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ON THE GROWTH OF THE HOMOLOGY OF A FREE LOOP SPACE II

by Yves FÉLIX, Steve HALPERIN & Jean-Claude THOMAS

ABSTRACT. — Controlled exponential growth is a stronger version of exponential growth. We prove that the homology of the free loop space $\mathcal{L}X$ has controlled exponential growth in two important situations : (1) when X is a connected sum of manifolds whose rational cohomologies are not monogenic, (2) when the rational homotopy Lie algebra L_X contains an inert element and $\rho(L_X) < \rho(L_X/[L_X, L_X])$, where $\rho(V)$ denotes the radius of convergence of V .

RÉSUMÉ. — La croissance exponentielle contrôlée est une version forte de la croissance exponentielle. Nous prouvons que les nombres de Betti de l'espace des lacets libres sur un espace X ont une croissance exponentielle contrôlée dans deux cas: lorsque X est la somme connexe de variétés dont la cohomologie n'est pas monogène, et lorsque l'algèbre de Lie L_X a une croissance exponentielle strictement plus grande que ses indécomposables.

1. Introduction

In this paper we are concerned with the growth of the homology $H_*(X^{S^1}; \mathbb{Q})$ of a free loop space on a simply connected space, X .

A graded vector space $V = V_{\geq 0}$ grows *exponentially* if there are constants $1 < C_1 < C_2$ such that for some N ,

$$C_1^k \leq \sum_{i \leq k} \dim V_i \leq C_2^k, \quad k \geq N.$$

In particular, if X is a simply connected CW complex of finite type and finite Lusternik–Schnirelmann category then [3] either $\dim \pi_*(X) \otimes \mathbb{Q} < \infty$ (X is rationally elliptic) or $\pi_*(X) \otimes \mathbb{Q}$ grows exponentially (X is rationally hyperbolic). The first examples of elliptic spaces are given by compact homogeneous spaces, but the generic situation is given by hyperbolic spaces.

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For instance if the Euler characteristic $\chi(X) < 0$ then X is hyperbolic (see [4] for other examples of elliptic or hyperbolic spaces)

In [7] Gromov conjectured that $H_*(X^{S^1}; \mathbb{Q})$ grows exponentially for almost all cases when X is a closed manifold. This would have an important consequence in Riemannian geometry, due to a theorem of Gromov, improved by Ballmann and Ziller:

THEOREM 1.1 ([7], [2]). — *Let $N_g(t)$ denote the number of geometrically distinct closed geodesics of length $\leq t$ on a simply connected closed Riemannian manifold (M, g) . Then, for generic metrics g , there are constants $K > 0$ and $\beta > 0$ such that for k sufficiently large,*

$$N_g(k) \geq K \cdot \max_{\ell \leq \beta k} \dim H_\ell(M^{S^1}; \mathbb{Q}).$$

One of the first applications of Sullivan's minimal models $(\wedge V, d)$ of a space X was the construction [16] (when X is simply connected) of the minimal model $(\wedge W, d)$ of X^{S^1} where $W^k = V^k \oplus V^{k-1}$. Since X is elliptic if and only if $\dim V < \infty$ it follows that in that case $H_*(X^{S^1}; \mathbb{Q})$ grows at most polynomially. In [16] Vigué-Poirrier conjectures that in the hyperbolic case, $H_*(X^{S^1}; \mathbb{Q})$ should grow exponentially, a conjecture which would give Gromov's conjecture as a special case.

The Vigué-Poirrier conjecture has been proved for a finite wedge of spheres [16], for a non-trivial connected sum of closed manifolds [11] and in the case X is coformal [12].

For simplicity we write $H(X)$ and $H^*(X)$ respectively for the rational homology and cohomology of a space X , and denote the free loop space of maps $S^1 \rightarrow X$ by $\mathcal{L}X$. If X is simply connected and $\dim \pi_*(X) \otimes \mathbb{Q} < \infty$ then it is immediate from Sullivan's model of $\mathcal{L}X$ [15] that $H(\mathcal{L}X)$ grows at most polynomially. However, even in the case when X is a rationally hyperbolic finite simply connected complex it is not known if $H(\mathcal{L}X)$ grows exponentially.

Next, for a graded vector space V denote by

$$V(z) := \sum_{k \geq 0} \dim V_k z^k$$

the formal *Hilbert series* of V and denote by ρ_V or $\rho(V)$ the radius of convergence of $V(z)$. If X is a topological space we denote by $X(z)$ and by ρ_X or by $\rho(X)$ the Hilbert series of $H(X)$ and its radius of convergence.

In [5] we introduced a much stronger version of exponential growth: V has *controlled exponential growth* if $0 < \rho_V < 1$ and for each $\lambda > 1$ there

is an infinite sequence $n_1 < n_2 < \dots$ such that $n_{i+1} < \lambda n_i$, $i \geq 1$, and

$$\lim_i \frac{\log \dim V_{n_i}}{n_i} = -\log(\rho_V).$$

As usual, ΩX denotes the (based) loop space on a space X . We recall [14] or [4] that if X is simply connected, then $H(\Omega X)$ is the universal enveloping algebra of the graded Lie algebra $L_X = \pi_*(\Omega X) \otimes \mathbb{Q}$; L_X is called the *homotopy Lie algebra* of X . According to [5, Lemma 4],

$$(1.1) \quad \rho_{\Omega X} = \rho(L_X).$$

If X has rational homology of finite type and infinite dimensional rational homotopy, then Sullivan’s model for $\mathcal{L}X$ gives

$$(1.2) \quad \rho_{\mathcal{L}X} \leq \rho_{\Omega X}.$$

Our objective here is to establish new classes of spaces X (Theorems 1.3 and 1.4 below) for which $H(\mathcal{L}X)$ has controlled exponential growth and

$$\rho_{\mathcal{L}X} = \rho_{\Omega X}.$$

Our approach is by constructing maps

$$F \rightarrow X \xrightarrow{p} Y$$

in which F is the homotopy fibre of p .

THEOREM 1.2. — *With the above notations if F is rationally a wedge of spheres, and if $0 < \rho_{\Omega F} < \rho_{\Omega Y}$ then $H(\mathcal{L}X)$ has controlled exponential growth and $\rho_{\mathcal{L}X} = \rho_{\Omega X}$.*

Proof. — This follows from [5, formula (4)], together with Theorems 1.2 and 1.4. □

One method for constructing other maps $p : X \rightarrow Y$ is via inert elements $\alpha \in L_X$, where L_X is the homotopy Lie algebra of X . Any $\alpha \in (L_X)_k$ corresponds up to a scalar multiple to a map $\sigma : S^{k+1} \rightarrow X$ and α is called *inert* if the map

$$p : X \rightarrow X \cup_{\sigma} D^{k+2}$$

is surjective in rational homotopy. In Lemma 2.2 we recall the proof that if α is inert then the homotopy fibre of p is a wedge of spheres with homology isomorphic to $H(\Omega(X \cup_{\sigma} D^{k+2})) \otimes \mathbb{Q}\alpha$. For instance the attaching map of the top cell in a simply connected manifold whose cohomology is not monogenic is inert [8]. (Recall that a graded algebra $A = \mathbb{Q} \oplus A^{\geq 1}$ is *monogenic* if it is generated by a single element $a \in A^{\geq 1}$). Also, every nonzero element α in a free Lie algebra generated by elements of even degrees is inert ([8]).

A key condition in our theorems is the hypothesis

$$(1.3) \quad \Omega X(\rho_{\Omega X}) := \lim_{z \rightarrow \rho_{\Omega X}} \Omega X(z) = \infty.$$

There are no examples where this is known to fail if X is a rationally hyperbolic, finite, simply connected CW complex. In fact (Proposition 2.1) this follows from the condition

$$\rho(L_X) < \rho\left(\frac{L_X}{[L_X, L_X]}\right),$$

which is not known to fail for such X . When $\dim L_X/[L_X, L_X] < \infty$, Proposition 2.1 follows from a result of Anick [1].

With this preamble we can state our two theorems:

THEOREM 1.3. — *Suppose X is a simply connected CW complex with rational homology of finite type. If L_X contains an inert element γ and if $\rho(L_X) < \rho(L_X/[L_X, L_X])$ then $H(\mathcal{L}X)$ has controlled exponential growth and $\rho_{\mathcal{L}X} = \rho_{\Omega X}$.*

THEOREM 1.4. — *Suppose $M\#N$ is the connected sum of two closed simply connected n -manifolds with $H^*(N)$ not monogenic and M not rationally a sphere. If $\rho_{\Omega N} \leq \rho_{\Omega M}$ and if $\Omega N(\rho_{\Omega N}) = \infty$ then $H(\mathcal{L}(M\#N))$ has controlled exponential growth and $\rho_{\mathcal{L}(M\#N)} = \rho_{\Omega(M\#N)}$.*

Remarks 1.5.

- (1) Theorem 1.3 is proved in [5] under the considerably stronger hypothesis that

$$\dim L_X/[L_X, L_X] < \infty.$$

- (2) If $H^*(M)$ and $H^*(N)$ are monogenic, but of dimension > 2 then $M\#N$ is elliptic and so $H(\mathcal{L}(M\#N))$ grows at most polynomially.
- (3) Theorem 1.4 strengthens a result of Lambrechts [10], which asserts that $H(\mathcal{L}(M\#N))$ grows exponentially unless both $H^*(M)$ and $H^*(N)$ are monogenic.

2. Proposition 2.1 and Theorem 1.3

Suppose $A = \mathbb{Q}1 \oplus A_{\geq 1}$ is a finitely generated graded algebra satisfying $\rho_A < 1$. Then it follows from a result of Anick [1] that

$$A(\rho_A) = \infty.$$

We generalize this with

PROPOSITION 2.1. — *Let $L = L_{\geq 1}$ be a graded Lie algebra of finite type such that $0 < \rho_{UL} < 1$. If L is generated by a subspace V with $\rho_{UL} < \rho_V$ then $UL(\rho_{UL}) = \infty$.*

Proof. — We assume $UL(\rho_{UL}) < \infty$, and deduce a contradiction. By Anick’s result we have $\dim V = \infty$. Choose some σ with $\rho_{UL} < \sigma < \rho_V$. Then $V(\sigma) < \infty$ and so $V_{\geq r}(\sigma) \rightarrow 0$ as $r \rightarrow \infty$. In particular, we may choose r so that

$$UL(\rho_{UL}) \cdot V_{\geq r}(\sigma) < 1.$$

Now let E be the sub Lie algebra generated by $V_{< r}$ and note that by Anick’s result, $E \neq L$. In particular, $UE(\rho_{UL}) < UL(\rho_{UL})$. Clearly $\rho_{UE} \geq \rho_{UL}$. If $\rho_{UE} = \rho_{UL}$, then $0 < \rho_{UE} < 1$. Then by Anick’s result $UE(\rho_{UE}) = \infty$, and $UL(\rho_{UL}) = \infty$. It follows that $\rho_{UE} > \rho_{UL}$. Thus for some τ with $\rho_{UL} < \tau < \rho_{UE}$ we have $UE(\tau) < UL(\rho_{UL})$.

Choose ρ so that $\rho_{UL} < \rho < \tau$ and $\rho < \sigma$. Then

$$UE(\rho) \cdot V_{\geq r}(\rho) < UE(\tau) \cdot V_{\geq r}(\sigma) < UL(\rho_{UL}) \cdot V_{\geq r}(\sigma) < 1.$$

Now let $W = UE \circ V_{\geq r}$ where “ \circ ” denotes the adjoint action and note that $W(\rho) < 1$. Then, let I be the sub Lie algebra generated by W . The inclusion of W in I extends to a surjection $TW \rightarrow UI$. Since $(TW)(\rho) = \frac{1}{1-W(\rho)} < \infty$, it follows that

$$\rho_{UI} \geq \rho_{TW} \geq \rho > \rho_{UL}.$$

On the other hand, since $W \supset V_{\geq r}$ and $[E, W] \subset W$, it follows that I is an ideal in L . The surjection $L \rightarrow L/I$ kills $V_{\geq r}$, and so it restricts to a surjection $E \rightarrow L/I$. Thus $\rho_{U(L/I)} \geq \rho_{UE} > \rho_{UL}$. But as graded vector spaces $UL \cong UI \otimes U(L/I)$ and so

$$\rho_{UL} = \min\{\rho_{UI}, \rho_{U(L/I)}\}.$$

This is the desired contradiction because $\rho_{UL} < \rho_{UI}$ and $\rho_{UL} < \rho_{U(L/I)}$. □

We also require the following lemma announced in the Introduction, and which is essentially proved, if not stated, in [8].

LEMMA 2.2. — *Let X be a simply connected CW complex that is not rationally a sphere. If $\alpha \in (L_X)_k$ is an inert element corresponding to $\sigma : S^{k+1} \rightarrow X$, then*

- (1) *The homotopy fibre $i : F \rightarrow X$ of $p : X \rightarrow X \cup_{\sigma} D^{k+2} = Y$ is rationally a wedge of spheres.*
- (2) *$H(\Omega i)$ restricts to an isomorphism $L_F \xrightarrow{\cong} I$, where $I \subset L_X$ is the ideal generated by α .*

- (3) I is a free Lie algebra and $I/[I, I] \cong U(L_X/I) \otimes \mathbb{Q}\alpha$.
- (4) $H_*(\Omega p)$ induces an isomorphism $U(L_X/I) \xrightarrow{\cong} H_*(\Omega Y)$.

Proof. — Since α is inert $\pi_*(p) \otimes \mathbb{Q}$ is surjective. Thus $\pi_*(\Omega p) \otimes \mathbb{Q}$ is surjective and

$$\pi_*(\Omega i) \otimes \mathbb{Q} : L_F = \pi_*(\Omega F) \otimes \mathbb{Q} \xrightarrow{\cong} \ker \pi_*(\Omega p) \otimes \mathbb{Q}.$$

Moreover, it follows from [8, Theorem 1.1], that $L_F = I$, and so $H_*(\Omega p) \otimes \mathbb{Q}$ induces an isomorphism $U(L_X/I) \xrightarrow{\cong} H_*(\Omega Y)$. Theorem 1.1 of [8] also asserts that I is a free Lie algebra, and that

$$I/[I, I] \cong U(L_X/I) \otimes \mathbb{Q}\alpha.$$

It remains to show that F is rationally a wedge of spheres. Let $\sigma_i : S^{n_i} \rightarrow F$ corresponding to elements $\alpha_i \in L_F$ which represent a basis of $I/[I, I]$. Then the map

$$\varphi = \vee_i \sigma_i : \vee S^{n_i} \rightarrow F$$

induces a map $\Omega\varphi : \Omega(\vee S^{n_i}) \rightarrow \Omega F$ and $\pi_*(\Omega\varphi) \otimes \mathbb{Q}$ is a morphism between free Lie algebras inducing an isomorphism $I/[I, I] \cong L_F/[L_F, L_F]$. Thus $\pi_*(\Omega\varphi) \otimes \mathbb{Q}$ is an isomorphism and φ is a rational homotopy equivalence. \square

Proof of Theorem 1.3. — Denote L_X simply by L , let $\alpha \in L_k$ be the inert element corresponding to $\sigma : S^{k+1} \rightarrow X$, and let $p : X \rightarrow X \cup_\sigma D^{k+2}$ be the map considered in Lemma 1. Then by Lemma 1, with I the ideal generated by α and $V = I/[I, I]$, we have isomorphisms

$$H_*(\Omega F) \cong UI \cong TV \quad \text{and} \quad H(\Omega(X \cup_\sigma D^{k+2})) \cong U(L/I).$$

Thus, as observed in the Introduction, Theorem 1.3 will be established once we prove

$$(2.1) \quad \rho_{UI} < \rho_{U(L/I)}.$$

Clearly $\rho_{UL} \leq \rho_{U(L/I)}$ and if $\rho_{UL} < \rho_{U(L/I)}$ then $\rho_{UI} < \rho_{U(L/I)}$ since $UL \cong UI \otimes U(L/I)$. It remains to consider the case that $\rho_{UL} = \rho_{U(L/I)}$. Since $UI \cong TV$ and since $\dim V \geq 2$ it follows that $\rho_{UL} \leq \rho_{UI} < 1$. Since $L/[L, L]$ maps surjectively to $(L/I)/[L/I, L/I]$, we obtain

$$\rho_{U(L/I)} = \rho_{UL} < \rho_{L/[L, L]} \leq \rho_{(L/I)/[L/I, L/I]}.$$

Thus by Proposition 2.1,

$$U(L/I)(\rho_{U(L/I)}) = \infty.$$

On the other hand, $UI \cong TV$ with $V \cong U(L/I) \otimes \mathbb{Q}\alpha$. Thus

$$UI(z) = \frac{1}{1 - z^k U(L/I)(z)}.$$

Since $\lim_{z \rightarrow \rho(U(L/I))} U(L/I)(z) = \infty$, it follows that $r^k U(L/I)(r) = 1$ for some $r < \rho(U(L/I))$. But then $r = \rho_{UI}$ and so again $\rho_{UI} < \rho(U(L/I))$. \square

3. Connected sums

The objective of this section is to prove Theorem 1.4, and we shall frequently rely on the *acyclic closure* [6] of a cdga, (A, d) in which $A^0 = \mathbb{Q}$ and $H^1(A) = 0$. This is a cdga of the form $(A \otimes \wedge U, d)$ containing (A, d) as a sub cdga, where the quotient $(\wedge U, \bar{d})$ is a minimal Sullivan algebra, and such that $H(A \otimes \wedge U, d) = \mathbb{Q}$. The acyclic closure is determined up to isomorphism ([6, Theorem 3.2]).

For the proof of Theorem 1.4 we establish a preliminary proposition to deal with the case that $H^*(M)$ is monogenic and $H^*(N)$ is not. Recall that a *model for a space X* is a connected commutative graded differential algebra whose minimal Sullivan model is also a minimal Sullivan model for the rational polynomial differential forms on X ([15], [4]).

Let (A, d) and (B, d) be finite dimensional models for the closed n -manifolds M and N of Theorem 1.4. We may suppose $A^0 = B^0 = \mathbb{Q}$, $A^1 = B^1 = 0$, $A^{>n} = B^{>n} = 0$, $A^n = \mathbb{Q}\alpha$ and $B^n = \mathbb{Q}\beta$.

LEMMA 3.1. — *A model for the connected sum $M \# N$ is given the cdga*

$$((A \oplus_{\mathbb{Q}} B) \oplus \mathbb{Q}w, d)$$

with $dw = \alpha - \beta$ and $w \cdot A^+ = w \cdot B^+ = 0$.

Proof. — By [4, §12], the cdga $A \oplus_{\mathbb{Q}} B$ is a model for the wedge $M \vee N$. Denote by $p : M \# N \rightarrow M \vee N$ the pinch map and $(\wedge X, d)$ a Sullivan minimal model for $M \vee N$. Since $H^{<n}(p)$ is an isomorphism and $H^n(p)$ simply identifies the classes α and β , a model of p is given by the inclusion $(\wedge X, d) \rightarrow (\wedge X \otimes \wedge u \otimes \wedge Z, d)$ where $du = \alpha - \beta$ with $[\alpha]$ and $[\beta]$ the fundamental classes of M and N , and where $Z = Z^{<n-1}$ is introduced to kill recursively all new cohomology classes. We then have clearly a commutative diagram, where the vertical maps are quasi-isomorphisms

$$\begin{array}{ccc} (\wedge X, d) & \xrightarrow{\varphi} & (\wedge Y, d) \\ \downarrow \simeq & & \downarrow \simeq \\ A \oplus_{\mathbb{Q}} B & \longrightarrow & ((A \oplus_{\mathbb{Q}} B) \oplus \mathbb{Q}w, d). \end{array} \quad \square$$

Now consider the case that $H^*(M)$ is monogenic. Then $H^*(M) = \wedge a/a^{n+1}$, where $\deg a = 2p$, $n = 2pk$, and $k \geq 2$ because M is not rationally

a sphere. In this case $(\wedge a/a^{n+1}, 0)$ is a model for M and we choose as model (B, d) for N a quotient of the minimal Sullivan model such that $B^{>n} = 0$ and $B^n = \mathbb{Q}\beta$. Then a represents a cohomology class in $H^{2p}(M\#N)$ and hence determines a map $p : M\#N \rightarrow K(2p, \mathbb{Q})$ with homotopy fibre F .

PROPOSITION 3.2. — *The homotopy fibre F has a model of the form*

$$(C, d) = (B/\beta, d) \oplus (B^{\geq 1}, d) \otimes \mathbb{Q}\bar{a}$$

where $\text{deg } \bar{a} = 2p - 1$, $(B/\beta, d)$ is the quotient cdga of (B, d) acting by multiplication on the left on $(B^{\geq 1}, d) \otimes \mathbb{Q}\bar{a}$, and $(B^{\geq 1} \otimes \mathbb{Q}\bar{a}) \cdot (B^{\geq 1} \otimes \mathbb{Q}\bar{a}) = 0$.

Proof. — As observed above, a model for $M\#N$ is given by $((\wedge a/a^{k+1} \times_{\mathbb{Q}} B) \oplus \mathbb{Q}w, d)$ with $dw = a^k - \beta$. Now a quasi-isomorphism

$$((\wedge a \otimes \wedge w) \times_{\mathbb{Q}} B, d) \xrightarrow{\simeq} (\wedge a/a^{k+1} \times_{\mathbb{Q}} B) \oplus \mathbb{Q}w$$

is given by dividing by the elements a^q and $a^r w$, $q \geq k + 1$ and $r \geq 1$; here on the left $dw = a^k - \beta$. (This follows by filtering by the degree in B .)

Thus it follows from Theorem 15.3 in [4] or Theorem 5.1 in [6] that the Sullivan fibre of the morphism $\wedge a \rightarrow ((\wedge a \otimes \wedge w) \times_{\mathbb{Q}} B)$ is a model for F . Let $(\wedge a \otimes \wedge \bar{a}, d\bar{a} = a)$ be the acyclic closure of $(\wedge a, 0)$. Then this Sullivan fibre is given by $((\wedge a \otimes \wedge w) \times_{\mathbb{Q}} B) \otimes_{\wedge a} (\wedge a \otimes \wedge \bar{a})$. Hence

$$\begin{aligned} (\wedge a \otimes \wedge w \otimes \wedge \bar{a}) \oplus (B^{\geq 1} \otimes \wedge \bar{a}) &= (\wedge a \otimes \wedge w \otimes \wedge \bar{a}) \times_{\wedge \bar{a}} (B \otimes \wedge \bar{a}) \\ &= [(\wedge a \otimes \wedge w) \times_{\mathbb{Q}} B] \otimes_{\wedge a} (\wedge a \otimes \wedge \bar{a}) \end{aligned}$$

is also a model for F .

Next note that $I = (\wedge^{\geq 2} a \oplus \wedge^{\geq 1} a \cdot \bar{a}) \otimes \wedge w \subset (\wedge a \otimes \wedge w \otimes \wedge \bar{a}) \oplus (B^{\geq 1} \otimes \wedge \bar{a})$ is an ideal preserved by d , and that $H(I, d) = 0$. Thus division by I produces another model for F , given explicitly by

$$(\mathbb{Q}(1 \oplus a \oplus \bar{a}) \otimes \wedge w) \oplus (B^{\geq 1} \otimes \wedge \bar{a})$$

with $a^2 = a\bar{a} = \bar{a}^2 = 0$, $d\bar{a} = a$ and, since $k \geq 2$, $dw = -\beta$. In this cdga, $d(\bar{a}w) = a\bar{w} + \bar{a}\beta$. Moreover, the subspace spanned by $\bar{a}w$ and $a\bar{w} + \bar{a}\beta$ is an ideal. Thus a quasi-isomorphism

$$(\mathbb{Q}(1 \oplus a \oplus \bar{a}) \otimes \wedge w) \oplus (B^{\geq 1} \otimes \wedge \bar{a}) \rightarrow \mathbb{Q}(1 \oplus a \oplus \bar{a} \oplus w) \oplus (B^{\geq 1} \otimes \wedge \bar{a})$$

is given by $\bar{a}w \mapsto 0$ and $aw \mapsto -\bar{a}\beta$.

Now the inclusion $\mathbb{Q} \oplus \mathbb{Q}w \oplus (B^{\geq 1} \otimes \wedge \bar{a})$ in $\mathbb{Q}(1 \oplus a \oplus \bar{a} \oplus w) \oplus (B^{\geq 1} \otimes \wedge \bar{a})$ is clearly a quasi-isomorphism. Since $dw = -\beta$, division by w and β then gives a quasi-isomorphism

$$\mathbb{Q} \oplus \mathbb{Q}w \oplus (B^{\geq 1} \otimes \wedge \bar{a}) \xrightarrow{\simeq} B/\beta \oplus (B^{\geq 1} \otimes \mathbb{Q}\bar{a}).$$

(Note that in the left hand cdga $\beta \otimes \bar{a}$ is not the product of β and \bar{a} , since \bar{a} is not an element in the cdga!). \square

Proof. — We consider separately the cases that $H^*(M)$ is monogenic and $H^*(N)$ is not, and that neither $H^*(M)$ nor $H^*(N)$ are monogenic. Note that since M and N are simply connected, and N is not a rational sphere, $n \geq 4$.

Case 1: $H^(M)$ is monogenic.* — We adopt the notation of Proposition 3.2, and for simplicity denote $-\otimes \mathbb{Q}\bar{a}$ simply by $-\otimes \bar{a}$. It is immediate from Theorem 3 and (4) in [5] that it is sufficient to prove that $H(\mathcal{L}F)$ has controlled exponential growth and that $\rho_{\mathcal{L}F} = \rho_{\Omega F}$. Let $(\wedge W, d) \rightarrow (B/\beta, d)$ be a minimal Sullivan model, and extend this to a Sullivan model $(\wedge W \otimes \wedge Z, d) \xrightarrow{\cong} (C, d)$. By Proposition 3.2, $(\wedge W \otimes \wedge Z, d)$ is a Sullivan model for F . Now, letting $(\wedge W \otimes \wedge U, d)$ be the acyclic closure of $(\wedge W, d)$, we have for the Sullivan fibre $(\wedge Z, \bar{d})$ that

$$\begin{aligned} (\wedge Z, \bar{d}) &\simeq (\wedge W \otimes \wedge Z \otimes_{\wedge W} \wedge W \otimes \wedge U, d) = (\wedge W \otimes \wedge Z \otimes \wedge U, d) \\ &\xrightarrow{\cong} (B/\beta \oplus (B^{\geq 1} \otimes \bar{a}) \otimes \wedge U, d) \\ &\xrightarrow{\cong} \mathbb{Q} \oplus (B^{\geq 1} \otimes \bar{a} \otimes \wedge U, d). \end{aligned}$$

Since products in $(B^{\geq 1} \otimes \bar{a})$ are zero it follows that $(\wedge Z, \bar{d})$ is the minimal Sullivan model of a wedge of spheres with cohomology $\mathbb{Q} \oplus H(B^{\geq 1} \otimes \bar{a} \otimes \wedge U, d)$.

Thus in this case Theorem 1.4 will follow from the Sullivan model version of Theorem 3 and (4) in [5] once we show that the Sullivan acyclic closure $(\wedge Z \otimes \wedge S, \bar{d})$ of $(\wedge Z, \bar{d})$ satisfies

$$(3.1) \quad \rho_{\wedge S} < \rho_{\wedge U}.$$

Denote $H(B^{\geq 1} \otimes \bar{a} \otimes \wedge U)$ simply by H . Since $(\wedge Z, \bar{d})$ is the model of a wedge of spheres, it follows that $\wedge S$ is the dual of a tensor algebra TE with $E_i \simeq H^{i+1}$. Thus

$$(3.2) \quad \wedge S(z) = \frac{1}{1 - E(z)} = \frac{1}{1 - \frac{1}{z}H(z)}.$$

It remains to estimate $H(z)$.

For this recall that the morphism $B \rightarrow B/\beta$ corresponds to the inclusion

$$N - D^n \rightarrow (N - D^n) \cup_{S^{n-1}} D^n,$$

where S^{n-1} is the boundary of a small disk $D^n \subset N$. Since $H(N)$ is not monogenic Theorem 5.1 of [8] asserts that the sphere S^{n-1} corresponds

to an inert element in the homotopy Lie algebra of $N - D^n$. Thus by [8, Theorem 1.1],

$$H(\Omega(N - D^n)) \cong TV \otimes H(\Omega N)$$

where $V \cong H(\Omega N) \otimes v$ and $\deg v = n - 2$. Since $V(z) = z^{n-2}\Omega N(z)$ it follows that $\rho_V = \rho_{\Omega N}$ and that $V(\rho_V) = \infty$. Since

$$TV(z) = \frac{1}{1 - V(z)}$$

it follows that $\rho_{TV} < \rho_V$ and that $TV(\rho_{TV}) = \infty$.

Moreover, the minimal Sullivan model $(\wedge W, d)$ of B/β has the form $(\wedge W_N \otimes \wedge P, d)$ in which $\wedge W_N$ is the minimal Sullivan model of N . Thus the acyclic closure $(\wedge W \otimes \wedge U, d)$ has the form

$$(\wedge W_N \otimes \wedge U_N \otimes \wedge P \otimes \wedge U_P, d)$$

in which $(\wedge W_N \otimes \wedge U_N, d)$ is the acyclic closure of $(\wedge W_N, d)$. In particular, $\wedge U \cong \wedge U_N \otimes \wedge U_P$, and there are linear isomorphisms

$$(3.3) \quad \wedge U_N \cong H^*(\Omega N) \quad \text{and} \quad \wedge U_P \cong TV^\#,$$

$V^\#$ denoting the dual of V . Thus

$$\rho_{\wedge U_P} = \rho_{TV} < \rho_V = \rho_{\wedge U_N}.$$

Since $\wedge U = \wedge U_N \otimes \wedge U_P$, it follows that

$$\rho_{\wedge U} = \rho(\wedge U_N \otimes \wedge U_P) = \rho_{\wedge U_P},$$

and that $\wedge U(\rho_{\wedge U}) = \infty$.

Now consider the short exact sequence

$$0 \rightarrow (B^{\geq 1} \otimes \bar{a} \otimes \wedge U, d) \rightarrow (B \otimes \bar{a} \otimes \wedge U, d) \rightarrow (\bar{a} \otimes \wedge U, 0) \rightarrow 0.$$

Since $(B \otimes \bar{a} \otimes \wedge U, d) = (B \otimes \bar{a} \otimes \wedge U_N \otimes \wedge U_P, d)$ it follows that

$$H = H(B \otimes \bar{a} \otimes \wedge U, d) \cong \bar{a} \otimes \wedge U_P.$$

It follows that $H(B^{\geq 1} \otimes \bar{a} \otimes \wedge U, d)$ contains a subspace T with

$$T^{i+\deg \bar{a}+1} \cong (\wedge^{\geq 1} U_N \otimes \wedge U_P)^i.$$

In particular, with \gg denoting coefficient-wise inequality, we have

$$E(z) \gg z^{\deg \bar{a}} \cdot (\wedge^{\geq 1} U_N)(z) \cdot (\wedge U_P)(z).$$

Thus $\rho_E \leq \rho_{\wedge U}$ and if $\rho_E = \rho_{\wedge U}$, then $E(\rho_E) = \infty$. Since

$$\wedge S(z) = \frac{1}{1 - E(z)}$$

it follows in either case that $\rho_{\wedge S} < \rho_{\wedge U}$, which completes the proof of Theorem 1.4 in this case.

Case 2: Neither $H(M)$ nor $H(N)$ is monogenic. — In this case Theorem 5.4 of [8] asserts that the collar sphere S^{n-1} joining $M - \{pt\}$ to $N - \{pt\}$ represents an inert element in $L_{M\#N}$. Attaching a disk to this sphere gives $M \vee N$ and thus by Theorem 1.1 in [8] the homotopy fibre F of the map $p : M\#N \rightarrow M \vee N$ is rationally a wedge of spheres with

$$H_i(F) \cong H_{i-n+2}(\Omega(M \vee N)).$$

Thus

$$H(\Omega F) = TV \quad \text{and} \quad V_i \cong H_{i-n+2}(\Omega(M \vee N)),$$

and so

$$\Omega F(z) = \frac{1}{1 - z^{n-2}(\Omega(M \vee N))(z)}.$$

On the other hand it is a classical fact that the homotopy fibre G of the map $q : M \vee N \rightarrow M \times N$ is the join $\Omega M * \Omega N$, (we sketch the proof in Lemma 3.3 below). Thus G is the suspension of $\Omega M \wedge \Omega N$ and therefore rationally a wedge of spheres. Since $\pi_*(q)$ is trivially surjective. It follows that

$$H(\Omega G) = TW \quad \text{with} \quad W_i \cong H_{i-1}(\Omega M * \Omega N).$$

By hypothesis, $\rho_{\Omega N} \leq \rho_{\Omega M}$ and $\rho_{\Omega N}(\Omega N) = \infty$. In particular, $W(\rho_{\Omega N}) = \infty$ and, since $\Omega G(z) = \frac{1}{1-W(z)}$, it follows that the radius of convergence, ρ , of $\Omega G(z)$ satisfies

$$\rho < \rho_{\Omega N} \leq \rho_{\Omega M} \quad \text{and} \quad W(\rho) = 1.$$

Moreover, since $\pi_*(q)$ is surjective,

$$H(\Omega(M \vee N)) = H(\Omega G) \otimes H(\Omega M) \otimes H(\Omega N)$$

and so ρ is also the radius of convergence of $\Omega(M \vee N)(z)$ and

$$\Omega(M \vee N)(\rho) = \infty.$$

Finally, since

$$\Omega F(z) = \frac{1}{1 - z^{n-2}\Omega(M \vee N)(z)}$$

it follows that $\rho_{\Omega F} < \rho = \rho_{\Omega(M \vee N)}$ and Theorem 1.4 follows from Theorem 1, Theorem 3 and (4) in [5]. □

LEMMA 3.3. — *The homotopy fiber G of the injection $q : M \vee N \rightarrow M \times N$ has the homotopy type of $\Omega M * \Omega N$.*

Proof. — Recall the Cube Lemma ([13]): In a homotopy commutative cube, if the vertical faces are homotopy pullbacks and the lower face an homotopy push-out, then the upper face is also an homotopy push-out.

Let $j : G \rightarrow M \vee N$ be the homotopy fibre of the inclusion q . Then we form the following cube by taking the pullbacks of j along the injections $M \rightarrow M \vee N$ and $N \rightarrow M \vee N$.

$$\begin{array}{ccccc}
 \Omega N \times \Omega M & \longrightarrow & \Omega M & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 & & \Omega N & \longrightarrow & G \\
 & & \downarrow & & \downarrow j \\
 \{*\} & \longrightarrow & N & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 & & M & \longrightarrow & M \vee N
 \end{array}$$

This shows that $G \cong \Omega M * \Omega N$. □

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