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## KLT SINGULARITIES OF HOROSPHERICAL PAIRS

by Boris PASQUIER (\*)

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ABSTRACT. — Let  $X$  be a horospherical  $G$ -variety and let  $D$  be an effective  $\mathbb{Q}$ -divisor of  $X$  that is stable under the action of a Borel subgroup  $B$  of  $G$  and such that  $D + K_X$  is  $\mathbb{Q}$ -Cartier. We prove, using Bott–Samelson resolutions, that the pair  $(X, D)$  is klt if and only if  $\lfloor D \rfloor = 0$ .

RÉSUMÉ. — Soient  $X$  une  $G$  variété horosphérique et  $D$  un  $\mathbb{Q}$ -diviseur de  $X$  stable sous l'action d'un sous-groupe de Borel  $B$  de  $G$  et tel que  $D + K_X$  est  $\mathbb{Q}$ -Cartier. Nous démontrons, en utilisant les résolutions de Bott–Samelson, que la paire  $(X, D)$  est klt si et seulement si  $\lfloor D \rfloor = 0$ .

### 1. Introduction

One of the biggest family of pairs we can consider in the log Minimal Model Program is the family of pairs with klt (Kawamata log terminal) singularities. Hence, it is useful to be able to characterize klt pairs. In this paper, we give a very simple characterization of klt pairs for horospherical varieties.

Let  $X$  be a normal algebraic variety over  $\mathbb{C}$  and let  $D$  be an effective  $\mathbb{Q}$ -divisor such that  $D + K_X$  is  $\mathbb{Q}$ -Cartier. If the pair  $(X, D)$  has klt singularities (see Definition 2.1) then  $\lfloor D \rfloor = 0$  (i.e.  $D = \sum_{D_i \text{ irreducible}} a_i D_i$  with  $a_i \in [0, 1[$ ). The inverse implication is false in general. In [1], V. Alexeev and M. Brion proved that, if  $X$  is a spherical  $G$ -variety and  $D$  is an effective  $\mathbb{Q}$ -divisor of  $X$  such that  $D + K_X$  is  $\mathbb{Q}$ -Cartier,  $\lfloor D \rfloor = 0$  and  $D = D_G + D_B$  where  $D_G$  is  $G$ -stable and  $D_B$  is stable under the action of a Borel subgroup  $B$  of  $G$ , then  $(X, D_G + D'_B)$  has klt singularities for general  $D'_B$  in  $|D_B|$ .

Here, we prove the following result.

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*Keywords:* klt pairs, flag varieties, horospherical varieties, Bott–Samelson resolutions.  
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(\*) I would like to thank Cédric Bonnafé for his shorter proof of Proposition 5.1.

**THEOREM 1.1.** — *Let  $X$  be a horospherical  $G$ -variety and  $D$  be an effective  $B$ -stable  $\mathbb{Q}$ -divisor of  $X$  such that  $D + K_X$  is  $\mathbb{Q}$ -Cartier and  $\lfloor D \rfloor = 0$ , then  $(X, D)$  has klt singularities.*

This result is even new in the particular case of flag varieties. Nevertheless, we can notice that S. Kumar and K. Schwede proved that for any Richardson variety  $X$ , there exists an effective  $\mathbb{Q}$ -divisor  $D$  of  $X$  (with  $\lfloor D \rfloor = 0$  and) such that  $(X, D)$  has klt singularities [6].

The strategy of the proof is the following. In Section 3, we recall the definitions and some properties of Bott–Samelson resolutions of any flag variety  $G/P$ . In particular, they are log resolutions and the klt singularity condition in the case of flag varieties becomes equivalent to some inequalities on the root systems of  $G$  and  $P \subset G$ , which we prove in Section 5. And in Section 4, we deduce the horospherical case from the case of flag varieties, using that any horospherical variety admits a desingularization that is a toric fibration over a flag variety (i.e. a fibration over a flag variety whose fiber is a smooth toric variety).

## 2. Notations and definitions

In all the paper, varieties are algebraic varieties over  $\mathbb{C}$ .

We first recall the definition of klt singularities.

**DEFINITION 2.1.** — *Let  $X$  be a normal variety and let  $D$  be an effective  $\mathbb{Q}$ -divisor such that  $K_X + D$  is  $\mathbb{Q}$ -Cartier. The pair  $(X, D)$  is said to be klt (Kawamata log terminal) if for any resolution  $f : V \rightarrow X$  of  $X$  such that  $K_V = f^*(K_X + D) + \sum_{i \in \mathcal{E}} a_i E_i$  where the  $E_i$ 's are distinct irreducible divisors, we have  $a_i > -1$  for any  $i \in \mathcal{E}$ .*

*Remarks 2.2.*

- (1) In fact, it is enough to check the above property for one log-resolution to say that a pair  $(X, D)$  is klt. A log-resolution of  $(X, D)$  is a resolution  $f$  such that, the exceptional locus  $\text{Exc}(f)$  of  $f$  is of pure codimension one and the divisor  $f_*^{-1}(D) + \sum_{E \subset \text{Exc}(f)} E$  has simple normal crossings (where  $f_*^{-1}(D)$  is the strict transform of  $D$  by  $f$ ).
- (2) The condition “ $a_i > -1$  for any  $i \in \mathcal{E}$ ” can be replaced by:  $\lfloor D \rfloor = 0$  and for any  $i \in \mathcal{E}$  such that  $E_i$  is exceptional for  $f$ ,  $a_i > -1$ .

In all the paper,  $G$  denotes a connected reductive algebraic group over  $\mathbb{C}$ .

Let  $T$  be a maximal torus in  $G$  and let  $B$  be a Borel subgroup of  $G$  containing  $T$ . We denote by  $\mathcal{R}$  the root system of  $(G, B, T)$ , by  $\mathcal{R}^+$  the

set of positive roots and by  $\mathcal{S}$  the set of simple roots. For any simple root  $\alpha \in \mathcal{S}$  we denote by  $s_\alpha$  the corresponding simple reflection of the Weyl group  $W = N_G(T)/T$ . By abuse of notation, for any  $w$  in  $W$ , we still denote by  $w$  one of its representative in  $G$ . We denote by  $w_0$  the longest element of  $W$ .

Let  $P$  be a parabolic subgroup of  $G$  that contains  $B$ . Denote by  $\mathcal{I}$  the set of simple roots of  $P$  (in particular, if  $P = B$  we have  $\mathcal{I} = \emptyset$  and, if  $P = G$  we have  $\mathcal{I} = \mathcal{S}$ ). Conversely, for any  $\mathcal{I} \subset \mathcal{S}$ , we denote by  $P_{\mathcal{I}}$  the parabolic subgroup of  $G$  containing  $B$  whose set of simple roots is  $\mathcal{I}$ .

Denote by  $W_P$  the subgroup of  $W$  generated by  $\{s_\alpha \mid \alpha \in \mathcal{I}\}$ . Also denote by  $W^P$  the quotient  $W/W_P$ . We identify  $W^P$  with the set of minimal length representatives in  $W$  and we denote by  $w_0^P$  the longest element of  $W^P$ .

The Bruhat decomposition of  $G$  in  $B \times B$ -orbits gives the following decomposition of  $G/P$ :

$$G/P = \bigsqcup_{w \in W^P} BwP/P.$$

Moreover the dimension of a cell  $BwP/P$  equals the length of  $w$ . In particular, the length of  $w_0^P$  is the dimension of  $G/P$  and irreducible  $B$ -stable divisors of  $G/P$  are the closures of the cells  $Bs_\alpha w_0^P P/P$  with  $\alpha \in \mathcal{S} \setminus \mathcal{I}$ . We denote them by  $D_\alpha$ .

A horospherical variety  $X$  is a normal  $G$ -variety with an open  $G$ -orbit isomorphic to a torus fibration  $G/H$  over a flag variety  $G/P$  (i.e.  $P/H$  is a torus). The irreducible divisors of such  $X$  that are  $B$ -stable but not  $G$ -stable, are the closures in  $X$  of the inverse images in  $G/H$  of the Schubert divisors  $D_\alpha$  of  $G/P$  defined above. We still denote them by  $D_\alpha$ , with  $\alpha \in \mathcal{S} \setminus \mathcal{I}$ .

If  $X$  and  $Y$  are varieties such that a parabolic subgroup  $P$  have a free right action on  $X$  and a left action on  $Y$ , we denote by  $X \times^P Y$  the quotient of the product  $X \times Y$  by the following equivalences:

$$\forall (x, y) \in X \times Y, \forall P \in P, (x, y) \sim (x \cdot p, p^{-1} \cdot y).$$

### 3. Bott–Samelson desingularizations and klt pairs of flag varieties

In this section, we prove the following result.

**THEOREM 3.1.** — *Let  $D = \sum_{\alpha \in \mathcal{S} \setminus \mathcal{I}} d_\alpha D_\alpha$  be a  $B$ -stable  $\mathbb{Q}$ -divisor of  $G/P$  such that  $\forall \alpha \in \mathcal{S} \setminus \mathcal{I}, d_\alpha \in [0, 1[$ .*

There exists a  $B$ -stable log-resolution  $\phi : Z/P \rightarrow G/P$  of  $(G/P, D)$ , where  $Z$  is a variety with a right action of  $P$  and a left action of  $B$ , such that the exceptional divisors of  $\phi$  are the quotient by  $P$  of irreducible divisors of  $Z$ , and such that  $K_{Z/P} - \pi^*(K_{G/P} + D) = \sum_{i \in \mathcal{E}} \alpha_i E_i$  where for any  $i \in \mathcal{E}$ ,  $\alpha_i > -1$  and  $E_i$  is an irreducible divisor of  $f$ .

In particular the pair  $(G/P, D)$  is klt.

Moreover, for any  $i \in \mathcal{E}$ ,  $E_i$  is the quotient of an exceptional  $B \times P$ -stable divisor  $F_i$  of  $Z$  by  $P$  (left action of  $B$  and right action of  $P$ ).

Remarks 3.2.

- (1) In general,  $\sum_{\alpha \in \mathcal{S} \setminus \mathcal{I}} D_\alpha$  is not a simple normal crossing  $\mathbb{Q}$ -divisor of  $G/P$ . Then, it is not enough to know that  $G/P$  is smooth to say that  $(G/P, D)$  is klt, when  $D \neq 0$ .
- (2) Since  $D$  is globally generated, then  $(G/P, D')$  is klt for a general  $D'$  in  $|D|$  (consequence of [8, Lemma 9.1.9]). This remark can be generalized to spherical pairs, see [1, Theorem 5.3].

To prove Theorem 3.1, we use a Bott–Samelson resolution of  $G/P$ . Bott–Samelson resolutions of Schubert varieties of  $G/B$  have been introduced by R. Bott and H. Samelson [2] and popularized by M. Demazure in [3] and H.C. Hansen [4]. Here, we use the easy (and well-known) generalization of these works to  $G/P$ . And we choose the equivalent definition of Bott–Samelson resolutions that is now used in almost all papers on the topic.

For any simple root  $\alpha$ , we denote by  $P_\alpha$  the minimal parabolic subgroup  $P_{\{\alpha\}}$ .

DEFINITION 3.3. — Let  $s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_N}$  be a reduced decomposition of  $w_0^P$  with  $\alpha_1, \dots, \alpha_N$  in  $\mathcal{S}$ . We define the Bott–Samelson variety  $BS$  to be the quotient of  $P_{\alpha_1} \times P_{\alpha_2} \times \cdots \times P_{\alpha_N}$  by the right action of  $B^N$  given by,

$$(p_1, p_2, \dots, p_N) \cdot (b_1, b_2, \dots, b_N) = (p_1 b_1, b_1^{-1} p_2 b_2, \dots, b_{N-1}^{-1} p_N b_N).$$

The map  $\phi' : BS \rightarrow G/P$  that sends  $(p_1, p_2, \dots, p_N)$  to  $p_1 p_2 \cdots p_N P/P$  is well-defined and birational (it is an isomorphism from the quotient of  $Bs_{\alpha_1} B \times Bs_{\alpha_2} B \times \cdots \times Bs_{\alpha_N} B$  by the right action of  $B^N$  to  $Bw_0^P/P$ ). Moreover, we can decompose this map by the usual map from  $BS$  to the Schubert variety  $Bw_0^P B/B$  of  $G/B$  and the projection map from  $G/B$  to  $G/P$ .

Hence, to get  $Z$  as in Theorem 3.1, we define  $Z$  to be the quotient of  $P_{\alpha_1} \times \cdots \times P_{\alpha_{N-1}} \times P_{\alpha_N \cup \mathcal{I}}$  by the right action of  $B^{N-1}$  given by,

$$(p_1, \dots, p_N) \cdot (b_1, b_2, \dots, b_{N-1}) = (p_1 b_1, \dots, b_{N-1}^{-1} p_N).$$

Then, since  $P_{\alpha_N \cup \mathcal{I}}/P = P_{\alpha_N}P/P \simeq P_{\alpha_N}/B$ , the  $B$ -varieties  $Z/P$  and  $BS$  are isomorphic and  $\phi : Z/P \rightarrow G/P$  that sends  $(p_1, \dots, p_N)$  to  $p_1 \cdots p_N P/P$  is well-defined and birational.

The lines bundles and divisors Bott–Samelson varieties are well-known, so that we can describe the lines bundles of  $Z/P$ , and the divisors of  $Z/P$  and  $Z$ .

PROPOSITION 3.4 ([7, Section 3.3]). — *For any  $i \in \{1, \dots, N - 1\}$ , we define  $F_i$  to be the  $B \times P$ -stable divisor of  $Z$  defined by the condition  $p_i \in B$ ; and we define  $F_N$  to be the  $B \times P$ -stable divisor of  $Z$  defined by  $p_N \in P$ .*

*Then, we can also define  $E_i$  to be the  $B$ -stable divisor  $F_i/P$  of  $Z/P$ . Moreover, the  $B$ -stable irreducible divisors of  $Z/P$  are the  $E_i$ 's with  $i \in \{1, \dots, N\}$ , and the family  $(E_i)_{i \in \{1, \dots, N\}}$  is a basis of the cone of effective divisors of  $Z/P$ .*

First remark that the divisor  $\sum_{i=1}^N E_i$  is clearly a simple normal crossing divisor. Also,  $G/P$  is smooth and  $\phi$  is  $B$ -equivariant, then by [5, VI.1, Theorem 1.5], we know that the exceptional locus of  $\phi$  is of pure codimension one, so it is the union of the  $E_i$ 's contracted by  $\phi$ .

Now, let  $\lambda$  be a character of  $P$ . It defines a line bundle  $\mathcal{L}_{G/P}(\lambda)$  on  $G/P$  (where  $P$  acts on the fiber over  $P/P$  by the character  $\lambda$ ). And by pull-back by  $\phi$ , it defines a line bundle  $\mathcal{L}_{Z/P}(\lambda)$  on  $Z/P$ .

The total space of  $\mathcal{L}_{Z/P}(\lambda)$  is the quotient of  $P_{\alpha_1} \times \cdots \times P_{\alpha_{N-1}} \times P_{\alpha_N \cup \mathcal{I}} \times \mathbb{C}$  by the right action of  $B^{N-1} \times P$  given by,

$$(p_1, \dots, p_N, z) \cdot (b_1, \dots, b_{N-1}, p) = (p_1 b_1, b_1^{-1} p_2 b_2, \dots, b_{N-1}^{-1} p_N p, \lambda(p)z).$$

By [3, Section 2.5, Proposition 1] adapted to our notation and by induction on  $N$ , we have the following result.

PROPOSITION 3.5. — *Let  $\lambda$  be a character of  $P$ . Then  $\mathcal{L}_{G/P}(\lambda)$  is the line bundle associated to the  $B$ -stable divisor  $D_\lambda := \sum_{\alpha \in \mathcal{S} \setminus \mathcal{I}} \langle \lambda, \alpha^\vee \rangle D_\alpha$ .*

*Moreover,  $\phi^*(D_\lambda) = \sum_{i=1}^N \langle \lambda, \beta_i^\vee \rangle E_i$ , where for any  $i \in \{1, \dots, N\}$ ,  $\beta_i = s_{\alpha_1} \cdots s_{\alpha_{i-1}}(\alpha_i)$ .*

If  $\mathcal{I} \subset \mathcal{S}$ , we denote by  $\mathcal{R}_{\mathcal{I}}^+$  the set of positive roots generated by simple roots of  $\mathcal{I}$ . Then we define  $\rho$  to be the half sum of positive roots, and  $\rho^P$  to be the half sum of positive roots that are not in  $\mathcal{R}_{\mathcal{I}}^+$  (in particular,  $\rho^B = \rho$ ).

It is well known that an anticanonical divisor of  $G/P$  is  $D_{2\rho^P}$ . Anticanonical divisors of Bott–Samelson resolutions are also well-known.

PROPOSITION 3.6 ([11, Proposition 2]). — *An anticanonical divisor of  $Z/P$  is  $\phi^*(D_\rho) + \sum_{i=1}^N E_i$ .*

COROLLARY 3.7. — *The pair  $(G/P, D)$  (with  $[D] = 0$  as in Theorem 3.1) is klt if and only if for any  $\beta$  in  $\mathcal{R}^+ \setminus \mathcal{R}_T^+$ ,*

$$\langle 2\rho^P - \rho - \sum_{\alpha \in \mathcal{S} \setminus \mathcal{I}} d_\alpha \varpi_\alpha, \beta^\vee \rangle > 0.$$

*Proof.* — By Propositions 3.5 and 3.6, we get

$$\begin{aligned} K_{Z/P} - \phi^*(K_{G/P} + D) &= -\phi^*(D_\rho) - \sum_{i=1}^N E_i + \phi^*(D_{2\rho^P}) - \phi^*(D) \\ &= \phi^*(D_{2\rho^P - \rho - \sum_{\alpha \in \mathcal{S} \setminus \mathcal{I}} d_\alpha \varpi_\alpha}) - \sum_{i=1}^N E_i \\ &= \sum_{i=1}^N (\langle 2\rho^P - \rho - \sum_{\alpha \in \mathcal{S} \setminus \mathcal{I}} d_\alpha \varpi_\alpha, \beta_i^\vee \rangle - 1) E_i. \end{aligned}$$

We conclude by remarking that, since  $s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_N}$  is a reduced expression of  $w_0^P$ , the set  $\{\beta_i \mid i = 1 \cdots N\}$  is  $\mathcal{R}^+ \setminus \mathcal{R}_T^+$ . □

The condition of Corollary 3.7 is always satisfied for  $[D] = 0$  (see Proposition 5.1). This proves Theorem 3.1.

### 4. Horospherical pairs

From the classification of horospherical  $G$ -varieties, the description of  $G$ -equivariant morphisms between horospherical  $G$ -varieties, the description of  $B$ -stable Cartier divisor of horospherical  $G$ -varieties and the description of a  $B$ -stable anticanonical divisor of horospherical  $G$ -varieties (see for example [9]), we have the following result.

PROPOSITION 4.1. — *Let  $X$  be a horospherical  $G$ -variety with open orbit  $G/H$  isomorphic to a torus fibration  $G/H \rightarrow G/P$  with  $P$  a parabolic subgroup of  $G$ . Then, there exists a smooth toric  $P/H$ -variety  $Y$  and a  $G$ -equivariant birational morphism  $f$  from the smooth horospherical  $G$ -variety  $V := G \times^P Y$  to  $X$ , such that the exceptional locus of  $f$  is of pure codimension one.*

*Let  $D$  be a  $B$ -stable effective  $\mathbb{Q}$ -divisor of  $X$  such that  $[D] = 0$ . Write  $K_V - f^*(K_X + D) = -f_*^{-1}(D) + \sum_{i \in \mathcal{E}} a_i V_i$ . Then, for any  $i \in \mathcal{E}$ ,  $V_i$  is exceptional and  $G$ -stable, in particular there exists a  $P$ -stable divisor  $Y_i$  of  $Y$  such that  $V_i = G \times^P Y_i$ . Moreover,  $a_i > -1$  for any  $i \in \mathcal{E}$ .*

We do not want here to recall the long theory of horospherical varieties. To get more details, see for example [9] or [10].

*Proof.* — With the description in terms of colored fans of horospherical  $G$ -varieties and  $G$ -equivariant morphisms between them,  $Y$  can be chosen as the toric  $P/H$ -variety associated to a smooth subdivision  $\mathbb{F}_Y$  of the fan associated to the colored fan  $\mathbb{F}_X$  of  $X$ . Then we clearly have that  $V := G \times^P Y$  is smooth and associated to the fan  $\mathbb{F}_Y$  considered as a colored fan without color. In particular, there exists a  $G$ -equivariant morphism from  $V := G \times^P Y$  to  $X$ .

Moreover, we can choose  $\mathbb{F}_Y$  such that:

- each image of a color of  $\mathbb{F}_X$  is in an edge of  $\mathbb{F}_Y$  and,
- each cone of  $\mathbb{F}_Y$  that is not a cone of  $\mathbb{F}_X$  contains an edge that is in  $\mathbb{F}_Y$  but not in  $\mathbb{F}_X$ .

These two conditions implies that the exceptional locus of  $f$  is of pure codimension one.

Any exceptional divisor  $V_i$  of  $f$  is  $G$ -stable and of the form  $G \times^P Y_i$  where  $Y_i$  is a  $P$ -stable divisor of  $Y$ .

It remains to prove that  $a_i > -1$  for any  $i \in \mathcal{E}$ . We use that  $-K_X = \sum_{i=1}^m X_i + \sum_{\alpha \in \mathcal{S} \setminus \mathcal{I}} a_\alpha D_\alpha$  where the  $X_i$ 's are the  $G$ -stable irreducible divisors of  $X$  and the  $a_\alpha$  are positive integers. Similarly, with our notation,  $K_V = -\sum_{i=1}^m f_*^{-1}(X_i) - \sum_{i \in \mathcal{E}} V_i - \sum_{\alpha \in \mathcal{S} \setminus \mathcal{I}} a_\alpha D_\alpha$ . In particular, by hypothesis on  $D$ , we remark that the divisor  $-K_X - D$  is strictly effective (ie,  $\sum_{i=1}^m b_i X_i + \sum_{\alpha \in \mathcal{S} \setminus \mathcal{I}} b_\alpha D_\alpha$ , with  $b_i > 0$  for any  $i \in \{1, \dots, m\}$  and  $b_\alpha > 0$  for any  $\alpha \in \mathcal{S} \setminus \mathcal{I}$ ) and then, by the description of pull-backs of  $B$ -stable divisors of horospherical varieties,  $f^*(-K_X - D)$  is also strictly effective. Hence, we have  $a_i > -1$  for any  $i \in \mathcal{E}$ . □

**THEOREM 4.2.** — *Let  $X$  be a horospherical  $G$ -variety. Let  $D$  be any  $B$ -stable  $\mathbb{Q}$ -divisor  $D$  of  $X$  such that  $[D] = 0$ , then  $(X, D)$  has klt singularities.*

*Proof.* — Let  $f$  be as in Proposition 4.1 and let  $Z$  be as in Theorem 3.1. Define  $V' := Z \times^P Y$  and let  $\pi : V' \rightarrow V$  the natural  $B$ -equivariant morphism defined from  $\phi$ .

We first prove that the  $B$ -equivariant morphism  $f \circ \pi : V' \rightarrow X$  is a log resolution of  $(X, D)$ . By composition, it is clearly a birational morphism and its exceptional locus is the union of the inverse images  $Z \times^P Y_i$  of the exceptional divisors of  $f$  and the exceptional divisors  $F_i \times^P Y$  of  $\pi$  (the exceptional locus of  $\pi$  is of pure codimension one because  $V$  is smooth).

The divisor  $(f \circ \pi)_*^{-1}(D) + \sum_{E \in \text{Exc}(f \circ \pi)} E$  is a  $B$ -stable divisor of  $V'$  and then has simple normal crossings. Indeed, a  $B$ -stable irreducible divisor of  $V'$  is either  $F_i \times^P Y$  where  $F_i$  is one of the  $B$ -stable irreducible divisors of



$Z$  described in Proposition 3.4, or  $Z \times^P Y_i$  where  $Y_i$  is a  $P$ -stable divisor of  $Y$ . (Recall that, any divisor of a smooth toric variety that is stable under the action of the torus has simple normal crossings, because such a variety is everywhere locally isomorphic to  $\mathbb{C}^n$  with the natural action of  $(\mathbb{C}^*)^n$ .)

Since  $D$  is  $B$ -stable, we have  $D = \sum_{i=1}^m d_i X_i + \sum_{\alpha \in \mathcal{S} \setminus \mathcal{I}} d_\alpha D_\alpha$  where the  $X_i$ 's are the  $G$ -stable irreducible divisors of  $X$ . We denote by  $D_B$  the  $B$ -stable but not  $G$ -stable part  $\sum_{\alpha \in \mathcal{S} \setminus \mathcal{I}} d_\alpha D_\alpha$  of  $D$ . Then we decompose  $K_{V'} - (f \circ \pi)^*(K_X + D)$  as follows:

$$(K_{V'} - \pi^*(K_V + f_*^{-1}(D_B))) + \pi^*(K_V - f^*(K_X + D) + f_*^{-1}(D_B)).$$

By Proposition 4.1,

$$K_V - f^*(K_X + D) + f_*^{-1}(D_B) = \sum_{i \in \mathcal{E}} a_i V_i - \sum_{i=1}^m d_i f_*^{-1}(X_i),$$

where for any  $i \in \mathcal{E}$ ,  $a_i > -1$  and  $V_i = G \times^P Y_i$  with some  $P$ -stable irreducible divisor  $Y_i$  of  $Y$ . Moreover, for any  $i \in \{1, \dots, m\}$ ,  $-d_i > -1$  and the  $G$ -stable divisor  $f_*^{-1}(X_i)$  can be written  $V_i = G \times^P Y_i$  with some  $P$ -stable irreducible divisor  $Y_i$  of  $Y$ . Thus,  $K_V - f^*(K_X + D) + f_*^{-1}(D_B) = \sum_{i \in \mathcal{E}'} a_i V_i$ , where for any  $i \in \mathcal{E}'$ ,  $a_i > -1$ .

We remark that the inverse image of  $V_i$  by  $\pi$  is the irreducible divisor  $Z \times^P V_i$  so that  $\pi^*(V_i) = Z \times^P Y_i$ . Hence,  $\pi^*(K_V - f^*(K_X + D) + f_*^{-1}(D)) = \sum_{i \in \mathcal{E}'} a_i Z \times^P Y_i$ .

To compute  $K_{V'} - \pi^*(K_V + f_*^{-1}(D_B))$ , we use the fibrations  $p : V = G \times^P Y \rightarrow G/P$  and  $p' : V' = Z \times^P Y \rightarrow Z/P$ , which have the same fiber. To summarize, we get the following commutative diagram.

$$\begin{CD} V' = Z \times^P Y @>\pi>> V = G \times^P Y @>f>> X \\ @Vp'VV @VVpV \\ Z/P @>>\phi>> G/P \end{CD}$$

In particular, we have  $K_V = p^*(K_{G/P}) + K_p$  and  $K_{V'} = p'^*(K_{Z/P}) + K_{p'}$ . Moreover, the relative canonical divisors  $K_{p'}$  and  $K_p$  satisfy  $K_{p'} = \pi^*(K_p)$ .

Moreover, for any  $B$ -stable irreducible divisor  $D$  of  $V$  that is not  $G$ -stable,  $D$  is the pull-back by  $p$  of a Schubert divisor of  $G/P$ , in particular  $D = p^*(p_*(D))$ .

Hence, we get

$$\begin{aligned}
 K_{V'} - \pi^*(K_V + f_*^{-1}(D_B)) &= p'^*(K_{Z/P}) + K_{p'} - \pi^*p^*(K_{G/P}) \\
 &\quad - \pi^*(K_p) - \pi^*(f_*^{-1}(D_B)) \\
 &= p'^*(K_{Z/P}) + \pi^*(K_p) - p'^*\phi^*(K_{G/P}) \\
 &\quad - \pi^*(K_p) - \pi^*(p^*p_*(f_*^{-1}(D_B))) \\
 &= p'^*(K_{Z/P} - \phi^*(K_{G/P} + p_*(f_*^{-1}(D_B))).
 \end{aligned}$$

Remark that  $[p_*(f_*^{-1}(D_B))] = [D_B]$ , so that by Theorem 3.1, we get  $K_{Z/P} - \phi^*(K_{G/P} + p_*(f_*^{-1}(D_B))) = \sum_{i \in \mathcal{E}''} a_i F_i/P$ , where for any  $i \in \mathcal{E}''$ , we have  $a_i > -1$  and  $F_i$  is a  $B \times P$ -stable irreducible divisor of  $Z$ .

Hence, we have  $K_{V'} - \pi^*(K_V + f_*^{-1}(D_B)) = \sum_{i \in \mathcal{E}''} a_i F_i \times^P Y$ .

And finally, we have

$$K_{V'} - (f \circ \pi)^*(K_X + D) = \sum_{i \in \mathcal{E}''} a_i F_i \times^P Y + \sum_{i \in \mathcal{E}' } a_i Z \times^P Y_i,$$

with, for any  $i \in \mathcal{E}'' \cup \mathcal{E}'$ ,  $a_i > -1$ . □

### 5. A result on root systems

This section is independent of the rest of the paper. We prove a result enabling to deduce Theorem 3.1 from Corollary 3.7. We keep notation of Section 2. If we have a parabolic subgroup  $P$  containing  $B$ , we denote by  $\mathcal{I}$  the subset of  $\mathcal{S}$  such that  $P = P_{\mathcal{I}}$ . Recall that, we denote by  $\mathcal{R}_{\mathcal{I}}^+$  the set of positive roots generated by simple roots of  $\mathcal{I}$ ,  $\rho$  denotes the half sum of positive roots, and  $\rho^P$  denotes the half sum of positive roots that are not in  $\mathcal{R}_{\mathcal{I}}^+$ .

PROPOSITION 5.1. — *For any (proper) parabolic subgroup  $P$  of  $G$  containing  $B$ , and for any  $\beta$  in  $\mathcal{R}^+ \setminus \mathcal{R}_{\mathcal{I}}^+$ ,*

$$(5.1) \quad \langle 2\rho^P - \rho - \sum_{\alpha \in \mathcal{S} \setminus \mathcal{I}} \varpi_{\alpha}, \beta^{\vee} \rangle \geq 0.$$

Note that  $\rho = \sum_{\alpha \in \mathcal{S}} \varpi_{\alpha}$  and that  $2\rho^P = 2\rho - \sum_{\gamma \in \mathcal{R}_{\mathcal{I}}^+} \gamma = 2 \sum_{\alpha \in \mathcal{S}} \varpi_{\alpha} - \sum_{\gamma \in \mathcal{R}_{\mathcal{I}}^+} \gamma$ . Hence, equation 5.1 is equivalent to

$$(5.2) \quad \langle \sum_{\alpha \in \mathcal{I}} \varpi_{\alpha} - \sum_{\gamma \in \mathcal{R}_{\mathcal{I}}^+} \gamma, \beta^{\vee} \rangle \geq 0.$$

Remarks 5.2.

- (1) If  $\mathcal{I} = \emptyset$  (i.e. if  $P = B$ ), equations (5.1) and (5.2) are trivially satisfied.
- (2) If  $\beta \in \mathcal{R}_{\mathcal{I}}^+$  then  $\langle \sum_{\alpha \in \mathcal{I}} \varpi_{\alpha} - \sum_{\gamma \in \mathcal{R}_{\mathcal{I}}^+} \gamma, \beta^{\vee} \rangle = -\langle \sum_{\alpha \in \mathcal{I}} \varpi_{\alpha}, \beta^{\vee} \rangle$  is negative.

LEMMA 5.3. — Denote by  $w_{0,P}$  the longest element of  $W_P$ . Then, we have

$$w_{0,P}(\sum_{\alpha \in \mathcal{I}} \varpi_{\alpha}) = \sum_{\alpha \in \mathcal{I}} \varpi_{\alpha} - \sum_{\gamma \in \mathcal{R}_{\mathcal{I}}^+} \gamma.$$

Proof. — First note that, for any character  $\lambda$  of  $T$  and for any  $w \in W_P$ ,  $w(\lambda) - \lambda$  is in the lattice  $\bigoplus_{\alpha \in \mathcal{I}} \mathbb{Z}\alpha$  (it can be easily proved by induction on the length of  $w$ ).

Moreover,  $\rho_P := \sum_{\gamma \in \mathcal{R}_{\mathcal{I}}^+} \gamma$  satisfies  $\langle \rho_P, \alpha^{\vee} \rangle = 2$  for any  $\alpha \in \mathcal{I}$  (same result as the well-known result:  $\langle \rho, \alpha^{\vee} \rangle = 2$  for any  $\alpha \in \mathcal{S}$ ). And, since  $w_{0,P}$  is the longest element of  $W_P$ , we see that for any  $\alpha \in \mathcal{I}$ , the root  $w_{0,P}(\alpha)$  is the opposite of a simple root in  $\mathcal{I}$ .

Hence, if we denote by  $\lambda$  the character  $w_{0,P}(\sum_{\alpha \in \mathcal{I}} \varpi_{\alpha}) - \sum_{\alpha \in \mathcal{I}} \varpi_{\alpha} + \sum_{\gamma \in \mathcal{R}_{\mathcal{I}}^+} \gamma$ , we get  $\lambda \in \bigoplus_{\alpha \in \mathcal{I}} \mathbb{Z}\alpha$  and, for any  $\alpha \in \mathcal{I}$ , we have

$$\langle \lambda, \alpha^{\vee} \rangle = \langle \sum_{\alpha \in \mathcal{I}} \varpi_{\alpha}, w_{0,P}(\alpha^{\vee}) \rangle - \langle \sum_{\alpha \in \mathcal{I}} \varpi_{\alpha}, \alpha^{\vee} \rangle + \langle \rho_P, \alpha^{\vee} \rangle = -1 - 1 + 2 = 0.$$

Then, we deduce that  $\lambda = 0$ , which proves the lemma. □

Proof of Proposition 5.1. — By Lemma 5.3, we get for any  $\beta \in \mathcal{R}^+ \setminus \mathcal{R}_{\mathcal{I}}^+$ ,

$$\langle \sum_{\alpha \in \mathcal{I}} \varpi_{\alpha} - \sum_{\gamma \in \mathcal{R}_{\mathcal{I}}^+} \gamma, \beta^{\vee} \rangle = \langle \sum_{\alpha \in \mathcal{I}} \varpi_{\alpha}, w_{0,P}(\beta^{\vee}) \rangle.$$

But, the positive roots that are sent to negative roots by  $w_{0,P}$  are exactly the roots in  $\mathcal{R}_{\mathcal{I}}^+$ . In particular, for any  $\beta \in \mathcal{R}^+ \setminus \mathcal{R}_{\mathcal{I}}^+$ ,  $w_{0,P}(\beta^{\vee})$  is a positive coroot, and

$$\langle \sum_{\alpha \in \mathcal{I}} \varpi_{\alpha}, w_{0,P}(\beta^{\vee}) \rangle \geq 0. \quad \square$$

Remark 5.4. — Let  $\beta \in \mathcal{R}^+ \setminus \mathcal{R}_{\mathcal{I}}^+$ . Then  $\langle \sum_{\alpha \in \mathcal{I}} \varpi_{\alpha} - \sum_{\gamma \in \mathcal{R}_{\mathcal{I}}^+} \gamma, \beta^{\vee} \rangle = 0$  if and only if  $w_{0,P}(\beta) \in \mathcal{R}_{\mathcal{S} \setminus \mathcal{I}}^+$ . In particular, if  $\mathcal{I} \neq \mathcal{S}$ , there exists  $\beta \in \mathcal{R}^+ \setminus \mathcal{R}_{\mathcal{I}}^+$  such that  $\langle \sum_{\alpha \in \mathcal{I}} \varpi_{\alpha} - \sum_{\gamma \in \mathcal{R}_{\mathcal{I}}^+} \gamma, \beta^{\vee} \rangle = 0$ .

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