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GLOBAL REGULARITY FOR MINIMAL SETS NEAR A UNION OF TWO PLANES

by Xiangyu LIANG

ABSTRACT. — We discuss the global regularity of 2 dimensional minimal sets that are near a union of two planes, and prove that every global minimal set in $\mathbb{R}^4$ that looks like a union of two almost orthogonal planes at infinity is a cone. The main point is to use the topological properties of a minimal set at a large scale to control its behavior at smaller scales.

RÉSUMÉ. — On traite la régularité globale des ensembles minimaux 2-dimensionnels qui sont proches d'une union de deux plans, et on démontre que tout ensemble minimal proche d'une union de deux plans presque orthogonaux à l'infini dans $\mathbb{R}^4$ est un cône. L'enjeu est de contrôler le comportement d'un ensemble minimal à petite échelle par la topologie à grande échelle.

1. Introduction

This paper deals with the local (resp. global) regularity of two-dimensional minimal sets in $\mathbb{R}^4$ that look like the union of two almost orthogonal planes locally (resp. at infinity). The motivation is that we want to decide whether all global minimal sets in $\mathbb{R}^n$ are cones.

This Bernstein type of problem is of typical interest for all kinds of minimizing problems in geometric measure theory and calculus of variations. It is natural to ask how does a global minimizer look like, as soon as we know already the local regularity for minimizers. Well known examples are the global regularity for complete 2-dimensional minimal surfaces in $\mathbb{R}^3$, area or size minimizing currents in $\mathbb{R}^n$, or global minimizers for the Mumford-Shah functional. Some of them admit very good descriptions. See [3, 5, 13, 14] for further information.

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Here our notion of minimality is defined in the setting of sets. Roughly speaking, we say that a set E is minimal when there is no deformation $F=\varphi(E)$, where φ is Lipschitz and $\varphi(x)-x$ is compactly supported, for which the Hausdorff measure $H^2(F)$ is smaller than $H^2(E)$. More precisely,

Definition 1.1 (Almgren competitor (Al competitor for short)). — *Let E be a closed set in an open subset U of $\mathbb{R}^n$ and $d\leqslant n-1$ be an integer. An Almgren competitor for E is a closed set $F\subset U$ that can be written as $F=\varphi_1(E)$, where $\varphi_t:U\to U$ is a family of continuous mappings such that*

$$\varphi_0(x)=x \text{ for } x\in U; \tag{1.1}$$

$$\text{the mapping } (t,x)\to\varphi_t(x) \text{ of } [0,1]\times U \text{ to } U \text{ is continuous;} \tag{1.2}$$

$$\varphi_1 \text{ is Lipschitz,} \tag{1.3}$$

and if we set $W_t=\{x\in U\ ;\ \varphi_t(x)\neq x\}$ and $\widehat{W}=\bigcup_{t\in[0.1]}[W_t\cup\varphi_t(W_t)]$, then

$$\widehat{W} \text{ is relatively compact in } U. \tag{1.4}$$

Such a φ_1 is called a deformation in U, and F is also called a deformation of E in U.

Definition 1.2 ((Almgren) minimal sets). — *Let $0<d<n$ be integers, U an open set of $\mathbb{R}^n$. A closed set E in U is said to be (Almgren) minimal of dimension d in U if*

$$H^d(E\cap B)<\infty \text{ for every compact ball } B\subset U, \tag{1.5}$$

and

$$H^d(E\backslash F)\leqslant H^d(F\backslash E) \tag{1.6}$$

for all Al competitors F for E.

This notion was introduced by Almgren [2] to modernize Plateau's problem, which aims at understanding physical objects, such as soap films, that minimize the area while spanning a given boundary. The study of regularity and existence for these sets is one of the canonical interests in geometric measure theory.

The Bernstein type problem aims at deciding whether every minimal set in $\mathbb{R}^n$ is a cone. The general idea is the following.

Let E be a $d-$dimensional reduced Almgren minimal set in $\mathbb{R}^n$. Reduced means that there is no unnecessary points. More precisely, we say that E

is reduced when

$$H^d(E \cap B(x,r)) > 0 \text{ for } x \in E \text{ and } r > 0. \tag{1.7}$$

Recall that the definition of minimal sets is invariant modulo sets of measure zero, and it is not hard to see that for each Almgren minimal set E, its closed support E^* (the reduced set $E^* \subset E$ with $H^2(E \backslash E^*) = 0$) is a reduced Almgren minimal set. Hence we can restrict ourselves to discussing only reduced minimal sets.

Now fix any $x \in E$, and set

$$\theta_x(r) = r^{-d} H^d(E \cap B(x,r)). \tag{1.8}$$

This density function θ_x is nondecreasing for $r \in]0,\infty[$ (cf. e.g. [6, Proposition 5.16]). In particular the two values

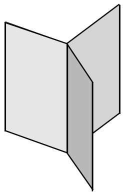
$$\theta(x) = \lim_{t \to 0^+} \theta_x(t) \text{ and } \theta_\infty(x) = \lim_{t \to \infty} \theta_x(t) \tag{1.9}$$

exist, and are called density of E at x, and density of E at infinity respectively. It is easy to see that $\theta_\infty(x)$ does not depend on x, hence we shall denote it by θ_∞. Also, by the global Ahlfors regularity of minimal sets (cf. [8, Proposition 4.1], with $\delta = \infty, k = 1, U = \mathbb{R}^n$), (1.8) is bounded on r, hence θ_∞ is always finite.

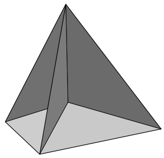
It is known that if E is a minimal set, $x \in E$, and $\theta_x(r)$ is a constant function of r, then E is a minimal cone centered on x (cf. e.g. [6, Theorem 6.2]). Thus by the monotonicity of the density functions $\theta_x(r)$ for any $x \in E$, if we can find a point $x \in E$ such that $\theta(x) = \theta_\infty$, then E is a cone and we are done.

On the other hand, the possible values for $\theta(x)$ and θ_∞ for any E and $x \in E$ are not arbitrary. By [6, Proposition 7.31], for each x, $\theta(x)$ is equal to the density at the origin of a $d-$dimensional Al-minimal cone in $\mathbb{R}^n$. Also, since θ_∞ is finite, an argument around [6, (18.33)], which is similar to the proof of [6, Proposition 7.31], gives that θ_∞ is also equal to the density at the origin of a $d-$dimensional Al-minimal cone in $\mathbb{R}^n$. In other words, if we denote by $\Theta_{d,n}$ the set of all possible numbers that could be the density at the origin of a $d-$dimensional Almgren-minimal cone in $\mathbb{R}^n$, then $\theta_\infty \in \Theta_{d,n}$, and for any $x \in E$, $\theta(x) \in \Theta_{d,n}$.

Thus we restrict the range of θ_∞ and $\theta(x)$. Recall that the set $\Theta_{d,n}$ is possibly very small for any d and n. For example, $\Theta_{2,3}$ contains only three values: 1 (the density of a plane), 1.5 (the density of a $\mathbb{Y}$ set, which is the union of three closed half planes with a common boundary L, and that meet along the line L with 120° angles), and d_T (is the density of a $\mathbb{T}$ set, i.e., the cone over the 1-skeleton of a regular tetrahedron centered at 0). (See the figure below).



a $\mathbb{Y}$ set



a $\mathbb{T}$ set

Recall that the reason why θ_∞ has to lie in $\Theta_{d,n}$ is that, for any Al-minimal set E, all its blow-in limits have to be Al-minimal cones (cf. Argument around [6, (18.33)]). A blow-in limit of E is the limit of any converging (for the Hausdorff distance) subsequence of

$$E_r = r^{-1}E, r \to \infty. \tag{1.10}$$

Hence the value of θ_∞ implies that at sufficiently large scales, E looks like an Al-minimal cone of density θ_∞.

This is the same reason why $\theta(x) \in \Theta_{d,n}$. Here we look at the behavior of E_r when $r \to 0$, and the limit of any converging subsequence is called a blow-up limit (it is unknown whether it is unique). Such a limit is also an Al-minimal cone C (cf. [6, Proposition 7.31]). This means, at some very small scales around each x, E looks like (i.e. very near with respect to the Hausdorff distance) some Al-minimal cone C of density $\theta(x)$. In this case we call x a C type point of E.

After the discussion above, our problem will be solved if we can prove that every minimal cone C satisfies the following property:

$$\begin{aligned}&\text{There exists } \epsilon = \epsilon_C > 0, \text{ such that for every minimal set } E,\\ &\text{if } d_{0,1}(C,E) < \epsilon, \text{ then there exists } x \in E \cap B(0,1) \text{ whose}\\ &\text{density } \theta(x) \text{ is the same as that of } C \text{ at the origin.}\end{aligned} \tag{1.11}$$

Here $d_{x,r}$ stands for the relative distance in the ball $B(x,r)$: for any closed sets E and F,

$$d_{x,r}(E,F) = \frac{1}{r}\max\Big\{\sup\{d(y,F) : y \in E \cap B(x,r)\},\\ \sup\{d(y,E) : y \in F \cap B(x,r)\}\Big\}. \tag{1.12}$$

The discussion above uses only the values of densities at small scale and at infinity. A geometric intepretation is: there exists $x \in E \cap B(0,1)$ such that a blow-up limit C_x of E at x admits the same density as C at the origin.

So far we know that (1.11) is true for the planes and $\mathbb{Y}$ sets (see [6, Proposition 16.24]). We do not know any minimal cone that does not verify the property (1.11). But there are at least two minimal cones for which we do not know whether (1.11) holds, either: the $\mathbb{T}$ set, and the sets $Y\times Y \in \mathbb{R}^4$, whose minimality has recently been proved in [11]. The topology of the set $Y \times Y$ is more complicated than that of $\mathbb{T}$ sets, and the situation of $\mathbb{T}$ sets is already tricky, see [12] for more detail.

In this paper we prove the property (1.11) for the unions of two almost orthogonal planes. Recall that in [10], we have proved the following

Theorem 1.3 (minimality of the union of two almost orthogonal planes, cf. [10, Thm 1.24]). — *There exists $0 < \theta_0 < \frac{\pi}{2}$, such that if P^1 and P^2 are two planes in $\mathbb{R}^4$ whose characteristic angles (α_1, α_2) satisfy $\alpha_2 \geqslant \alpha_1 \geqslant \theta$, then their union $P^1 \cup P^2$ is a minimal cone in $\mathbb{R}^4$.*

Here the characteristic angles describe the relative position between planes. Two planes P^1 and P^2 have characteristic angles (α_1, α_2) means that there exists an orthonormal basis $\{e_i\}_{1\leqslant i\leqslant 4}$ of $\mathbb{R}^4$ such that P^1_α is generated by e_1 and e_2, and P^2_α is generated by $\cos\alpha_1 e_1 + \sin\alpha_1 e_3$ and $\cos\alpha_2 e_2 + \sin\alpha_2 e_4$. Each pair of $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_2 \geqslant \alpha_1 \geqslant \theta_0$ gives a minimal cone $P_\alpha = P^1 \cup_\alpha P^2$, and the origin is called a singularity of type $\mathbb{P}_\alpha$ in the set P_α. These gives a continuous family of minimal cones with the same density at the origin, any two of which are not C^1 equivalent to each other. But still, we give them a general name, that is, each singularity of type $\mathbb{P}_\alpha$ is a singular point of type $2\mathbb{P}$.

So let us state our main results.

Theorem 1.4. — *There is an angle $\theta_1 \in [\theta_0, \frac{\pi}{2})$, (where θ_0 is the θ_0 in Theorem 1.3), and $\lambda > 0$, such that for any $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_2 \geqslant \alpha_1 \geqslant \theta_1$, if E is a 2-dimensional reduced Almgren minimal set in $U \subset \mathbb{R}^4$, $B(x, r) \subset U$, and there is a reduced minimal cone $P_\alpha + x$ of type $\mathbb{P}_\alpha$ centered at x such that $d_{x,r}(E, P_\alpha + x) \leqslant \lambda$, then $E \cap B(x, r/100)$ contains (at least) one $2\mathbb{P}$ type point.*

A direct corollary to this is the expected global regularity for minimal sets that look like a union of two plane at the infinity:

Theorem 6.1. — *Let θ_1 be as in Theorem 1.4. Then for any $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_2 \geqslant \alpha_1 \geqslant \theta_1$, if E is a 2-dimensional reduced Almgren minimal set in $\mathbb{R}^4$ such that one blow-in limit of E at infinity is P_α (i.e., there* *exists a sequence of numbers $r_n \to \infty$, and the sequence of sets $r_n^{-1}(E)$ converge to P_α under the Hausdorff distance as $n \to \infty$), then E is a $\mathbb{P}_\alpha$ set.*

Besides the global regularity, the property (1.11) helps also to control the the relative distances $d_{x,r}$ between a minimal set and minimal cones in the balls $B(x,r)$ and the local speed of decay of the density function $\theta_x(r)$, because this property gives a lower bound of $\theta_x(r)$. When we prove (1.11) for a minimal cone C, we can get nicer local regularity results, that is, if a minimal set is very near C in a ball, then it should be equivalent to C in a smaller ball through a bi-Hölder homeomorphism (C^1 diffeomorphism in good cases). So here Theorem 1.4 has another useful corollary:

THEOREM 6.2. — *Let θ_1 be as in Theorem 1.4. Then there exists a $\epsilon > 0$ such that for any $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_2 \geqslant \alpha_1 \geqslant \theta_1$, if E is a 2-dimensional reduced Almgren minimal set in $U \subset \mathbb{R}^4$, $B(x, 300r) \subset U$ ($r < 1$), and there is a reduced minimal cone $P_\alpha + x$ of type $\mathbb{P}_\alpha$ centered at x such that $d_{x,200r}(E, P_\alpha + x) \leqslant \epsilon$, then there exists C^1 diffeomorphism $\Phi : B(x, 2r) \to \Phi(B(x, 2r))$, such that $|\Phi(y) - y| \leqslant 10^{-2}r$ for $y \in B(x, 2r)$, and $E \cap B(x,r) = \Phi(P_\alpha + x) \cap B(x,r)$.*

The proof of Theorem 1.4 will keep us busy until the end of Section 6, but let us already try to explain how it goes.

First notice that Theorem 1.4 is invariant under translation with respect to x, and homogenous with respect to r, so we can only restrict to the case where $x = 0$ and $r = 1$.

Section 2 is devoted to giving some regularity properties for a minimal set E that is close to P_α, but does not contain any point of type $2\mathbb{P}$. In particular, we use a stopping time argument to find a critical region, outside of which everything goes fine, and inside of which things begin to go bad. Here "bad" means that the set begins to get far away from P_α. The main idea is to control the measure of E in the good region by finer estimates, since there we have good regularity properties; and for the bad region we only control its measure roughly by projections. Part of the argument will be similar to the proof of minimality of P_α.

Section 3 is quite short, where we sum up a little what happens, and give a competitor for E, using minimal graphs. We also state a basic estimate for minimal graphs, for later use.

Section 4 is devoted to giving some useful control on the measure of the competitor defined in Section 3.

In Section 5 we conclude, using harmonic extensions and projection properties of the competitor.

We discuss the global regularity and local C^1 regularity of minimal sets that are near a P_α cone in Section 6.

In this article, some of the results and arguments cited in [6] exist also in some other (earlier) references, e.g. [17]. But for simplify the article, the author will cite [6] systematically throughout this article.

Some useful notation

In all that follows, minimal set means Almgren minimal set;

- $[a,b]$ is the line segment with end points a and b;
- $B(x,r)$ is the open ball with radius r and centered on x;
- $\overline{B}(x,r)$ is the closed ball with radius r and center x;
- $\overrightarrow{ab}$ is the vector $b-a$;
- H^d is the Hausdorff measure of dimension d ;
- $d_H(E,F)=\max\{\sup\{d(y,F):y\in E\},\sup\{d(y,E):y\in F\}\}$ is the Hausdorff distance between two sets E and F.
- $d_{x,r}$: the relative distance with respect to the ball $B(x,r)$, is defined by

$$d_{x,r}(E,F)=\frac{1}{r}\max\{\sup\{d(y,F):y\in E\cap B(x,r)\},\\ \sup\{d(y,E):y\in F\cap B(x,r)\}\}.$$

2. A stopping time argument, and regularity and projection properties for minimal sets near P_α

In this section we use a stopping time argument to control some large scale behavior for minimal sets that near P_α. Let us first introduce some notation.

Fix an orthonormal basis $\{e_i\}_{1\leqslant i\leqslant 4}$ of $\mathbb{R}^4$. For each $\alpha=(\alpha_1,\alpha_2)\in[0,\frac{\pi}{2}]^2$ and $i=1,2$, denote by $P_\alpha=P_\alpha^1\cup P_\alpha^2$, where P_α^1 is the plane generated by e_1 and e_2, and P_α^2 is the plane generated by $\cos\alpha_1 e_1+\sin\alpha_1 e_3$ and $\cos\alpha_2 e_2+\sin\alpha_2 e_4$. (Hence P_α^1 and P_α^2 are two planes in $\mathbb{R}^4$ with characteristic angles (α_1,α_2)). Set

$$C_\alpha^i(x,r)=(p_\alpha^i)^{-1}(B(0,r)\cap P_\alpha^i)+x, \tag{2.1}$$

where p_α^i is the orthogonal projection on P_α^i, and

$$D_\alpha(x,r)=C_\alpha^1(x,r)\cap C_\alpha^2(x,r). \tag{2.2}$$

So $C_\alpha^i(x,r)$ is a cylinder and $D_\alpha(x,r)$ is the intersection of two cylinders. It is not hard to see that $D_\alpha(x,r)\supset B(x,r)$ and $D_\alpha(0,1)\cap P_\alpha=B(0,1)\cap P_\alpha$.

We say that two sets E,F are ϵr near each other in an open set U if

$$d_{r,U}(E,F)<\epsilon, \tag{2.3}$$

where

$$d_{r,U}(E,F)=\frac{1}{r}\max\{\sup\{d(y,F):y\in E\cap U\},\\ \sup\{d(y,E):y\in F\cap U\}\}. \tag{2.4}$$

We set also

$$\begin{aligned} d_{x,r}^\alpha(E,F)&=d_{r,D_\alpha(x,r)}(E,F)\\ &=\frac{1}{r}\max\{\sup\{d(y,F):y\in E\cap D_\alpha(x,r)\},\\ &\qquad \sup\{d(y,E):y\in F\cap D_\alpha(x,r)\}\}. \end{aligned} \tag{2.5}$$

Remark 2.1. — We should be clear about the fact that

$$d_{r,U}(E,F)\neq\frac{1}{r}d_H(E\cap U,F\cap U). \tag{2.6}$$

To see this, we can take $U=D_\alpha(x,r)$, and set $E_n=\partial D_\alpha(x,r+\frac{1}{n})$ and $F_n=\partial D_\alpha(x,r-\frac{1}{n})$. Then we have

$$d_{x,r}^\alpha(E_n,F_n)\to 0 \tag{2.7}$$

and

$$\frac{1}{r}d_H(E_n\cap D_\alpha(x,r),F_n\cap D_\alpha(x,r))=\frac{1}{r}d_H(E_n\cap D_\alpha(x,r),\emptyset)=\infty. \tag{2.8}$$

So $d_{r,U}$ measures rather how the part of one set in the open set U could be approximated by the other set, and vice versa. However we always have

$$d_{x,r}^\alpha(E,F)\leqslant\frac{1}{r}d_H(E\cap D_\alpha(x,r),F\cap D_\alpha(x,r)). \tag{2.9}$$

Now we give the proposition below, obtained by a stopping time argument.

Proposition 2.2. — *There exists $\epsilon_0>0$, such that for any $\epsilon<\epsilon_0$, and $\alpha>\frac{\pi}{3}$, if E is a closed reduced set which is minimal in $D_\alpha(0,1)$, $d_{0,1}^\alpha(E,P_\alpha)<\frac{\epsilon}{10}$, and E contains no $2\mathbb{P}$ point in $B(0,\frac{1}{100})$, then there exists $r_E\in]0,\frac{1}{2}[$ and $o_E\in B(0,12\epsilon)$ such that E is $2\epsilon r_E$ near $P_\alpha+o_E$ in $D_\alpha(o_E,2r_E(1-12\epsilon))$, but not ϵr_E near $P_\alpha+q$ in $D_\alpha(o_E,r_E)$ for any $q\in\mathbb{R}^4$.*

Remark 2.3. — The construction and estimates in the proof will also be used later.

Proof of Proposition 2.2. — We fix any ϵ and $\alpha = (\alpha_1, \alpha_2) > \frac{\pi}{3}$, and set $s_i = 2^{-i}$ for $i \geqslant 0$. Set $D(x,r) = D_\alpha(x,r), d_{x,r} = d^\alpha_{x,r}$ for short.

We proceed in the following way.

– *Step 1:* Denote by $q_0 = q_1 = O$, then in $D(q_0, s_0)$, E is ϵs_0 near $P_\alpha + q_1$ by hypothesis.

– *Step 2:* If in $D(q_1, s_1)$, the set E is not ϵs_1 near $P_\alpha + q$ for any q, we stop; if not, there exists a q_2 such that E is ϵs_1 near $P_\alpha + q_2$ in $D(q_1, s_1)$. Here we also ask ϵ to be small enough (say, $\epsilon < \frac{1}{100}$) so that $q_2 \in D(q_1, \frac{1}{2} s_1)$, thanks to the conclusion of Step 1. Then in $D(q_1, s_1)$, we have simultaneously :

$$\begin{aligned} d_{q_1,s_1}(E, P_\alpha + q_1) &\leqslant s_1^{-1} d_{q_0,s_0}(E, P_\alpha + q_1) \\ &\leqslant 2\epsilon \ ; \ d_{q_1,s_1}(E, P_\alpha + q_2) \leqslant \epsilon. \end{aligned} \tag{2.10}$$

Let us verify that (2.10) implies that $d_{q_1,\frac{1}{2}s_1}(P_\alpha + q_1, P_\alpha + q_2) \leqslant 12\epsilon$ when ϵ is small, say, $\epsilon < \frac{1}{100}$. In fact, for each $z \in D(q_1, \frac{1}{2}s_1) \cap (P_\alpha + q_1)$, we have $d(z, E) \leqslant d_{q_0,s_0}(E, P_\alpha + q_1) \leqslant \epsilon$, hence there exists $y \in E$ such that $d(z, y) \leqslant \epsilon$. But since $z \in D(q_1, \frac{1}{2}s_1)$, we have $y \in D(q_1, \frac{1}{2}s_1 + \epsilon) \subset D(q_1, s_1)$, and hence $d(y, P_\alpha + q_2) \leqslant s_1^{-1} d_{q_1,s_1}(E, P_\alpha + q_2) \leqslant 2\epsilon$, therefore $d(z, P_\alpha + q_2) \leqslant d(z, y) + d(y, P_\alpha + q_2) \leqslant 3\epsilon$.

On the other hand, suppose $z \in D(q_1, \frac{1}{2}s_1) \cap (P_\alpha + q_2)$, we have $d(z, E) \leqslant s_1^{-1} d_{q_1,s_1}(P_\alpha + q_2, E) \leqslant 2\epsilon$, hence there exists $y \in E$ such that $d(z, y) \leqslant 2\epsilon$. But since $z \in D(q_1, \frac{1}{2}s_1)$, we have $y \in D(q_1, \frac{1}{2}s_1 + 2\epsilon) \subset D(q_0, s_0)$, and hence $d(y, P_\alpha + q_1) \leqslant d_{q_0,s_0}(E, P_\alpha + q_1) \leqslant \epsilon$, which implies $d(z, P_\alpha + q_1) \leqslant d(z, y) + d(y, P_\alpha + q_1) \leqslant 3\epsilon$.

As a result

$$d_{q_1,\frac{1}{2}s_1}(P_\alpha + q_1, P_\alpha + q_2) \leqslant (\frac{1}{2}s_1)^{-1} \times 3\epsilon = 12\epsilon, \tag{2.11}$$

hence $d_{q_1,\frac{1}{2}s_1}(q_1, q_2) \leqslant 24\epsilon$, and therefore $d(q_1, q_2) \leqslant 6\epsilon = 12\epsilon s_1$.

Now we define our iteration process (notice that it depends on ϵ, so we also call it a ϵ-process).

Suppose that all $\{q_i\}_{i\leqslant n}$ have already been defined, with

$$d(q_i, q_{i+1}) \leqslant 12 s_i \epsilon = 12 \times 2^{-i}\epsilon \tag{2.12}$$

for $0 \leqslant i \leqslant n-1$, and hence

$$d(q_i, q_j) \leqslant 24\epsilon s_{\min(i,j)} = 2^{-\min(i,j)} \times 24\epsilon \tag{2.13}$$

for $0 \leqslant i, j \leqslant n$. Moreover, for all $i \leqslant n-1$, E is ϵs_i near $P_\alpha + q_{i+1}$ in $D(q_i, s_i)$. We say that the process does not stop at step n. In this case

– *Step n+1:* We look at the situation in $D(q_n, s_n)$.

If E is not ϵ near any $P_\alpha + q$ in this ball of radius s_n, we stop, since we have found the $o_k = q_n$, $r_k = s_n$ as desired. In fact, since $d(q_{n-1}, q_n) \leqslant 12\epsilon s_{n-1}$, we have $D(q_n, 2s_n(1-12\epsilon)) = D(q_n, s_{n-1}(1-12\epsilon)) \subset D(q_{n-1}, s_{n-1})$, and hence

$$(2.14)\qquad \begin{aligned} d_{q_n,2s_n(1-12\epsilon)}(P_\alpha+q_n,E) &\leqslant (1-12\epsilon)^{-1} d_{q_{n-1},s_{n-1}}(P_\alpha+q_n,E) \\ &\leqslant \frac{\epsilon}{1-12\epsilon}. \end{aligned}$$

Moreover

$$(2.15)\qquad d(o_k,O) = d(q_n,q_1) \leqslant 2^{-\min(1,n)} \times 24\epsilon = 12\epsilon.$$

Otherwise, we can find a $q_{n+1} \in \mathbb{R}^4$ such that E is still ϵs_n near $P_\alpha + q_{n+1}$ in $D(q_n, s_n)$. Then since ϵ is small, $q_{n+1} \in D(q_n, \frac{1}{2}s_n)$. Moreover we have as before $d(q_{n+1}, q_n) \leqslant 12\epsilon s_n$, and for $i \leqslant n-1$,

$$(2.16)\qquad \begin{aligned} d(q_i,q_{n+1}) &\leqslant \sum_{j=i}^{n} d(q_j,q_{j+1}) \\ &\leqslant \sum_{j=i}^{n} 12 \times 2^{-j}\epsilon \leqslant 2^{-j} \times 24\epsilon = 2^{-\min(i,n+1)} \times 24\epsilon. \end{aligned}$$

Thus we have obtained our q_{n+1}.

Now all we have to do is to prove that for every ϵ small enough, this process has to stop at a finite step. For this purpose we need the following proposition.

Proposition 2.4. — *There exists $\theta_1' \in [\theta_0, \frac{\pi}{2})$, and for any $l \in]0, \frac{1}{2}]$, there exists $\epsilon_l \in]0, \frac{1}{2}[$, such that for any $\alpha > \theta_1'$, $\epsilon \leqslant \epsilon_l$, and E as in Proposition 2.2, if the $\epsilon-$process does not stop before the step n, then*

(1) The part $E \cap (D_\alpha(0, \frac{39}{40}) \backslash D_\alpha(q_n, \frac{1}{10}s_n))$ is composed of two disjoint pieces $G^i, i = 1, 2$, such that:

(2.17) *G^i is the graph of a C^1 map*

$$g^i : C_\alpha^i(0, \frac{39}{40}) \backslash C_\alpha^i(q_n, \frac{1}{10}s_n) \cap P_\alpha^i \to P_\alpha^{i\perp}$$

with

$$(2.18)\qquad ||\nabla g^i||_\infty < l \leqslant \frac{1}{2};$$

(2) For every $\frac{1}{10}s_n \leqslant t \leqslant s_n$

$$(2.19)\qquad E \cap (D_\alpha(0,1) \backslash D_\alpha(q_n,t)) = G_t^1 \cup G_t^2$$

where G_t^1, G_t^2 do not meet each other. Moreover

$$(2.20)\qquad P_\alpha^i \cap (D_\alpha(0,1) \backslash C_\alpha^i(q_n,t)) \subset p_\alpha^i(G_t^i)$$

where p_α^i is the orthogonal projection on $P_\alpha^i, i = 1, 2$;

Remark 2.5. — If we take the optimal ϵ_l for each l such that Proposition 2.4 holds, then obviously for any $l \leqslant l'$, $\epsilon_l \leqslant \epsilon_{l'}$.

We will not prove this proposition, see [10, Proposition 6.1(1)-(2)] for the proof. But we'll use it to finish our Proposition 2.2.

Remark 2.6. — In fact we need all the properties stated in [10, Proposition 6.1] for our set E. For (1) and (2) in [10, Proposition 6.1], the arguments there can be applied directly here to our set E with no change. But for (3) and (4), the proof in [10, Proposition 6.1] uses some special property of E_k, which are not necessarily true for our set E here. Hence we will treat the property of surjective projections ([10, Proposition 6.1(4)]) later in a different way.

So let ϵ_0 be the $\epsilon_{\frac{1}{2}}$ in Proposition 2.4. Suppose that the $\epsilon-$process does not stop at any finite step, and we'll try to get a contradiction. By Proposition 2.4(1), for any n, $E \cap (D_\alpha(0,1) \backslash D_\alpha(q_n, \frac{1}{10}s_n))$ is composed of two disjoint graphs G^i on $[C^i_\alpha(0,1) \backslash C^i_\alpha(q_n, \frac{1}{10}s_n)] \cap P^i_\alpha, i = 1, 2$. Denote by $\Delta_n = D_\alpha(q_n, s_n)$.

Notice that by (2.16), with $\epsilon < \frac{1}{100}$, the sets $\Delta_n = D_\alpha(q_n, s_n)$ are in fact a sequence of non degenerate compact balls, with

$$\Delta_n \subset \Delta_{n-1}, n \in \mathbb{N}, \lim_{n\to\infty} \operatorname{diam}(\Delta_n) \to 0, \tag{2.21}$$

Hence there exists a point $p \in B(0, \frac{1}{2})$, such that $\{p\} = \cap_n \Delta_n$. Then p is also the limit of q_n, hence it lies in $B(0, \frac{1}{100})$. By Proposition (2.4)(1), for any $r \in (0, \frac{1}{2})$, $E \cap D(p, \frac{1}{2}) \backslash D(p, r)$ is composed of the union of two disjoint graphs on $P^i_\alpha \cap C^i_\alpha(p, \frac{1}{2}) \backslash C^i_\alpha(p, r)$. As a result , $E \cap D(p, \frac{1}{2}) \backslash \{p\}$ is composed of two C^1 graphs on $P^i_\alpha \cap C^i_\alpha(p, \frac{1}{2}) \backslash \{p\}$. Denote by G^i these two graphs. By (2.18), they are both $\frac{1}{2}$-Lipschitz. Now E is closed hence $p \in E$. Then for each $i = 1, 2$, $G^i \cup \{p\}$ is a $\frac{1}{2}$-Lipschitz graph on $P^i_\alpha \cap C^i_\alpha(p, \frac{1}{2})$, and hence $E \cap D_\alpha(p, \frac{1}{2})$ is composed of the disjoint union of these two $\frac{1}{2}$-Lipschitz graphs. Now we define $\varphi : E \cap D_\alpha(p, \frac{1}{2}) \to P_\alpha + p$, where the restriction of φ to each $G^i \cup \{p\}$ is just the orthogonal projection to $P^i_\alpha + p$. Then it is easy to check that φ is a Lipschitz homeomorphism. That is, E is bi-Lipschitz homeomorphic to P_α in $D_\alpha(p, \frac{1}{2})$.

We want to prove that p is a point of type $2\mathbb{P}$. Take any blow-up limit C of E at the point p. Then C is a minimal cone. By the bi-Hölder regularity for 2-dimensional minimal sets, near the point p, E is locally bi-Hölder equivalent to C. But E is also bi-Lipschitz equivalent to p_α near p, hence the two minimal cones P_α and C are topologically the same. As a consequence, $P_\alpha \cap \partial B(0,1)$ and $C \cap \partial B(0,1)$ are topologically the same, therefore,

$C \cap \partial B(0,1)$ is the union of two topological circles. But by the description of 2-dimensional minimal cones (cf. [6, Proposition 14.1]), the intersection of any minimal cone with the unit sphere is a finite union of great circles and arcs of great circles that meet at their extremities by group of three with 120° angles. Here in our case, we can deduce that $C \cap \partial B(0,1)$ is the union of two circles. Hence C is a minimal cone of type $2\mathbb{P}$.

Hence the point p is a point of type $2\mathbb{P}$. This contradicts the fact that $E \cap B(0, \frac{1}{100})$ contains no point of type $2\mathbb{P}$, because $p \in B(0, \frac{1}{100})$.

Thus we complete the proof of Proposition 2.2. □

Next we still have to prove some property of surjective projection, as remarked in Remark 2.6.

Proposition 2.7. — *Take $\epsilon \leqslant \epsilon_0$, and take α and E as in Proposition 2.4. Then for any $n \geqslant 1$, if the $\epsilon-$process does not stop before the step n, then the orthogonal projections $p^i_\alpha : E \cap \overline{D}_\alpha(q_n, t) \to P^i_\alpha \cap \overline{C}^i_\alpha(q_n, t), i = 1, 2$ are surjective, for all $\frac{1}{9}s_n \leqslant t \leqslant s_n$.*

Proof. — Fix a such n. Set $s_i = 2^{-i}$ for $i \geqslant 0$. Set $D(x,r) = D_\alpha(x,r)$, $C^i(x,r) = C^i_\alpha(x,r)$, $d_{x,r} = d^\alpha_{x,r}$ for short. By Proposition 2.4(1), the part $E \cap (D_\alpha(0, \frac{39}{40}) \backslash D_\alpha(q_n, \frac{1}{10}s_n))$ is composed of two disjoint pieces $G^i, i = 1, 2$, such that:

(2.22) G^i is the graph of a C^1 map

$$g^i : C^i_\alpha(0, \frac{39}{40}) \backslash C^i_\alpha(q_n, \frac{1}{10}s_n) \cap P^i_\alpha \to {P^i_\alpha}^\perp$$

with

$$||\nabla g^i||_\infty < \frac{1}{2}. \tag{2.23}$$

Thus $G^i \cap \partial C^i(0, \frac{39}{40})$ is a nice C^1 curve, which is the graph of g^i on $P^i_\alpha \cap \partial C^i(0, \frac{39}{40})$, and g^i is $\frac{1}{2}$-Lipschitz. Denote by $\gamma^i = g^i|_{P^i_\alpha \cap \partial C^i(0, \frac{39}{40})}$. Then $||\gamma^i||_\infty \leqslant \frac{\epsilon}{10}$ by hypothesis.

Now we define a set Q as follows. First, $Q \subset \overline{B}(0,1)$, and $Q \backslash D(0, \frac{39}{40}) = E \backslash D(0, \frac{39}{40})$. Inside $D(0, \frac{3}{4})$, $Q \cap \overline{D}(0, \frac{3}{4}) = P_\alpha \cap \overline{D}(0, \frac{3}{4})$, the union of two planes. For the part on the annulus $D(0, \frac{39}{40}) \backslash \overline{D}(0, \frac{3}{4})$, we just use two graphs of affine functions to join $P^i_\alpha \cap \partial D(0, \frac{3}{4})$ and γ^i. That is, we define $h^i : P^i_\alpha \cap D(0, \frac{39}{40}) \backslash \overline{D}(0, \frac{3}{4}) \to {P^i_\alpha}^\perp$, for any $x \in P^i_\alpha \cap D(0, \frac{39}{40}) \backslash \overline{D}(0, \frac{3}{4}) (\frac{3}{4}, \frac{39}{40})$, $h^i(x) = \frac{|x| - \frac{3}{4}}{\frac{39}{40} - \frac{3}{4}} \gamma^i(\frac{39x}{40|x|})$.

Thus for any $x \in D(0, \frac{39}{40}) \backslash \overline{D}(0, \frac{3}{4})$, $|\frac{\partial}{\partial r} h^i(x)| = \frac{1}{\frac{39}{40} - \frac{3}{4}} |\gamma^i(\frac{39x}{40|x|})| \leqslant \frac{40}{9} \frac{\epsilon}{100} \leqslant \frac{\epsilon}{20} \leqslant \frac{1}{2000}$, and $|\frac{\partial}{\partial \theta}(x)| \leqslant Lip(\gamma^i) \leqslant \frac{1}{2}$, hence the tangent direction derivative is less than

$$\frac{1}{|x|}|\frac{\partial}{\partial\theta}(x)| \leqslant \frac{1}{2}/\frac{3}{4} = \frac{2}{3}. \tag{2.24}$$

Hence we have

$$\text{Lip } h^i \leqslant \max\{\frac{1}{2000}, \frac{2}{3}\} = \frac{2}{3}. \tag{2.25}$$

Thus the map $H^i : P_\alpha^i \cap D(0, \frac{39}{40})\backslash\overline{D}(0, \frac{3}{4}) \to \mathbb{R}^4 : x \mapsto (x, h^i(x))$ is $(1 + (\frac{2}{3})^2)^{\frac{1}{2}} = \frac{\sqrt{13}}{3}$-Lipschitz. So if we denote by Σ^i the graph of h^i, then

$$\begin{aligned} H^2(\Sigma^i) &= H^2(H^i(P_\alpha^i \cap D(0, \frac{39}{40})\backslash\overline{D}(0, \frac{3}{4})) \\ &\leqslant (\frac{\sqrt{13}}{3})^2)H^2(P_\alpha^i \cap D(0, \frac{39}{40})\backslash\overline{D}(0, \frac{3}{4})) \\ &= \frac{897}{1600}\pi \leqslant \frac{9\pi}{16}, i = 1, 2. \end{aligned} \tag{2.26}$$

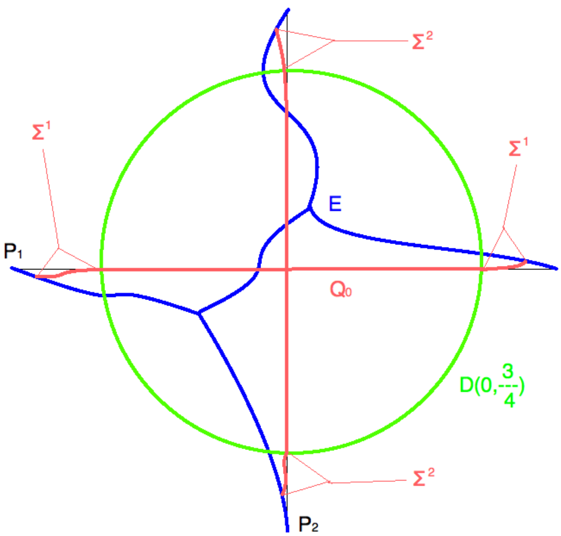


Figure 2.1

Let $Q = [E\backslash D(0, \frac{39}{40})] \cup \Sigma^1 \cup \Sigma^2 \cup [P_\alpha \cap D(0, \frac{3}{4})]$, and $Q_0 = Q \cap D(0, \frac{39}{40})$. (See Figure 2.1) Set $Q^i = \Sigma^i \cup [P_\alpha^i \cap D(0, \frac{3}{4})]$, then Q_0 is the almost disjoint union $Q^1 \cup Q^2$. For each $i = 1, 2$,

$$H^2(Q^i) = H^2(\Sigma^i) + H^2(P_\alpha^i \cap D(0, \frac{3}{4})) \leqslant \frac{9\pi}{16} + \frac{9\pi}{16} = \frac{9\pi}{8}. \tag{2.27}$$

Notice that the set Q_0 is a C^1 version of $P_\alpha\cap D(0,\frac{3}{4})$, and $Q^i, i=1,2$ are its two flat parts as P_α^i.

Now suppose that for some $t\in[\frac{1}{9}s_n,s_n)$, for example the projection $p_\alpha^1: E\cap D(q_n,t)\to P_\alpha^1\cap C^1(q_n,t)$ is not surjective. Then we are going to prove that we can deform E to $[Q\backslash Q_0]\cup Q^2$, and deduce a contradiction.

So take a point $p\in P_\alpha^1\cap\overline{C}^1(q_n,t)$ which does not admit a pre-image in $E\cap\overline{D}(q_n,t)$. Since the set $E_t:=E\cap\overline{D}(q_n,t)$ is compact, its projection $p_\alpha^1(E_t)$ is also compact, which means that we can pick $p\in P_\alpha^1\cap C^1(q_n,t)\backslash p_\alpha^1(E_t)$ and $r\in(0,\frac{t}{10})$ such that $B(p,r)\cap P_\alpha^1\subset P_\alpha^1\cap C^1(q_n,t)\backslash p_\alpha^1(E_t)$, and moreover $0\notin B(p,3r)$.

Now the set $E_t\subset\overline{D}(q_n,t)\backslash {p_\alpha^1}^{-1}(B(p,r)\cap P_\alpha^1)$. Take an orthogonal union of two planes $P_0=P_0^1\cup_\perp P_0^2$ in $\mathbb{R}^4$, denote by p_0^i the orthogonal projection on $P_0^i, k=1,2$, take a point $p_0\in P_0^1$ such that $d(p_0,o)=\frac{1}{2}$.

Then we can easily find a Bi-Lipschitz mapping

$$\varphi:\overline{D}(q_n,t)\backslash {p_\alpha^1}^{-1}(B(p,r)\cap P_\alpha^1)\to\overline{D}(0,1)\backslash {p_0^1}^{-1}(B(p_0,\frac{1}{4})\cap P_0^1),$$

such that $\varphi(E_t\cap D(q_n,t)\backslash D(q_n,\frac{1}{10}s_n))=P_0\cap D(0,1)\backslash D(0,\frac{3}{4})$ (because in the annulus $D(q_n,t)\backslash D(q_n,\frac{1}{10}s_n)$, the set E is still a C^1 graph of P_α).

For any point $x\in D(0,1)$, write $x=(x_1,x_2)$, where $x_i=p_0^i(x)\in B^i(0,1), i=1,2$ ($B^i(0,1)$ is the unit ball of the plane P_0^i). We define $\psi: D(0,1)\backslash {p_0^1}^{-1}(B((p_0,\frac{1}{4})\cap P_0^1)\to D(0,1)\cap P_0\backslash {p_0^1}^{-1}(B((p_0,\frac{1}{4})\cap P_0^1)$ as follows:

$$\psi(x)=\begin{cases} p_0^1(x), & x_2<\frac{3}{4};\\ (x_1,4x_2-3), & x_2\geqslant\frac{3}{4}.\end{cases}\tag{2.28}$$

Then ψ is a Lipschitz map, which maps $[C^1(0,1)\cap C^2(0,\frac{3}{4})]\cup[P_0\cap D(0,1)]$ to $P_0\cap D(0,1)$, and $\psi|_{P_0\cap\partial D(0,1)}=Id$. In particular, $\psi(\varphi(E_t))\subset P_0\cap D(0,1)\backslash {p_0^1}^{-1}(B(p_0,\frac{1}{4})\cap P_0^1)$.

Thus the map $f_1=\varphi^{-1}\circ\psi\circ\varphi$ maps E_t to $P_\alpha\cap D(q_n,t)\backslash D(q_n,\frac{1}{10}s_n)$, and $f_1|_{E\cap\partial D(q_n,t)}=id$.

We can extend f_1 to a Lipschitz map from $D(0,\frac{39}{40})\to D(0,\frac{39}{40})$, such that $f_1|_{E\cap D(0,\frac{39}{40})\backslash D(q_n,t)}=id$ and $f_1|_{D(0,\frac{39}{40})\backslash D(0,\frac{1}{2})}=id$.

Then f_1 is a deformation of E in $D(0,\frac{39}{40})$, which sends $E\cap D(0,\frac{39}{40})$ to $Q_0\backslash[B(p,r)\cap P_\alpha^1]$, this is the union of Q^2 and Q^1 minus a hole $B(p,r)\cap P_\alpha^1$. So we can keep on the deformation, and take the map f_2 which deforms $Q^1\backslash[B(p,r)\cap P_\alpha^1]$ to a set $E^1=\{0\}\cup\partial Q^1\cup C$ of measure zero, where C is a segment that connects the origin and ∂Q^1 and keeps Q^2 fixed. Then the map $f=f_2\circ f_1$ sends $E\backslash D(0,\frac{39}{40})$ to $Q^2\cup E^1$, hence the measure

$$H^2(E\cap D(0,\frac{39}{40}))=H^2(Q^2)\leqslant\frac{9\pi}{8}. \tag{2.29}$$

The map f is Lipschitz, and its restriction to $Q_0\cap\partial D(0,\frac{39}{40})$ is the identity. We extend f to a Lipschitz map on $D(0,1)$, still denoted by f, such that $f=id$ near the boundary of $D(0,1)$. Thus by the minimality of E, and since f does not move $E\backslash D(0,\frac{39}{40})$, we have

$$H^2(E\cap D(0,\frac{39}{40}))\leqslant H^2(f(E\cap D(0,\frac{39}{40}))\leqslant\frac{9\pi}{8}. \tag{2.30}$$

However since $n>1$, we have $s_n<\frac{1}{2}$. By Proposition 2.4(1), we have

$$\begin{aligned}H^2(E\cap D(0,\frac{39}{40}))&\geqslant H^2(G^1)+H^2(G^2)\\&\geqslant H^2(p^1_\alpha(G^1))+H^2(p^2_\alpha(G^2))\\&=\sum_{i=1,2}H^2(P^i_\alpha\cap C^i(0,\frac{39}{40})\backslash C^i(q_n,\frac{1}{10}s_n))\\&\geqslant\sum_{i=1,2}H^2(P^i_\alpha\cap C^i(0,\frac{39}{40})\backslash C^i(q_n,\frac{1}{20}))\\&=2\times\pi((\frac{39}{40})^2-(\frac{1}{20})^2)=\frac{1517}{800}\pi>\frac{9\pi}{8},\end{aligned} \tag{2.31}$$

which leads to a contradiction.

This completes the proof of Proposition 2.7. □

3. A competitor, and estimates for minimal graphs

Let θ_1', α be as in Proposition 2.4, let $\epsilon=\epsilon_0$, μ be chosen later, and let E be as in Proposition 2.2, that is, $d^\alpha_{0,1}<\frac{\epsilon}{10}$, and E contains no $2\mathbb{P}$ type point in $B(0,\frac{1}{100})$. We want to construct a competitor for E, and show that if $d^\alpha_{0,1}$ is sufficiently small, this competitor admits necessarily less measure than E, and thus leads to a contradiction.

Let us point out that the condition $d^\alpha_{0,1}<\frac{\epsilon}{10}$ is a general qualitative one, which guarantees that E satisfies the regularity properties in Proposition 2.4 and 2.7. To make the necessary finer estimates for measures of E and its competitor, we still have to get the"λ-near" condition as in Theorem 1.4.

So by Proposition 2.2, there is a $r_E \in]0, \frac{1}{2}[$, $o_E \in B(0, \frac{1}{2}\epsilon_0)$ such that the conclusion in Proposition 2.2 holds for E. Denote by $\gamma^i : \partial B(0, \frac{1}{2}) \cap P_\alpha^i \to {P_\alpha^i}^\perp$ the C^1 curve $g^i|_{\partial B(0,\frac{1}{2})\cap P_\alpha^i}$. Suppose that $||\gamma^i|_{\partial B(0,\frac{1}{2})\cap P_\alpha^i}||_{C^1} \leqslant \mu$.

The idea of the construction of the competitor is not complicated. We take, for each i, a minimal graph Σ^i which is the graph of a function $f^i : B(0, \frac{1}{2}) \cap P_\alpha^i \to {P_\alpha^i}^\perp$ such that $f^i|_{\partial B(0,\frac{1}{2})\cap P_\alpha^i} = \gamma^i$. Take $\Sigma = \Sigma^1 \cup \Sigma^2$. Then hopefully when μ is small enough, these two graphs are very flat at the center, so that Σ is very similar to P_α. Thus we can deform $E \cap D_\alpha(0, \frac{1}{2})$ to a subset of Σ in a Lipschitz manner, while keeping $E \cap \partial D_\alpha(0, \frac{1}{2})$ unchanged. Hence Σ contains a competitor of E in $D_\alpha(0, \frac{1}{2})$. By the minimality of E, the measure of Σ has to be larger than that of $E \cap D_\alpha(0, 1)$. But we are going to show that when μ is small enough, this is not true.

Before we go down to the following two sections, which will be devoted to giving some estimates for minimal graphs, let us already explain what happens.

We want to compare the measures of $E \cap D_\alpha(0, \frac{1}{2})$ and Σ. Outside $D(o_E, \frac{1}{10}r_E)$, by Proposition 2.4, E is also composed of two C^1 graphs G^i on the two annuli $P_\alpha^i \cap B(0, \frac{1}{2}) \backslash C^i((o_E, \frac{1}{10}r_E)$. So in this part, our goal is to compare the surface measure of Σ^i and G^i, that is, the graph of f^i and g^i. Notice that f^i and g^i coincide on $P_\alpha^i \cap \partial B(0, \frac{1}{2})$; while on $P_\alpha^i \cap \partial B(o_E, \frac{1}{10}r_E)$, g^i is supposed to be ϵ-far from any plane, while f^i is almost a plane (see Proposition 3.1 below). Then Section 5 will be devoted to estimating the difference between these two graphs.

So this will help estimate the difference between measures of E and Σ on the annulus region $D_\alpha(0, \frac{1}{2}) \backslash D(o_E, \frac{1}{10}r_E)$. For the part of $E \cap D(o_E, \frac{1}{10}r_E)$, we estimate its measure by using projections.

In the rest of the section, let us state some well known results for minimal graphs.

Denote by $B = B(0, 1) \cap \mathbb{R}^2$ the unit disc in $\mathbb{R}^2$. Let γ be a C^1 function from ∂B to $\mathbb{R}^2$. Now by [16, Theorem 7.2], there exists a function $f : \overline{B} \to \mathbb{R}^2$, whose graph $\Sigma_f = \{(x, f(x)) : x \in \overline{B}\} \subset \mathbb{R}^4$ is a minimal surface, $f|_{\partial B} = \gamma$, and $f \in C^0(\overline{B}) \cap C^\infty(B)$. In particular, by the convex hull property for minimal surfaces, we have

$$||f||_\infty \leqslant ||\gamma||_{L^\infty(\partial B)}. \tag{3.1}$$

Note that f is a minimal graph means that it is a solution of the following system

$$div\left(\frac{\nabla f + det(\nabla f)(\nabla f)^*}{\sqrt{1 + S(f)}}\right) = (0, 0), \tag{3.2}$$

where for any C^2 function $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$,

$$\nabla\varphi = \begin{pmatrix} \varphi_x^1 & \varphi_x^2 \\ \varphi_y^1 & \varphi_y^2 \end{pmatrix}; \quad (\nabla\varphi)^* = \begin{pmatrix} \varphi_y^2 & -\varphi_y^1 \\ -\varphi_x^2 & \varphi_x^1 \end{pmatrix}, \tag{3.3}$$

and

$$S(\varphi) = |\nabla(\varphi)|^2 + (det\nabla\varphi)^2. \tag{3.4}$$

Now suppose that $\mu = \max\{||\gamma||_{L^\infty(\partial B)}, ||D\gamma||_{L^\infty(\partial B)}\}$ is small, then by (3.1), $||f||_\infty \leqslant \mu$ is small. Then the following proposition states that $|\nabla f|, |\nabla^2 f|, |\nabla^3 f|$ are also small in a neighborhood of 0, and are controlled by μ.

The following result is well known. So we will give the proof in detail, but only a brief brief explanation, as well as some references.

Proposition 3.1. — *There exists $\mu_0 > 0$, such that for any $\mu < \mu_0$, there exists a constant $C(\mu)$, with $\lim_{\mu\to 0} C(\mu) = 0$, such that if f is a minimal graph on $B(0,1)$, with*

$$\max\{||f|_{\partial B(0,1)}||_\infty, ||Df|_{\partial B(0,1)}||_\infty\} \leqslant \mu, \tag{3.5}$$

then

$$\max_{0\leqslant i\leqslant 3} ||\nabla^i f||_{L^\infty(B(0,\frac{3}{4}))} \leqslant C(\mu). \tag{3.6}$$

For the proof, first we apply Allard's regularity theorem ([1]) on stationary varifolds to get the initial estimate for ∇f:

Theorem 3.2 ([1, Regularity theorem, §8]). — *Suppose $2 \leqslant d < p < \infty$, $q = \frac{p}{p-1}$. Corresponding to every $\epsilon \in]0,1[$ there is $\eta > 0$ with the following property:*

Suppose $0 < R < \infty$, $0 < \lambda < \infty$, $V \in \mathbb{V}_d(\mathbb{R}^n)$, $a \in spt||V||$ and

1. *$\theta^d(||V||, x) \geqslant \lambda$ for $||V||$ almost all $x \in B(a,R)$;*
2. *$||V||B(a,R) \leqslant (1+\eta)\lambda\alpha(d)R^d$;*
3. *$\delta V(g) \leqslant \eta\lambda^{\frac{1}{p}} R^{\frac{d}{p-1}} (\int |g|^q \lambda ||V||)^{\frac{1}{q}}$ whenever $g \in \mathfrak{X}(\mathbb{R}^n)$ and $spt\ g \subset B(a,R)$.*

Then there are $T \in G(n,d)$ and a continuously differentiable function $F : T \to \mathbb{R}^n$, such that $\pi_T \circ F = 1_T$,

$$||DF(y) - DF(z)|| \leqslant \epsilon(|y-z|/R)^{1-\frac{d}{p}} \text{ whenever } y, z \in T, \tag{3.7}$$

and

$$B(a,(1-\epsilon)R) \cap spt||V|| = B(a,(1-\epsilon)R) \cap \text{ image } F. \tag{3.8}$$

Minimal surfaces are naturally stationary varifolds. We apply Theorem 3.2 to our set Σ_f, on taking $\lambda = 1, a = (0, f(0)), R = 1$, then (1) and (3) are automatically true; for (2), we just apply a isoperimetric inequality for minimal surface (cf. [4]), and get

$$H^2(\Sigma_f \cap B(a,R)) \leqslant H^2(\Sigma_f) \leqslant (1+\mu^2)^2\pi. \tag{3.9}$$

Hence we can take μ small enough such that (2) holds for some η, such that (3.7) and (3.8) are true for some ϵ small, which give us that

$$||f||_{C^{1,\sigma}(B(0,\frac{8}{9}))} \leqslant C_1(\mu), \tag{3.10}$$

with $\lim_{\mu\to 0} C_1(\mu) = 0$.

Now we have the $C^{1,\alpha}$ estimate of our minimal graph Σ_f. Then (3.6) follows naturally from the standard bootstrap method, which gives an control on higher order derivatives by lower derivatives of solutions of elliptic systems, in particular minimal surfaces. See e.g. [15], the treat of non linear second order systems in Section 9, with corresponding estimates in Section 4, or [9, 5.2.14, 5.2.15] for more general settings.

4. Estimates for perturbations around a minimal graph

Denote by $B = B(0,1) \cap \mathbb{R}^2$ the unit disc in $\mathbb{R}^2$. Let $q \in B(0, \frac{1}{100})$, and set $B_r = B(q,r)$ for $r > 0$. Fix any ϵ and l less than 10^{-4}, let $\mu < 10^{-4}$ be small. (Here in this section the three are independent; in the next section, l will be chosen first, and then ϵ will depend on l, and both will be fixed at the beginning, while μ will be supposed to be much smaller than these two, and will be determined later.) Let f be a function from $\overline{B}$ to $\mathbb{R}^2$ whose graph $\{(x, f(x)); x \in \overline{B}\} \subset \mathbb{R}^4$ is a minimal submanifold in $\mathbb{R}^4$, with $||f|_{\partial B}||_{C^1} \leqslant \mu$. Let h be a C^1 function from $A_r := \overline{B}\backslash B_r$ to $\mathbb{R}^2$ with $h|_{\partial B} = 0$, Lip $h \leqslant l$, and there exists a vector $M \in \mathbb{R}^2$ such that for any $x \in \partial B_r$, $|h(x) - M| \leqslant \epsilon r$. Denote by Σ_f and Σ_{f+h} the graphs of f and $f + h$ respectively on the annulus A_r.

Proposition 4.1. — *Take all the notations and assumptions above, then*

$$H^2(\Sigma_{f+h}) - H^2(\Sigma_f) \geqslant \frac{1}{4}\int_{A_r} |\nabla h|^2 - Cr^2(\mu + \mu\epsilon + C_0(\mu)), \tag{4.1}$$

where $\lim_{\mu\to 0} C_0(\mu) = 0$.

Proof. — Now let us compare Σ_{f+h} and Σ_f above A_r. We have

$$(4.2)\quad \begin{aligned} H^2(\Sigma_{f+h}) - H^2(\Sigma_f) &= \int_{A_r} \sqrt{1+S(f+h)} - \sqrt{1+S(f)} \\ &= \int_{A_r} \sqrt{1+S(f)}\Big(\sqrt{\frac{1+S(f+h)}{1+S(f)}} - 1\Big) \\ &= \int_{A_r} \sqrt{1+S(f)}\Big(\sqrt{1+\frac{S(f+h)-S(f)}{1+S(f)}} - 1\Big). \end{aligned}$$

But

$$(4.3)\quad \begin{aligned} &S(f+h) - S(f) \\ &\quad = [|\nabla(f+h)|^2 - |\nabla f|^2] + [(\det \nabla(f+h))^2 - (\det \nabla f)^2] \\ &\quad = [2<\nabla f, \nabla h> + |\nabla h|^2] + [<(\nabla f)^*, \nabla h> + \det \nabla h)] \\ &\qquad\qquad \cdot[2 \det \nabla f + \det \nabla h + <(\nabla f)^*, \nabla h>]. \end{aligned}$$

Notice that $|\nabla f| < 2\mu$, $|(\nabla f)^*| < 2\mu$ is small, and $|\det \nabla f| \leqslant |\nabla f|^2$, $|\det \nabla h| \leqslant |\nabla h|^2$, therefore $|S(f+h) - S(f)| < 1$ since $|\nabla h| < l$ is small. But $S(f) > 0$, hence $|\frac{S(f+h)-S(f)}{1+S(f)}| < 1$. For any $|x| < 1$ we have

$$(4.4)\quad 1 + x = (1+\frac{x}{2})^2 - \frac{x^2}{4} \geqslant (1+\frac{x}{2}-\frac{x^2}{4})^2,$$

hence

$$(4.5)\quad \begin{aligned} &\sqrt{1+\frac{S(f+h)-S(f)}{1+S(f)}} \\ &\qquad \geqslant 1 + \frac{1}{2}\frac{S(f+h)-S(f)}{1+S(f)} - \frac{1}{4}\Big(\frac{S(f+h)-S(f)}{1+S(f)}\Big)^2, \end{aligned}$$

which gives

$$(4.6)\quad \begin{aligned} &H^2(\Sigma_{f+h}) - H^2(\Sigma_f) \\ &\quad \geqslant \int_{A_r} \sqrt{1+S(f)}\Big(\frac{1}{2}\frac{S(f+h)-S(f)}{1+S(f)} - \frac{1}{4}\Big(\frac{S(f+h)-S(f)}{1+S(f)}\Big)^2\Big) \\ &\quad = \frac{1}{2}\int_{A_r} \frac{S(f+h)-S(f)}{\sqrt{1+S(f)}} - \frac{1}{4}\int_{A_r} \frac{(S(f+h)-S(f))^2}{(1+S(f))^{\frac{3}{2}}}. \end{aligned}$$

For the first term, by (4.3),

$$
\begin{aligned}
&\frac{1}{2}\int_{A_r}\frac{S(f+h)-S(f)}{\sqrt{1+S(f)}}\\
&=\frac{1}{2}\int_{A_r}\frac{2<\nabla f,\nabla h>+|\nabla h|^2+2\det\nabla f<(\nabla f)^*,\nabla h>}{\sqrt{1+S(f)}}\\
&\quad+\frac{1}{2}\int_{A_r}\frac{2<(\nabla f)^*,\nabla h>\det\nabla h+<(\nabla f)^*,\nabla h>^2+2\det\nabla h\det\nabla f+|\det\nabla h|^2}{\sqrt{1+S(f)}}\\
&\geqslant\int_{A_r}\frac{<\nabla f,\nabla h>+\frac{1}{2}|\nabla h|^2+\det\nabla f<(\nabla f)^*,\nabla h>}{\sqrt{1+S(f)}}-(2\mu+l^2)\int_{A_r}|\nabla h|^2
\end{aligned}
\tag{4.7}
$$

But $S(f)\leqslant 5\mu^2$, hence $\frac{1}{1+S(f)}\geqslant\frac{8}{9}$, hence we have

$$
\begin{aligned}
\frac{1}{2}\int_{A_r}&\frac{S(f+h)-S(f)}{\sqrt{1+S(f)}}\\
&\geqslant\int_{A_r}<\frac{\nabla f+\det\nabla f(\nabla f)^*}{\sqrt{1+S(f)}},\nabla h>+\frac{1}{3}\int_{A_r}|\nabla h|^2.
\end{aligned}
\tag{4.8}
$$

By (3.2), and the hypothesis that $h|_{\partial B}=0$, we have

$$
\begin{aligned}
&\int_{A_r}<\frac{\nabla f+\det\nabla f(\nabla f)^*}{\sqrt{1+S(f)}},\nabla h>\\
&\quad=\int_{\partial A_r}<h,[\vec{n}\cdot\frac{\nabla f+\det\nabla f(\nabla f)^*}{\sqrt{1+S(f)}}]>\\
&\quad\quad-\int_{A_r}<div(\frac{\nabla f+\det\nabla f(\nabla f)^*}{\sqrt{1+S(f)}}),h>\\
&\quad=-\int_{\partial B_r}<h,[\vec{n}\cdot\frac{\nabla f+\det\nabla f(\nabla f)^*}{\sqrt{1+S(f)}}]>\\
&\quad\quad-\int_{\partial B_r}<(M+h-M),[\vec{n}\cdot\frac{\nabla f+\det\nabla f(\nabla f)^*}{\sqrt{1+S(f)}}]>\\
&\quad=-<M,\int_{\partial B_r}[\vec{n}\cdot\frac{\nabla f+\det\nabla f(\nabla f)^*}{\sqrt{1+S(f)}}]>\\
&\quad\quad+\int_{\partial B_r}<(M-h),[\vec{n}\cdot\frac{\nabla f+\det\nabla f(\nabla f)^*}{\sqrt{1+S(f)}}]>.
\end{aligned}
\tag{4.9}
$$

For the second term of (4.9), since $|M-h|\leqslant\epsilon r$, $\mathrm{Lip} f\leqslant\mu$, and $|\det\nabla f|\leqslant 2|\nabla f|^2\leqslant 2\mu^2\leqslant\mu$ since μ is small, we have

$$
\left|\int_{\partial B_r}<(M-h),[\vec{n}\cdot\frac{\nabla f+\det\nabla f(\nabla f)^*}{\sqrt{1+S(f)}}]>\right|\leqslant\int_{\partial B_r}\epsilon r(2\mu)\leqslant 4\pi\mu\epsilon r^2.
\tag{4.10}
$$

For the first term of (4.9), first by Taylor expansion at the point 0, we have, for any $x \in \partial B_r$,

$$\nabla f(x) = \nabla f(0) + x \cdot \nabla^2 f(0) + o_1(r), \tag{4.11}$$

$$(\nabla f)^*(x) = (\nabla f)^*(0) + x \cdot \nabla(\nabla)^* f(0) + o_2(r), \tag{4.12}$$

$$\det(\nabla f)(x) = \det(\nabla f)(0) + x \cdot \nabla \det(\nabla f)(0) + o_3(r), \tag{4.13}$$

$$\frac{1}{\sqrt{1+S(f)}}(x) = \frac{1}{\sqrt{1+S(f)}}(0) + x \cdot \nabla(\frac{1}{\sqrt{1+S(f)}})(0) + o_4(r) \tag{4.14}$$

where $|o_1(r)| \leqslant r^2 ||\nabla^3 f||_{L^\infty(B(0,r))}$, $|o_2(r)| \leqslant r^2 ||\nabla^3 f||_{L^\infty(B(0,r))}$, $|o_3(r)| \leqslant r^2 ||\nabla^2 \det(\nabla f)||_{L^\infty(B(0,r))}$, $|o_4(r)| \leqslant r^2 ||\nabla^2(\frac{1}{\sqrt{1+S(f)}})||_{L^\infty(B(0,r))}$.

Hence we have

$$\begin{aligned}
&\frac{\nabla f + \det \nabla f (\nabla f)^*}{\sqrt{1+S(f)}} \\
&= \{\nabla f(0) + x \cdot \nabla^2 f(0) + o_1(r) + [\det(\nabla f)(0) + x \cdot \nabla \det(\nabla f)(0) + o_3(r)] \\
&\qquad [(\nabla f)^*(0) + x \cdot \nabla(\nabla)^* f(0) + o_2(r)]\} \\
&\quad [\frac{1}{\sqrt{1+S(f)}}(0) + x \cdot \nabla(\frac{1}{\sqrt{1+S(f)}})(0) + o_4(r)] \\
&= \{[\nabla f(0) + \det(\nabla f)(0)(\nabla f)^*(0)] + x \cdot [\nabla^2 f(0) + \nabla \det(\nabla f)(0)(\nabla f)^*(0) \\
&\qquad + \det(\nabla f)(0) \nabla(\nabla)^* f(0)] + o(r)\} \\
&\quad [\frac{1}{\sqrt{1+S(f)}}(0) + x \cdot \nabla(\frac{1}{\sqrt{1+S(f)}})(0) + o(r)] \\
&= [\nabla f(0) + \det(\nabla f)(0)(\nabla f)^*(0)] \frac{1}{\sqrt{1+S(f)}}(0) \\
&\quad + x \cdot \frac{1}{\sqrt{1+S(f)}}(0)[\nabla^2 f(0) + \nabla \det(\nabla f)(0)(\nabla f)^*(0) \\
&\qquad + \det(\nabla f)(0) \nabla(\nabla)^* f(0)] \\
&\quad + [\nabla f(0) + \det(\nabla f)(0)(\nabla f)^*(0)][x \cdot \nabla(\frac{1}{\sqrt{1+S(f)}})(0)] + o(r),
\end{aligned} \tag{4.15}$$

where all the $o(r)$ in (4.15) satisfied that $|o(r)| \leqslant C_0 r^2$, where

$$C_0 = C(||\nabla f||_{L^\infty B(0,r)}, ||\nabla^2 f||_{L^\infty B(0,r)}, ||\nabla^3 f||_{L^\infty B(0,r)}) \tag{4.16}$$

tends to 0 as $||\nabla f||_{L^\infty B(0,r)}, ||\nabla^2 f||_{L^\infty B(0,r)}, ||\nabla^3 f||_{L^\infty B(0,r)}$ tend to 0.

Therefore,

$$(4.17)\quad |-<M,\int_{\partial B_r}[\vec{n}\cdot\frac{\nabla f+\det\nabla f(\nabla f)^*}{\sqrt{1+S(f)}}]>|$$
$$\leqslant |<M,\int_{\partial B_r}[\vec{n}\cdot[\nabla f(0)+\det(\nabla f)(0)(\nabla f)^*(0)]\frac{1}{\sqrt{1+S(f)}}(0)]>|$$
$$+|<M,\int_{\partial B_r}[\vec{n}\cdot(x\cdot\frac{1}{\sqrt{1+S(f)}}(0)[\nabla^2 f(0)+\nabla\det(\nabla f)(0)(\nabla f)^*(0)$$
$$+\det(\nabla f)(0)\nabla(\nabla)^* f(0)])]>|$$
$$+|<M,\int_{\partial B_r}\{\vec{n}\cdot[\nabla f(0)+\det(\nabla f)(0)(\nabla f)^*(0)][x\cdot\nabla(\frac{1}{\sqrt{1+S(f)}})(0)]\}>|$$
$$+|<M,\int_{\partial B_r}o(r)>|.$$

For the first term of (4.17), since $[\nabla f(0)+\det(\nabla f)(0)(\nabla f)^*(0)]\frac{1}{\sqrt{1+S(f)}}(0)$ is a constant matrix, which we denote by V, and hence we have

$$(4.18)\quad <M,\int_{\partial B_r}\vec{n}\cdot[\nabla f(0)+\det(\nabla f)(0)(\nabla f)^*(0)]\frac{1}{\sqrt{1+S(f)}}(0)>$$
$$=<M,(\int_{\partial B_r}\vec{n})\cdot V>=0$$

because $\int_{\partial B_r}\vec{n}=0$.

For the second and third term of (4.17), notice that $|x|=r$, $\nabla f\leqslant\mu$, hence their sum is less than

$$(4.19)\quad C\mu r^2+C|\nabla^2 f(0)|r^2\leqslant(C\mu+CC_0)r^2,$$

where C_0 is as in (4.16) and C does not depend on μ,ϵ.

For the last, by the previous control on $o(r)$, this term is less than $C_0 r^3$.

Altogether we have

$$(4.20)\quad |-<M,\int_{\partial B_r}[\vec{n}\cdot\frac{\nabla f+\det\nabla f(\nabla f)^*}{\sqrt{1+S(f)}}]>|\leqslant Cr^2(\mu+\ C_0).$$

Combining with (4.10) and (4.8), we have

$$(4.21)\quad \frac{1}{2}\int_{A_r}\frac{S(f+h)-S(f)}{\sqrt{1+S(f)}}\geqslant\frac{1}{3}\int_{A_r}|\nabla h|^2-Cr^2(\mu+\mu\epsilon+\ C_0),$$

where C does not depend on μ,l and ϵ.

Recall that this is the estimation for the first term of the last line in (4.6). Now we treat its second term.

By (4.3), we have

$$\begin{aligned}(4.22)\quad &|S(f+h)-S(f)|\\ &=|[2<\nabla f,\nabla h>+|\nabla h|^2]\\ &\quad+[<(\nabla f)^*,\nabla h>+\det\nabla h)][2\det\nabla f+\det\nabla h+<(\nabla f)^*,\nabla h>]|\\ &\leqslant 2|\nabla f||\nabla h|+|\nabla h|^2+[|(\nabla f)^*||\nabla h|+|\nabla h|^2|][2|\nabla f|^2+|\nabla h|^2+|(\nabla f)^*||\nabla h|]\\ &\leqslant C(|\nabla f||\nabla h|+|\nabla h|^2)\leqslant C\mu|\nabla h|+C|\nabla h|^2,\end{aligned}$$

therefore the second term of (4.6) verifies

$$\begin{aligned}-\frac{1}{4}\int_{A_r}\frac{(S(f+h)-S(f))^2}{(1+S(f))^{\frac{3}{2}}}&\geqslant-\frac{1}{4}\int_{A_r}(S(f+h)-S(f))^2\\ &\geqslant-\frac{1}{4}\int_{A_r}(C\mu|\nabla h|+C|\nabla h|^2)\\ &\geqslant -C(\mu^2+||\nabla h||_\infty^2)\int_{A_r}|\nabla h|^2.\end{aligned}\tag{4.23}$$

On combining (4.6), (4.21) and (4.23) we get

$$\begin{aligned}&H^2(\Sigma_{f+h})-H^2(\Sigma_f)\\ &\geqslant\frac{1}{3}\int_{A_r}|\nabla h|^2-Cr^2(\mu+\mu\epsilon+C_0)-C(\mu^2+||\nabla h||_\infty^2)\int_{A_r}|\nabla h|^2\\ &\geqslant(\frac{1}{3}-C\mu^2-Cl^2)\int_{A_r}|\nabla h|^2-Cr^2(\mu+\mu\epsilon+C_0).\end{aligned}\tag{4.24}$$

But Lip $h<l$ is small, hence we have

$$H^2(\Sigma_{f+h})-H^2(\Sigma_f)\geqslant\frac{1}{4}\int_{A_r}|\nabla h|^2-Cr^2(\mu+\mu\epsilon+C_0).\tag{4.25}$$

Now we apply Proposition 3.1, and get that when $r<\frac{3}{4}$ and μ is small enough,

$$\begin{aligned}C_0&=C_0(||\nabla f||_{L^\infty B(0,r)},||\nabla^2 f||_{L^\infty B(0,r)},||\nabla^3 f||_{L^\infty B(0,r)})\\ &=C_0(C(\mu))=C_0(\mu),\end{aligned}\tag{4.26}$$

with $\lim_{\mu\to 0}C_0(\mu)=0$. Thus we have

$$H^2(\Sigma_{f+h})-H^2(\Sigma_f)\geqslant\frac{1}{4}\int_{A_r}|\nabla h|^2-Cr^2(\mu+\mu\epsilon+C_0(\mu)).\tag{4.27}$$

This completes the proof of Proposition 4.1. □

5. Conclusion

Now return to our set E. Recall that α is a pair of angles larger than $\theta_1' > \frac{\pi}{3}$. E is a reduced closed set that is minimal in $B(0,1)$, which contains no $2\mathbb{P}$ type point in $B(0, \frac{1}{100})$.

Set $l = 10^{-3}$, and suppose that $d_{0,1}^{\alpha} < \mu < \min\{\frac{\epsilon_0}{10}, \frac{l}{2}\}$, μ is to be decided later.

We apply Proposition 2.2 to E, with $\epsilon' = \min\{\epsilon_{\frac{l}{2}}, 10^{-4}\}$, (where $\epsilon_{\frac{l}{2}}$ corresponds to $\frac{l}{2}$ in Proposition 2.4), and get our o_E and r_E. Then $r_E < \frac{1}{4}$.

Let γ^i, g^i, as in Section 3. Suppose that

$$||\gamma^i||_{C^1} \leqslant \mu, i = 1, 2. \tag{5.1}$$

By [16, Theorem 7.2], for each i there exists a function $f^i : \overline{B}(0, \frac{1}{2}) \cap P_\alpha^i \to {P_\alpha^i}^\perp$, whose graphs $\Sigma^i = \Sigma_{f^i} = \{(x, f(x)) : x \in \overline{B}(0, \frac{1}{2}) \cap P_\alpha^i\} \subset \mathbb{R}^4$ are minimal surfaces. Denote by $B^i(x, r) = B(x, r) \cap P_\alpha^i$.

On the other hand, we want to show the part of E in the annulus $D_\alpha(o_E, r_E) \backslash D_\alpha(o_E, \frac{1}{4} r_E)$ is far from any translation of P_α. Recall that Proposition 2.2 says that E is $\epsilon' r_E$ far from any translation of P_α in the ball $D_\alpha(o_E, r_E)$. So for having a relatively big distance in the annulus, we simply use a compactness argument, and can get the following proposition. (See [10] for the proof).

Proposition 5.1 (cf. [10, Corollary 8.24]). — *For every $\epsilon > 0$, there exists $0 < \delta < \epsilon$, and $0 < \theta_0 < \frac{\pi}{2}$, which do not depend on ϵ, with the following properties. If $\theta_0 < \theta < \frac{\pi}{2}$, and if E is minimal in $D_\theta(0,1)$ and is δ near P_θ in $D_\theta(0,1) \backslash D_\theta(0, \frac{1}{4})$, and moreover*

$$p_\theta^i(E) \supset P_\theta^i \cap B(0, \frac{3}{4}), \tag{5.2}$$

then E is ϵ near P_θ in $D_\theta(0,1)$.

Let δ' be the δ corresponding to ϵ' in Proposition 5.1, we know that E is not $\delta' r_E$ near any translation of P_α in $D_\alpha(o_E, r_E) \backslash D_\alpha(o_E, \frac{1}{4} r_E)$. On the other hand, by definition of o_E and r_E, we know that the ϵ'-process does not stop at the scale $2r_E$, thus by Proposition 2.4, $E \cap D_\alpha(o_E, r_E) \backslash D_\alpha(o_E, \frac{1}{4} r_E)$ is composed of two fine C^1 graphs G^1, G^2 of two functions $g^i, i = 1, 2$ on $P_\alpha^i \cap D_\alpha(o_E, r_E) \backslash D_\alpha(o_E, \frac{1}{4} r_E)$ respectively. Thus $G^1 \cup G^2$ is not $\delta' r_E$ near any translation of P_α, there exists $i = 1, 2$ such that G^i is not δ' near any translation of P_α^i in $D_\alpha(o_E, r_E) \backslash D_\alpha(o_E, \frac{1}{4} r_E)$. Suppose this is the case for $i = 1$.

Denote by $g = g^1, f = f^1$, and $h = g - f$. We want to apply Proposition 4.1 to f and h, with $B(q, r) = B^1(o_E, \frac{1}{4} r_E)$ (hence $q = o_E, r = \frac{1}{4} r_E$).

Recall that we have set $\epsilon' \leqslant \epsilon_{\frac{l}{2}}$, hence $|\nabla g|$ is smaller than $\frac{l}{2}$, which gives $|\nabla h| = |\nabla(g-f)|$ is smaller than $|\nabla g| + |\nabla f| < \frac{l}{2} + \mu < l$ because μ is supposed to be less than $\frac{l}{2}$.

Also, by Proposition 2.2, G^1 is still $2\epsilon' r_E$ near some translation of P^1_α, hence there exists $M_g \in P^{1\perp}_\alpha$ such that $|g(x) - M_g| \leqslant 2\epsilon' r_E = 8\epsilon' r$. But f is μ-Lipschitz, hence there exists M_f such that $|f(x) - M_f| \leqslant C\mu r$ on $\partial B(q,r)$, which gives $|h - (M_g + M_f)| \leqslant 9\epsilon' r < 10^{-3} r$ on $\partial B(q,r)$, when μ is small.

Now we can apply Proposition 4.1, and get

(5.3)
$$\begin{aligned} H^2(G^1) - H^2(\Sigma^1 \backslash C^1(o_E, \frac{1}{4} r_E)) &= H^2(\Sigma_{f+h}) - H^2(\Sigma_f) \\ &\geqslant \frac{1}{4} \int_{A_r} |\nabla h|^2 - Cr^2(\mu + \epsilon' \mu + C_0(\mu)), \end{aligned}$$

with $A_r = B^1(0, \frac{1}{2}) \backslash B(q,r)$.

Now we want to estimate $\int_{A_r} |\nabla h|^2$. Recall that on the one hand $B^1(o_E, r_E) \backslash B^1(o_E, \frac{1}{4} r_E)$, the graph of g is $\delta' r_E$ far from any translation of P^1_α. On the other hand f is μ-Lipschitz, hence when μ is small, the graph of $h = g - f$ is $\frac{1}{2}\delta' r_E$ far from any translation of P^1_α.

Firstly we cite here two lemmas for estimating the Dirichlet's energy of our perturbation function h.

Lemma 5.2 (cf. [10, Corollary 7.23]). — *Let $r_0 > 0$, $q \in \mathbb{R}^2$ be such that $r_0 < \frac{1}{2} d(q, \partial B(0,1))$, suppose $u_0 \in C^1(\partial B(q, r_0) \cap \mathbb{R}^2, \mathbb{R})$, and denote by $m(u_0) = \frac{1}{2\pi r_0} \int_{\partial B(q,r_0)} u_0$ its average.*

Then for all $u \in C^1((\overline{B(0,1)} \backslash B(q, r_0)) \cap \mathbb{R}^2, \mathbb{R})$ that satisfies

$$u|_{\partial B(q,r_0)} = u_0 \tag{5.4}$$

we have

$$\int_{B(0,1)\backslash B(q,r_0)} |\nabla u|^2 \geqslant \frac{1}{4} r_0^{-1} \int_{\partial B(q,r_0)} |u_0 - m(u_0)|^2. \tag{5.5}$$

Lemma 5.3 (cf. [10, Corollary 7.36]). — *For all $0 < \epsilon < 1$, there exists $C = C(\epsilon) > 100$ such that if $0 < r_0 < 1$, $u \in C^1(\ B(0,1) \backslash B(0, r_0), \mathbb{R})$ and*

$$u|_{\partial B(0,r_0)} > \delta r_0 - \frac{\delta r_0}{C} \text{ and } u|_{\partial B(0,1)} < \frac{\delta r_0}{C} \tag{5.6}$$

then

$$\int_{B(0,1)\backslash B(0,r_0)} |\nabla u|^2 \geqslant \epsilon \frac{2\pi \delta^2 r_0^2}{|\log r_0|}. \tag{5.7}$$

Then denote by $P = P^1_\alpha$ for short. Denote by $D = D_\alpha$. Then h is a map from P to $P^\perp$, and is therefore from $\mathbb{R}^2$ to $\mathbb{R}^2$. Write $h = (\varphi_1, \varphi_2)$, where $\varphi_i : \mathbb{R}^2 \to \mathbb{R}$. Then since the graph of h is $\frac{1}{2}\delta' r_E$ far from all translation of P, there exists $j \in \{1, 2\}$ such that

$$\sup_{x,y\in P\cap D(o_E,r_E)\setminus D(o_E,\frac{1}{4}r_E)} |\varphi_j(x) - \varphi_j(y)| \geqslant \frac{1}{4} r_E \delta'. \tag{5.8}$$

Suppose this is true for $j = 1$. Denote by

$$K = \{(z, \varphi_1(z)) : z \in (D(0, \frac{1}{2})\setminus D(o_k, \frac{1}{4} r_E)) \cap P\}, \tag{5.9}$$

then

$$K \text{ is the orthogonal projection of } G^1 \cap D(0, \frac{1}{2}) \text{ on a 3-dimensional subspace of } \mathbb{R}^4. \tag{5.10}$$

For $\frac{1}{4} r_E \leqslant s \leqslant r_E$, define

$$\Gamma_s = K \cap p^{-1}(\partial D(o_E, s) \cap P) = \{(x, \varphi_1(x)) | x \in \partial D(o_E, s) \cap P\} \tag{5.11}$$

the graph of φ_1 on $\partial D(o_E, s) \cap P$.

We know that in in $D(o_E, r_E)\setminus D(o_E, \frac{1}{4} r_E)$, the graph of φ_1 is $\frac{1}{4}\delta' r_E$ far from P; then there are two cases:

Case 1 — there exists $t \in [\frac{1}{4} r_E, r_E]$ such that

$$\sup_{x,y\in\Gamma_t} \{|\varphi_1(x) - \varphi_1(y)|\} \geqslant \frac{\delta'}{C} r_E, \tag{5.12}$$

where $C = 4C(\frac{1}{2})$ is the constant of Lemma 5.3.

Then there exists $a, b \in \Gamma_t$ such that $|\varphi_1(a) - \varphi_1(b)| > \frac{\delta'}{C} r_E \geqslant \frac{\delta'}{C} t$. Since $||\nabla\varphi_1||_\infty \leqslant ||\nabla\varphi||_\infty < 1$, we have

$$\int_{\Gamma_t} |\varphi_1 - m(\varphi_1)|^2 \geqslant \frac{t^3 \delta'^3}{4C^3} = (\frac{4}{3} t\delta')^3 (\frac{27}{4^4 C^3}). \tag{5.13}$$

Now in $D(0, \frac{1}{2})$ we have $d(0, o_E) < 6\epsilon' \leqslant 10\epsilon' \cdot \frac{1}{2}$, and $s < r_E < \frac{1}{8} < \frac{1}{2} \times \frac{1}{2}$, therefore we can apply Lemma 5.2 and obtain

$$\int_{(D(0,\frac{1}{2})\setminus D(o_E,t))\cap P} |\nabla\varphi_1|^2 \geqslant C(\delta') t^2 \geqslant C_1(\delta') r_E^2. \tag{5.14}$$

Case 2 — for all $\frac{1}{4} r_E \leqslant s \leqslant r_E$,

$$\sup_{x,y\in\Gamma_s} \{|\varphi_1(x) - \varphi_1(y)|\} \leqslant \frac{\delta'}{C} r_E. \tag{5.15}$$

However, since

$$\begin{aligned}(5.16)\quad \frac{1}{2}r_E\delta' &\leqslant \sup\{|\varphi_1(x)-\varphi_2(y)| : x,y\in P\cap D(o_E,r_E)\backslash D(o_E,\frac{1}{4}r_E)\}\\ &= \sup\{|\varphi_1(x)-\varphi_2(y)| : s,s'\in[\frac{1}{4}r_E,r_E], x\in\Gamma_s, y\in\Gamma_{s'}\},\end{aligned}$$

there exist $\frac{1}{4}r_E \leqslant t < t' \leqslant r_E$ such that

$$(5.17)\qquad \sup_{x\in\Gamma_t, y\in\Gamma_{t'}}\{|\varphi_1(x)-\varphi_1(y)|\} \geqslant \frac{1}{2}r_E\delta'.$$

Fix t and t', and without loss of generality, suppose that

$$(5.18)\qquad \sup_{x\in\Gamma_t, y\in\Gamma_{t'}}\{\varphi_1(x)-\varphi_1(y)\} \geqslant \frac{1}{4}r_E\delta'.$$

Then

$$\begin{aligned}(5.19)\quad \inf_{x\in\Gamma_t}\varphi_1(x) - \sup_{x\in\Gamma_{t'}}\varphi_1(x) &\geqslant \frac{1}{4}r_E\delta' - 2\frac{\delta'}{C}r_E = (1-\frac{2}{C(\frac{1}{2})})\frac{\delta'}{4}r_E\\ &\geqslant (1-\frac{2}{C(\frac{1}{2})})\frac{\delta'}{2}t'\end{aligned}$$

because $C = 4C(\frac{1}{2})$.

Now look at what happens in the ball $D(o_E,t')\cap P$. Apply Lemma 5.3 to the scale t', we get

$$(5.20)\qquad \int_{(D(o_E,t')\backslash D(o_E,t))\cap P}|\nabla\varphi_1|^2 \geqslant C(\delta',\frac{1}{2})\frac{\pi(\frac{\delta'}{2})^2t'^2}{\log\frac{t'}{t}}.$$

Then since $\frac{t'}{t}\leqslant 4, t'>t>\frac{1}{4}r_E$, we have

$$(5.21)\qquad \int_{((D(o_E,t')\backslash D(o_E,t))\cap P}|\nabla\varphi_1|^2 \geqslant C_2(\delta')r_E^2.$$

So in both cases, there exists a constant $C=C_5(\delta')=\min\{C_1(\delta'),C_2(\delta')\}$, which depends only on δ', such that

$$(5.22)\qquad \int_{(D(0,\frac{1}{2})\backslash D(o_E,t_E))\cap P}|\nabla\varphi_1|^2 \geqslant C_5(\delta')r_E^2.$$

On the other hand, since $|\nabla\varphi_1|\leqslant|\nabla h|<1$, we have

$$(5.23)\qquad \int_{A_r}|\nabla h|^2 = \int_{(D(0,\frac{1}{2})\backslash D(o_E,t_E))\cap P}|\nabla h|^2 \geqslant C_5(\delta')r_E^2.$$

Thus by (5.3),

$$(5.24)\quad H^2(G^1)-H^2(\Sigma^1\backslash C^1(o_E,\frac{1}{4}r_E)) \geqslant C_5(\delta')r_E^2-Cr_E^2(\mu+\epsilon'\mu+C_0(\mu)).$$

We apply also Proposition 4.1 to $i=2$, where all the verifications for g^2, f^2, $h^2=g^2-f^2$ are similar to that of g^1, f^1, g^1. Hence we have

$$
\begin{aligned}
&H^2(G^2)-H^2(\Sigma^2\backslash C^2(o_E,\frac{1}{4}r_E))\\
(5.25)\qquad &\geqslant \frac{1}{4}\int_{P_\alpha^2\cap D(0,\frac{1}{2})\backslash D(o_E,\frac{1}{4}r_E)}|\nabla h|^2 - Cr_E^2(\mu+\epsilon'\mu+C_0(\mu))\\
&\geqslant -Cr_E^2(\mu+\epsilon'\mu+C_0(\mu)).
\end{aligned}
$$

Now we still have to estimate the part inside $D(o_E,\frac{1}{4}r_E)$. For this purpose we need the following lemma.

Lemma 5.4 (cf. [10, Corollary 2.45]). — *Suppose $\xi>0$ is such that $\arccos(\xi/2)\leqslant\alpha_1\leqslant\alpha_2$, and P^1, P^2 are two planes with characteristic angles (α_1,α_2). Denote by p^i the orthogonal projection on $P^i, i=1,2$. Then if E is a closed 2- rectifiable set satisfying $p^i(E)\supset B(0,1)\cap P^i$, we have*

$$
(5.26)\qquad H^2(E)\geqslant\frac{2\pi}{1+\xi}.
$$

We apply Lemma 5.4 to the part $E\cap D_\alpha(o_E,\frac{1}{4}r_E)$, and by Proposition 2.7, we get

$$
(5.27)\qquad H^2(E\cap D_\alpha(o_E,\frac{1}{4}r_E))\geqslant 2\pi(\frac{1}{4}r_E)^2\frac{1}{1+2\cos\theta_1'}.
$$

On the other hand, notice that Lip $f^1<C_0(\mu)$ and Lip $f^2<C_0(\mu)$, we have

$$
\begin{aligned}
H^2(\Sigma^i\cap D_\alpha(o_E,\frac{1}{4}r_E)) &= \int_{P_\alpha^i\cap D_\alpha(o_E,\frac{1}{4}r_E)}\sqrt{1+S(f)}\\
&\leqslant \int_{P_\alpha^i\cap D_\alpha(o_E,\frac{1}{4}r_E)}\sqrt{1+C_0(\mu)^2+C_0(\mu)^4}\\
(5.28)\qquad &\leqslant \int_{P_\alpha^i\cap D_\alpha(o_E,\frac{1}{4}r_E)} 1+\frac{C_0(\mu)^2+C_0(\mu)^4}{2}\\
&= \pi(\frac{1}{4}r_E)^2(1+\frac{C_0(\mu)^2+C_0(\mu)^4}{2}),
\end{aligned}
$$

therefore

$$
(5.29)\qquad H^2(\Sigma\cap D_\alpha(o_E,\frac{1}{4}r_E))\leqslant 2\pi(\frac{1}{4}r_E)^2(1+\frac{C_0(\mu)^2+C_0(\mu)^4}{2}).
$$

Thus

$$H^2(\Sigma\cap D_\alpha(o_E,\frac{1}{4}r_E))-H^2(E\cap D_\alpha(o_E,\frac{1}{4}r_E))\leqslant 2\pi(\frac{1}{4}r_E)^2(\frac{C_0(\mu)^2+C_0(\mu)^4}{2}+2\cos\alpha_1). \tag{5.30}$$

We combine (5.30), (5.25) and (5.24), and get

$$\begin{aligned} &H^2(E\cap D(0,\frac{1}{2}))-H^2(\Sigma)\\ &\quad=\sum_{i=1,2}[H^2(G^i-H^2(\Sigma^1\backslash C^1(o_E,\frac{1}{4}r_E))]\\ &\qquad+[H^2(E\cap D_\alpha(o_E,r_E))-H^2(\Sigma\cap D_\alpha(o_E,r_E)]\\ &\quad\geqslant C_5(\delta')r_E^2-Cr_E^2(\mu+\epsilon'\mu+C_0(\mu))-Cr_E^2(\mu+\epsilon'\mu+C_0(\mu))\\ &\qquad-2\pi(\frac{1}{4}r_E)^2(\frac{C_0(\mu)^2+C_0(\mu)^4}{2}+2\cos\alpha_1). \end{aligned} \tag{5.31}$$

Notice that δ' is just a constant, depending on ϵ', where ϵ' is the parameter for the ϵ'-process, and guarantees the regularity for parts of minimal sets where the $\epsilon'-$process does not stop. Hence it does not depend on μ or α. Therefore when α is large enough and μ is small enough,

$$H^2(E\cap D_\alpha(0,\frac{1}{2}))-H^2(\Sigma)>0. \tag{5.32}$$

Recall that Σ contains a deformation of E in $D_\alpha(0,\frac{1}{2})$, hence (5.32) contradicts the fact that E is minimal.

This contradiction yields that there exists $\theta_1\in]0,\frac{\pi}{2}[$ and $\mu_0>0$ such that for any $\alpha>\theta_1$, if E is minimal in $B(0,1)$ with $d_{0,1}(E,P_\alpha)<\epsilon'$, and moreover (5.1) holds, then E contains a point of type $2\mathbb{P}$ in $B(0,\frac{1}{100})$.

Now for guarantee the condition (5.1), we apply Proposition 2.4 again. Set $\lambda=\epsilon_\mu$. Then when $d_{0,1}(E,P_\alpha)<\lambda$, our λ-process does not stop before step 1. Then by (2.18), the curves γ^i admits Lipschitz constants less than μ. Thus (5.1) holds.

Thus when $d_{0,1}(E,P_\alpha)\leqslant\lambda$, there exists a point of type $2\mathbb{P}$ in $B(0,\frac{1}{100})$. This completes the proof of Theorem 1.4. □

6. Global regularity and local C^1 regularity for minimal sets that are near $2\mathbb{P}$ type minimal cones

In this section we give two useful corollaries of Theorem 1.4, concerning global and local regularity for minimal sets that are near $2\mathbb{P}$ type minimal cones.

THEOREM 6.1. — *Let θ_1 be as in Theorem 1.4. Then for any $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_2 \geqslant \alpha_1 \geqslant \theta_1$, if E is a 2-dimensional reduced Almgren minimal set in $\mathbb{R}^4$ such that one blow-in limit of E at infinity is P_α (i.e., there exists a sequence of numbers $r_n \to \infty$, and the sequence of sets $r_n^{-1}(E)$ converge to P_α under the Hausdorff distance as $n \to \infty$), then E is a $\mathbb{P}_\alpha$ set.*

Proof. — By hypothesis, there exists $R > 0$ and a $\mathbb{P}_\alpha$ set P_α such that $d_{0,R}(E, P_\alpha) < \lambda$. Then by Theorem 1.4, there exists a $2\mathbb{P}$ type point $x \in E$. In particular, the density $\theta(x)$ of E at x is 2, which is equal to the density θ_∞ of E at infinity. By the monotonicity (cf. [6, Proposition 5.16]) of the density function $\theta_x(r) = r^{-d}H^d(E \cap B(x, r))$, it has to be constant for $r \in]0, \infty[$. By [6, Theorem 6.2], E is a minimal cone centered at x. As a result, $d_{x,r}(E, P_\alpha + x)$ is constant for $r \in]0, \infty[$, since $P_\alpha + x$ is also a cone centered at x. But by hypothesis, $d_{x,r}(E, P_\alpha + x) \to 0$ as $r \to \infty$, hence $d_{x,r}(E, P_\alpha + x) = 0$, which means that $E = P_\alpha + x$. □

THEOREM 6.2. — *Let θ_1 be as in Theorem 1.4. Then there exists a $\epsilon > 0$ such that for any $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_2 \geqslant \alpha_1 \geqslant \theta_1$, if E is a 2-dimensional reduced Almgren minimal set in $U \subset \mathbb{R}^4$, $B(x, 300r) \subset U$ ($r < 1$), and there is a reduced minimal cone $P_\alpha + x$ of type $\mathbb{P}_\alpha$ centered at x such that $d_{x,200r}(E, P_\alpha + x) \leqslant \epsilon$, then there exists C^1 diffeomorphism $\Phi : B(x, 2r) \to \Phi(B(x, 2r))$, such that $|\Phi(y) - y| \leqslant 10^{-2}r$ for $y \in B(x, 2r)$, and $E \cap B(x, r) = \Phi(P_\alpha + x) \cap B(x, r)$.*

To get this theorem as a corollary of Theorem 1.3, we need the following regularity theorem for minimal sets:

THEOREM 6.3 ([7, Corollary 12.25]). — *For each choice of $n \geqslant 3, C_1 \geqslant 1$, $\eta_1 > 0$ we can find $\beta > 0$ and $\epsilon_1 > 0$ such that the following holds. Let $U \subset \mathbb{R}^n$ be open and let $E \subset U$ be a reduced minimal set in U. Suppose that $x \in E, r_0 > 0$ is such that $B(x, 110r_0) \subset U$,*

$$\theta_x(110r_0) - \theta(x) \leqslant \epsilon_1, \tag{6.1}$$

and

$$d_{x,100r_0}(E, X) \leqslant \epsilon_1 \tag{6.2}$$

for some full length minimal cone (See the remark below) X centered at x such that

$$H^2(X \cap B(0,1)) \leqslant \theta(x) \tag{6.3}$$

and

(6.4) *X is a full length minimal cone,*

with constants $\eta_1 \leqslant \eta_0/10$ and $C_1 \geqslant 1$.

Then for $0 < r < r_0$ there is a $C^{1+\beta}$ diffeomorphism $\Phi : B(x, 2r) \to \Phi(B(x, 2r))$, such that $\Phi(x) = x, |\Phi(y) - y| < 10^{-2}r$ for $y \in B(x, 2r)$, and $E \cap B(x, r) = \Phi(X + x) \cap B(x, r)$.

Remark 6.4. — In particular, if X is a full-length minimal cone, then there exists $\epsilon_3 > 0$ such that for any minimal set E and any X-type point $x \in E$, if $d_{x,2r}(E, X + x) < \epsilon_3$, then we have the C^1 equivalence described in the theorem. Here the existence of an X type points guarantees the condition (6.6), and together with the following Lemma 6.5, (6.1) will also be satisfied automatically. The full length property (with constant C_1 and η_1) is defined in [7, Definition 2.10]. It is not necessary to come into details of the definition. In fact there is a stronger condition called "full-length because of angles" (FLBA, see [7] below (14.3)) with constant η_1. And there exists a constant C such that if X is a minimal cone that satisfies FLBA with constant η_1, then it is a full-length minimal cone with constant $C_1 = C, \eta_1$.

By [7, Lemma 14.4], planes in $\mathbb{R}^4$ satisfies FLBA with constant η_1 (and thus any constant smaller than η_1). [7, Lemma 14.40] tells us that a minimal cone X satisfies FLBA with constant η_1 if each of the connected components of $X \cap \partial B(0,1)$ satisfies it, and these connected components lie at distances at least $10\eta_1$ from each other. Hence we can take $\eta_2 < \eta_1$ small, such that for our family of unions of planes P_α, the two circles of $\mathbb{P}_\alpha \cap \partial B(0,1)$ are $10\eta_2$ far from each other. As a results, for $\alpha > \theta_1$, P_α is FLBA with constant η_2, and hence is full-length with constant η_2 and C.

Lemma 6.5 ([6] Lemma 16.43). — *For each small $\delta > 0$, there is a constant ϵ (that only depends on n and d), such that if E and F are reduced d-dimensional minimal sets in an open set $U \subset \mathbb{R}^n$, and if*

$$d_{x,10r/9}(E, F) < \epsilon, \tag{6.5}$$

then

$$H^d(E\cap B(x,r)) \leqslant H^d(F\cap B(x,(1+\delta)r)+\delta r^d. \tag{6.6}$$

Now let us prove Theorem 6.2

Proof of Theorem 6.2. — Let λ be the λ in Theorem 1.4. Let δ be small such that $10\delta<\epsilon_1$, where ϵ_1 is the one in Theorem 6.3. Let ϵ_4 be the $\epsilon(\delta)$ in Lemma 6.5 that corresponds to δ, with $n=4, d=2$.

Let $\epsilon=\min\{10^{-3}\lambda, \frac15\epsilon_1, \frac15\epsilon_4\}$. Then by Theorem 1.4, $d_{x,r}(E,P_\alpha)\leqslant 500d_{x,200r}(E,P_\alpha)\leqslant\frac12\lambda$ yields that there exists a point $x_0\in B(x,\frac{1}{100}r)$ of type $P_{\alpha'}$ for some angle α'. As a result, $\theta_{x_0}=2$, which is the density at origin of every union of two planes. And since $x_0\in B(x,\frac{1}{100}r)$, we have

$$d_{x_0,150r}(E,P_\alpha+x_0)<2\epsilon<\epsilon_4. \tag{6.7}$$

Then by Lemma 6.5,

$$H^2(E\cap B(x_0,110r))\leqslant H^2(P_\alpha\cap B(x_0,(1+\delta)110r)+\delta(110r)^2. \tag{6.8}$$

But P_α is a cone, and

$$H^2(P_\alpha\cap B(0,1))=\theta(x), \tag{6.9}$$

hence

$$H^2(E\cap B(x_0,110r))\leqslant\theta(x)[(1+\delta)110r]^2+\delta(110r)^2, \tag{6.10}$$

which yields

$$(110r)^{-2}H^2(E\cap B(x_0,110r))\leqslant\theta(x)[(1+\delta)]^2+\delta. \tag{6.11}$$

Hence

$$\theta(x,110r)-\theta(x)<8\delta<\epsilon_1. \tag{6.12}$$

On the other hand we also have

$$d_{x_0,150r}(E,P_\alpha+x_0)<2\epsilon<\epsilon_1. \tag{6.13}$$

Now by Remark 6.4, P_α is a full-length minimal cone. By (6.9), (6.12), (6.13), Theorem 6.3 applies, which yields the conclusion of Theorem 6.2. □

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