1}}$. The assertion iii) follows from this fact. □

For $b \in \mathcal{O}_K$ prime to p , we put

$$\wp_b(z) = (1 - b^2[b]^*)\wp(z)$$

and $\phi(t) = \wp_b(z)|_{z=\lambda(t)}$. Then $\wp_b(z)$ has no pole at $z = 0$ and

$$\wp_b(z) = \sum_{n \geq 2} (1 - b^{n+2}) \frac{BH(n+2)}{n+2} \frac{z^n}{n!}.$$

It is known that $\phi(t)$ is an integral power series. Similarly, for $c \in \mathcal{O}_K$ prime to p , we put

$$\zeta_c(z) = (c - [c]^*)\zeta(z), \quad \zeta_{b,c}(z) = (1 - b[b]^*)\zeta_c(z),$$

where $\zeta(z)$ is the Weierstrass zeta function and $\psi(t) = \zeta_{b,c}(z)|_{z=\lambda(t)}$. Note that $\zeta_c(z)$ is double periodic and $\zeta_{b,c}(z)$ has no pole at $z = 0$. Then

$$\zeta_{b,c}(z) = \sum_{n \geq 3} (c - c^n)(1 - b^{n+1}) \frac{BH(n+1)}{n+1} \frac{z^n}{n!}$$

and $\psi(t)$ is an integral power series.

LEMMA 5.4.

$$\sum_{z_0 \in \frac{1}{p}\Gamma/\Gamma} \wp_b(z + z_0) = p^2 \wp_b(pz), \quad \sum_{z_0 \in \frac{1}{p}\Gamma/\Gamma} \zeta_c(z + z_0) = p\zeta_c(pz).$$

Proof. — It is known that

$$\sum_{z_0 \in \frac{1}{p}\Gamma/\Gamma} \wp(z + z_0) = p^2 \wp(pz).$$

The first formula follows from this. The above formula also shows that for a set S of representatives of $\frac{1}{p}\Gamma/\Gamma$, there exists a constant $A(S)$ such that

$$\sum_{z_0 \in S} \zeta(z + z_0) = p\zeta(pz) + A(S).$$

We take S so that $S = -S$. Then since $\zeta(z)$ is an odd function, $A(S)$ should be zero. Therefore,

$$\sum_{z_0 \in S} \zeta_c(z + z_0) = p\zeta_c(pz).$$

Since $\zeta_c(z)$ is an elliptic function, the left-hand side does not depend on the choice of S . □

PROPOSITION 5.5. — We put $B(n) = BH(n+2)/(n+2)$ if $n \geq 2$ and 0 if $n = -1, 0, 1$. For $n \geq 0$, we have

$$\varpi_p^n \int_{\mathcal{O}_K^\times} x^n d\mu_\phi = (1 - p^n)(1 - b^{n+2})B(n),$$

$$\varpi_p^n \int_{\mathcal{O}_K^\times} x^n d\mu_\psi = (1 - p^{n-1})(c - c^n)(1 - b^{n+1})B(n - 1).$$

Proof. — Since $\wp_b(z)$ and $\zeta_{b,c}(z)$ are double periodic, for $t_0 \in \mathcal{G}[p]$ we have $\psi(t \oplus t_0) = \zeta_{b,c}(z+z_0)|_{z=\lambda(t)}$ and $\phi(t \oplus t_0) = \wp_b(z+z_0)|_{z=\lambda(t)}$ where z_0 is an image of t_0 by $\mathcal{G}[p] \rightarrow E[p] \rightarrow \frac{1}{p}\Gamma/\Gamma$ (see for example, [2], Lemma 2.18). From this fact and the previous lemma, we have

$$\begin{aligned} \phi(t) - \frac{1}{q} \sum_{t_0 \in \mathcal{G}[p]} \phi(t \oplus t_0) &= (\wp_b(z) - \wp_b(pz))|_{z=\lambda(t)}, \\ \psi(t) - \frac{1}{q} \sum_{t_0 \in \mathcal{G}[p]} \psi(t \oplus t_0) &= (\zeta_{b,c}(z) - p^{-1}\zeta_{b,c}(pz))|_{z=\lambda(t)}. \end{aligned}$$

Hence

$$\begin{aligned} \varpi_p^n \int_{\mathcal{O}_K^\times} x^n d\mu_\phi &= \partial_{\mathcal{G}}^n \left(\phi(t) - \frac{1}{q} \sum_{t_0 \in \mathcal{G}[p]} \phi(t \oplus t_0) \right) \Big|_{t=0} \\ &= \partial_z (\wp_b(z) - \wp_b(pz))|_{z=0} = (1 - p^n)(1 - b^{n+2})B(n). \end{aligned}$$

The other equality is also shown similarly. □

We put

$$c(n) = (1 - p^n)(1 - b^{n+2}) \frac{BH(n + 2)}{n + 2}.$$

COROLLARY 5.6.

i) We have

$$\left| \frac{c(n)}{\varpi_p^n} \right| \leq p.$$

Furthermore, if $n \equiv 0 \pmod{q - 1}$, then

$$\left| \frac{c(n)}{\varpi_p^n} \right| \leq p^{\frac{p}{q-1}}.$$

ii) Suppose that $m \equiv n \pmod{p^l(q - 1)}$. Then

$$\frac{c(m)}{\varpi_p^m} \equiv \frac{c(n)}{\varpi_p^n} \pmod{p^{l - \frac{p}{q-1}} \mathcal{O}_{\mathbb{C}_p}}.$$

Furthermore, if $n \not\equiv 0 \pmod{q - 1}$, then

$$\frac{c(m)}{\varpi_p^m} \equiv \frac{c(n)}{\varpi_p^n} \pmod{p^l \mathcal{O}_{\mathbb{C}_p}}.$$

If $n \equiv 0 \pmod{q - 1}$, then

$$\frac{c(m)}{m\varpi_p^m} \equiv \frac{c(n)}{n\varpi_p^n} \pmod{p^{l+1 - \frac{2p}{q-1}}}.$$

Proof. — For i), the first inequality follows from Proposition 5.3 i) for μ_ϕ . The second inequality follows from Proposition 5.3 ii) for $l = 0$. Note that $\int_{\mathcal{O}_K^\times} d\mu_\phi = 0$. For ii), the first and third congruences follow from Proposition 5.3 for ϕ , and the second inequality for ψ . □

Next, we compare $c(n)$ with $L(n)$.

LEMMA 5.7. — We choose $u \in \mathbb{C}_p$ so that $\varpi_p^{q-1} = p^p u^{q-1}$. Then u is a unit of $\mathcal{O}_{\mathbb{C}_p}$ and

$$\left(\frac{u}{\gamma}\right)^{q-1} \equiv 1 \pmod{p}.$$

Proof. — Simple calculation shows the valuation of u is zero. We have $\lambda(t) = t + \theta t^q + \dots$ with $\theta = 1/\epsilon(p^q - p)$. The q -th coefficient of the integral power series $\exp(\varpi_p \lambda(t))$ is

$$\frac{\varpi_p^q}{q!} + \varpi_p \theta = \varpi_p \theta \left(\frac{\varpi_p^{q-1}}{\theta q!} + 1\right).$$

Since $\varpi_p \theta$ is not integral, the valuation $v_p((\varpi_p^{q-1}/\theta q!) + 1) \geq 1$. Thus

$$\frac{\varpi_p^{q-1}}{\theta q!} + 1 \equiv \left(\frac{u}{\gamma}\right)^{q-1} \frac{(1 - p^{q-1})}{(q-1)} + 1 \equiv -\left(\frac{u}{\gamma}\right)^{q-1} + 1 \pmod{p}.$$

must be congruent to zero. □

Let $n = n'(q-1) + r$ with $0 \leq r < q-1$ and put $c_r = u^{-r} p^{-[pr/(q-1)]} \varpi_p^r$. Then

$$\varpi_p^n = c_r p^{[pn/(q-1)]} u^n.$$

Hence we have

$$L(n) = c_r \left(\frac{u}{\gamma}\right)^n \frac{c(n)}{\varpi_p^n}.$$

Therefore by Corollary 5.6 i), we have $|L(n)| < p$ (note that if $n \not\equiv 0 \pmod{q-1}$, then $|c_r| < 1$). Since $L(n)$ is contained in the unramified field K , we have $L(n) \in \mathcal{O}_K$. Similarly, for $m \equiv n \pmod{p^l(q-1)}$, the fact $L(n) \in \mathcal{O}_K$, Lemma 5.7 and Corollary 5.6 ii) imply the congruence

$$L(m) \equiv L(n) \left(\frac{u}{\gamma}\right)^{m-n} \equiv L(n) \pmod{p^{l - \frac{p}{q-1}}}.$$

Since this is a congruence between elements of \mathcal{O}_K , we have

$$L(m) \equiv L(n) \pmod{p^l}.$$

Similarly, from Corollary 5.6, we obtain the congruences originally proved by Katz [7, Theorem 3.1] and Chellali [4, Théorème 1.1].

THEOREM 5.8.

- i) We have $L(n) \in \mathcal{O}_K$.
- ii) Suppose that $m \equiv n \pmod{p^l(q-1)}$. Then

$$L(m) \equiv L(n) \pmod{p^l}.$$

Furthermore, if $n \not\equiv 0 \pmod{q-1}$, then

$$L(m) \equiv L(n) \pmod{p^{l+1}}.$$

If $n \equiv 0 \pmod{q-1}$, then

$$L'(m) \equiv L'(n) \pmod{p^{l+1}}.$$

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