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# SCHATTEN PROPERTIES OF TOEPLITZ OPERATORS ON THE PALEY–WIENER SPACE

**by R. V. BESSONOV (*)**

Abstract. — We collect several old and new descriptions of Schatten class Toeplitz operators on the Paley–Wiener space and answer a question on discrete Hilbert transform commutators posed by Richard Rochberg.

Résumé. — Nous présentons plusieurs descriptions anciennes et nouvelles des opérateurs de Toeplitz de classe de Schatten sur l'espace de Paley-Wiener et répondons à une question de Richard Rochberg sur les commutateurs discrets de la transformée de Hilbert.

## 1. Introduction

Given a bounded function $\varphi$ on the real line, $\mathbb{R}$, consider the Toeplitz operator $T_\varphi$ on the classical Paley–Wiener space $\mathrm{PW}_a$,

$$T_\varphi\colon f \mapsto P_a(\varphi f), \qquad f \in \mathrm{PW}_a. \tag{1.1}$$

The space $\mathrm{PW}_a$ could be regarded as the subspace in $L^2(\mathbb{R})$ of functions with Fourier spectrum in the interval $[-a, a]$, symbol $P_a$ above denotes the orthogonal projection in $L^2(\mathbb{R})$ to $\mathrm{PW}_a$. Basic theory of Toeplitz operators on $\mathrm{PW}_a$ can be found in paper [9] by R. Rochberg.

We are interested in description of Schatten class Toeplitz operators on $\mathrm{PW}_a$ in terms of their standard symbols. By the standard symbol of an operator in (1.1) we mean the entire function $\varphi_{st} = \mathcal{F}^{-1}\chi_{2a}\,\mathcal{F}\varphi$, where $\mathcal{F}$ denotes the Fourier transform on the Schwartz space of tempered distributions, and $\chi_{2a}$ is the indicator function of the interval $(-2a, 2a)$. As we

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will see, a Toeplitz operator $T_\varphi$ on $\mathrm{PW}_a$ belongs to the Schatten class $\mathcal{S}^p$, $0 < p < \infty$, if and only if $\mathrm{e}^{2iax}\,\varphi_{st}$ belongs to a discrete oscillation Besov space introduced in 1987 by R. Rochberg [9]. Its definition we now recall.

For a measure $\mu$ on $\mathbb{R}$ and a function $f \in L^1_{\mathrm{loc}}(\mu)$, the oscillation of order $n$ of $f$ on an interval $I \subset \mathbb{R}$ with respect to $\mu$ is defined by

$$\mathrm{osc}(f, I, \mu, n) = \inf_{P_n} \frac{1}{\mu(I)} \int_I |f(x) - P_n(x)|\,\mathrm{d}\mu(x),$$

where the infimum is taken over all polynomials $P_n$ of degree at most $n$. If $\mu(I) = 0$, we put $\mathrm{osc}_I(f, I, \mu, n) = 0$. Define the family $\mathcal{I}_a$ of closed intervals

$$I_{a,j,k} = \left[\frac{2\pi}{a}\,k\,2^j, \frac{2\pi}{a}(k+1)2^j\right], \qquad j, k \in \mathbb{Z}, \quad j \geqslant 0.$$

Note that endpoints of intervals in $\mathcal{I}_a$ belong to the lattice $\mathbb{Z}_a = \left\{\frac{2\pi}{a}k,\ k \in \mathbb{Z}\right\}$. Let $p$ be a positive real number, and let $[\frac{1}{p}]$ be the integer part of $\frac{1}{p}$. The discrete oscillation Besov space $\mathbb{B}_p(a, \mathrm{osc}) = \mathbb{B}^{1/p}_{p,p}(\mathbb{Z}_a, \mu_a, \mathrm{osc})$ is defined by

$$\mathbb{B}_p(a, \mathrm{osc}) = \left\{ f \in L^1_{\mathrm{loc}}(\mu_a) : \|f\|^p_{\mathbb{B}_p(a,\mathrm{osc})} = \sum_{I \in \mathcal{I}_a} \mathrm{osc}\left(f, I, \mu_a, \left[\frac{1}{p}\right]\right)^p < \infty \right\},$$

where $\mu_a = \frac{2\pi}{a}\sum_{x\in\mathbb{Z}_a}\delta_x$ is the normalized counting measure on $\mathbb{Z}_a$.

Our main result is the following theorem.

Theorem 1.1. — *Let $a$, $p$ be positive real numbers, let $\varphi$ be a bounded function on $\mathbb{R}$, and let $\varphi_{st}$ be the standard symbol of the Toeplitz operator $T_\varphi$ on $\mathrm{PW}_a$. Then we have $T_\varphi \in \mathcal{S}^p(\mathrm{PW}_a)$ if and only if $\mathrm{e}^{2iax}\,\varphi_{st} \in \mathbb{B}_p(4a, \mathrm{osc})$. Moreover, $\|T_\varphi\|_{\mathcal{S}^p}$ is comparable to $\|\mathrm{e}^{2iax}\,\varphi_{st}\|_{\mathbb{B}_p(4a,\mathrm{osc})}$ with constants depending only on $p$.*

Theorem 1.1 complements a classical description of Toeplitz operators in $\mathcal{S}^p(\mathrm{PW}_a)$ given by R. Rochberg [9] for $1 \leqslant p < \infty$ and extended by V. Peller [5] to the whole range $0 < p < \infty$. To formulate the result, consider a system $\{\nu_j\}_{j\leqslant -1}$ of infinitely smooth functions on $\mathbb{R}$ such that $\mathrm{supp}\,\nu_j \subset [2^{j-1}, 2^j]$,

$$0 \leqslant \nu_j \leqslant 1, \quad \nu_{j-1}(x) = \nu_j(x/2), \quad \sum \nu_j = 1 \text{ on } \left(0, \frac{1}{3}\right].$$

Define $\nu_j(x) = \nu_{-j}(1 - x)$ for real $x \geqslant \frac{1}{2}$ and integer $j \geqslant 1$, put $\nu_0 = 1 - \sum_{j\neq 0}\nu_j$ for $j = 0$. Finally, let $\nu_{a,j}(x) = \nu_j((x+a)/2a)$ for all $x \in [-a, a]$ and $j \in \mathbb{Z}$. Observe that system $\{\nu_{a,j}\}$ provides a resolution of unity on the interval $[-a, a]$ by functions supported on subintervals $I_j$ whose lengths are comparable to the distance from $I_j$ to the endpoints of $[-a,a]$. Rochberg–Peller theorem says that $T_\varphi$ is in $\mathcal{S}^p(\mathrm{PW}_a)$ for $0 < p < \infty$ if and only if

$$a\sum_{j\in\mathbb{Z}} 2^{-|j|}\cdot \|\mathcal{F}^{-1}(\nu_{2a,j}\cdot\mathcal{F}\varphi)\|^p_{L^p(\mathbb{R})} < \infty,$$

with control of the norms. R. Rochberg gives yet another characterization of Toeplitz operators in class $\mathcal{S}^p(\mathrm{PW}_a)$, $1 \leqslant p < \infty$, in terms of a reproducing kernel decomposition of their standard symbols, see Theorem 5.3 in [9]. Both the statement and the proof of his result for $p = 1$ contain errors that we correct in Section 3.

As a consequence of Theorem 1.1, we obtain the following result.

Theorem 1.2. — *Let $a > 0$. The discrete Hilbert transform commutator*

$$C_\psi : f \mapsto \frac{1}{\pi}\int_{\mathbb{Z}_a} \frac{\psi(x)-\psi(t)}{x-t} f(t)\,\mathrm{d}\mu_a(t), \qquad f \in L^2(\mu_a),$$

*belongs to the trace class $\mathcal{S}^1(L^2(\mu_a))$ if and only if $\psi \in \mathbb{B}_1(a,\mathrm{osc})\cap L^\infty(\mathbb{Z}_a)$.*

This answers the question posed by R. Rochberg in 1987. See Section 6 for a summary of results on discrete Hilbert transform commutators and an analogue of Theorem 1.2 for the case $0 < p < 1$.

We would like to mention papers [11, 12] by R. Torres for readers interested in wavelet characterizations and interpolation theory of discrete Besov spaces. The problem of membership in Schatten classes $\mathcal{S}^p$ for general truncated Toeplitz operators has been recently studied by P. Lopatto and R. Rochberg [3], see also Section 4.3 in author's paper [1].

## 2. Proof of Theorem 1.1 for $1 < p < \infty$

Theorem 1.1 for $1 < p < \infty$ follows from known results. Let $\mathbb{B}_p(\mathbb{R}) = \dot{\mathbb{B}}^{1/p}_{p,p}(\mathbb{R})$ be the standard homogeneous Besov space on the real line $\mathbb{R}$, see, e.g., Chapter 3 in [4] for definition and basic properties. Given a Toeplitz operator $T_\varphi$ on $\mathrm{PW}_a$ with symbol $\varphi \in L^\infty(\mathbb{R})$, we denote

$$\varphi^-_{st} = \mathcal{F}^{-1}\chi_{(-2a,0)}\mathcal{F}\varphi, \qquad \varphi^+_{st} = \mathcal{F}^{-1}\chi_{[0,2a)}\mathcal{F}\varphi,$$

where $\chi_S$ is the indicator function of a set $S$. As usual, $\mathcal{F}$ stands for the Fourier transform on the Schwartz space of tempered distributions. The following result is a combination of Theorem 5.1 and its Corollary in [9].

Theorem (R. Rochberg). — *Let $1 < p < \infty$ and let $a > 0$. Then a Toeplitz operator $T_\varphi$ on $\mathrm{PW}_a$ belongs to $\mathcal{S}_p(\mathrm{PW}_a)$ if and only if*

$$\|\,\mathrm{e}^{2iax}\,\varphi_{st}^-\|_{\mathbb{B}_p(\mathbb{R})} + \|\,\mathrm{e}^{-2iax}\,\varphi_{st}^+\|_{\mathbb{B}_p(\mathbb{R})} < \infty,$$

*in which case $\|T_\varphi\|_{\mathcal{S}^p}$ is comparable to $\|\,\mathrm{e}^{2iax}\,\varphi_{st}^-\|_{\mathbb{B}_p(\mathbb{R})} + \|\,\mathrm{e}^{-2iax}\,\varphi_{st}^+\|_{\mathbb{B}_p(\mathbb{R})}$ with constants depending only on $p$.*

Denote by $\mathcal{E}_a$ the set of tempered distributions whose Fourier transforms are supported on the interval $[-a, a]$. Next result is Theorem 1 in [12].

Theorem (R. Torres). — *Let $1 < p < \infty$ and let $f$ be a function in $\mathcal{E}_a \cap \mathbb{B}_p(\mathbb{R})$ for some $a > 0$. Then its restriction to $\mathbb{Z}_{2a}$ belongs to $\mathbb{B}_p(2a, \mathrm{osc})$ and $\|f\|_{\mathbb{B}_p(2a,\mathrm{osc})}$ is comparable to $\|f\|_{\mathbb{B}_p(\mathbb{R})}$ with constants depending only on $p$. Moreover, every sequence in $\mathbb{B}_p(a, \mathrm{osc})$ is the restriction to $\mathbb{Z}_a$ of a unique function (modulo polynomials) in $\mathcal{E}_a \cap \mathbb{B}_p(\mathbb{R})$.*

*Proof of Theorem 1.1* ($1 < p < \infty$). — Let $\varphi$ be a bounded function of $\mathbb{R}$ and let $\varphi_{st} = \mathcal{F}^{-1}\chi_{(-2a,2a)}\mathcal{F}\varphi$ be the standard symbol of the Toeplitz operator $T_\varphi \in \mathcal{S}^p(\mathrm{PW}_a)$. Then functions $\mathrm{e}^{2iax}\,\varphi_{st}^-$, $\mathrm{e}^{-2iax}\,\varphi_{st}^+$ belong to $\mathcal{E}_{2a} \cap \mathbb{B}_p(\mathbb{R})$ by R. Rochberg's theorem above. From theorem by R. Torres we see that $\mathrm{e}^{2iax}\,\varphi_{st}^- \in \mathbb{B}_p(4a, \mathrm{osc})$ and $\mathrm{e}^{-2iax}\,\varphi_{st}^+ \in \mathbb{B}_p(4a, \mathrm{osc})$ with control of the norms. Now observe that $\mathrm{e}^{4iax} = 1$ and $\mathrm{e}^{2iax}\,\varphi_{st} = \mathrm{e}^{2iax}\,\varphi_{st}^- + \mathrm{e}^{-2iax}\,\varphi_{st}^+$ on $\mathbb{Z}_{4a}$, hence $\mathrm{e}^{2iax}\,\varphi_{st} \in \mathbb{B}_p(4a, \mathrm{osc})$.

Conversely, assume that the restriction of $\mathrm{e}^{2iax}\,\varphi_{st}$ to $\mathbb{Z}_{4a}$ is in $\mathbb{B}_p(4a, \mathrm{osc})$. Using theorem by R. Torres, find a function $f \in \mathcal{E}_{2a} \cap \mathbb{B}_p(\mathbb{R})$ such that its restriction to $\mathbb{Z}_{4a}$ agrees with $\mathrm{e}^{2iax}\,\varphi_{st}$. Put $f^- = \mathcal{F}^{-1}\chi_{(-2a,0)}\mathcal{F}f$ and $f^+ = \mathcal{F}^{-1}\chi_{[0,2a)}\mathcal{F}f$. Observe that $\tilde{\varphi} = \mathrm{e}^{-2iax}\,f^+ + \mathrm{e}^{2iax}\,f^-$ is an entire function of exponential type at most $2a$ coinciding with $\varphi_{st}$ on $\mathbb{Z}_{4a}$. Since $\varphi_{st}$, $\tilde{\varphi}$ are the first order distributions supported on the finite interval $[-2a, 2a]$, we have $|\tilde{\varphi}(x)| + |\varphi(x)| \leqslant c + c|x|$ for all $x \in \mathbb{R}$ and a constant $c \geqslant 0$. It follows that the entire function $\frac{\tilde{\varphi}-\varphi}{z}$ of exponential type at most $2a$ is bounded on $\mathbb{R}$ and vanishes on $\mathbb{Z}_{4a} \setminus \{0\}$, hence $\tilde{\varphi} - \varphi_{st} = p\sin(2az)$ for a polynomial $p$ of degree at most 1. Therefore, we have $T_\varphi = T_{\varphi_{st}} = T_{\tilde{\varphi}}$ on $\mathrm{PW}_a$, see Section 2.D in [9]. Since $f^\pm \in \mathbb{B}_p(\mathbb{R})$, we can use R. Rochberg's theorem and conclude that $T_{\tilde{\varphi}} \in \mathcal{S}^p(\mathrm{PW}_a)$ with control of the norms: $\|T_{\tilde{\varphi}}\|_{\mathcal{S}^p}$ is controllable by $\|\,\mathrm{e}^{2iax}\,\tilde{\varphi}^-\|_{\mathbb{B}_p(\mathbb{R})} + \|\,\mathrm{e}^{-2iax}\,\tilde{\varphi}^+\|_{\mathbb{B}_p(\mathbb{R})} \leqslant c_p\|f\|_{\mathbb{B}_p(\mathbb{R})} \leqslant \tilde{c}_p\|\,\mathrm{e}^{2iax}\,\varphi_{st}\|_{\mathbb{B}_p(4a,\mathrm{osc})}$. □

## 3. Reproducing kernel decomposition of standard symbols

In this section we show that the standard symbol of a Toeplitz operator on $\mathrm{PW}_a$ from class $\mathcal{S}^p$ could be represented as a linear combination of normalized reproducing kernels of $\mathrm{PW}_{2a}$ with coefficients $c_k$ such that $\sum |c_k|^p < \infty$. We consider only the case $0 < p \leqslant 1$. Proposition 3.1 below is a corrected version of Theorem 5.3 in [9]. In the original statement the author of [9] forgot to normalize the exponentials in formula (5.6) of [9]. More importantly, he used the fact that the Fourier multiplier $f \mapsto \mathcal{F}^{-1}\chi_{[0,1]}\mathcal{F}f$ is bounded on $\mathbb{B}_p(\mathbb{R})$. This is not the case for $p = 1$. Here is a more accurate implementation of the ideas from [9].

Let $\psi$ be a bounded function on the real line $\mathbb{R}$. Consider the standard Hardy space $H^2$ in the upper half-plane $\mathbb{C}^+ = \{\lambda \in \mathbb{C} : \ \operatorname{Im}\lambda > 0\}$ of the complex plane $\mathbb{C}$. Denote by $H^2_-$ the anti-analytic subspace $\{f \in L^2(\mathbb{R}) : \bar{f} \in H^2\}$ of $L^2(\mathbb{R})$. Recall that the classical Hankel operator $H_\psi : H^2 \to H^2_-$ is defined by

$$H_\psi : f \mapsto P_-(\psi f), \qquad f \in H^2,$$

where $P_-$ denotes the orthogonal projection from $L^2(\mathbb{R})$ to $H^2_-$. The operator $H_\psi$ is completely determined by its standard anti-analytic symbol $\psi_{st} = \mathcal{F}^{-1}\chi_{(-\infty,0)}\mathcal{F}\psi$. The latter means that $H_\psi f = H_{\psi_{st}} f$ for all $f \in H^2$ such that $\sup_{x\in\mathbb{R}} |xf(x)| < \infty$. Take a positive number $\varepsilon > 0$ and define the sets $\mathcal{U}_\varepsilon^+$, $\mathcal{U}_\varepsilon^-$ by

$$\mathcal{U}_\varepsilon^\pm = \{\lambda \in \mathbb{C} : \ \lambda = (1+\varepsilon)^m(\varepsilon x \pm i); \ x, m \in \mathbb{Z}\}.$$

For $\lambda \in \mathbb{C}^+$, let $k_\lambda = -\frac{1}{2\pi i}\frac{1}{z-\bar{\lambda}}$ denote the reproducing kernel of $H^2$ at $\lambda$.

Theorem (R. Rochberg [8]). — *There exists a number $\varepsilon > 0$ such that $H_\psi \in \mathcal{S}^p(H^2)$ if and only if $\psi_{st} = \sum_{\lambda\in\mathcal{U}_\varepsilon^+} c_\lambda \frac{\overline{k_\lambda}}{\|k_\lambda\|^2}$, where $\sum |c_\lambda|^p$ is finite and the infimum of $\sum |c_\lambda|^p$ over all possible representations of $\psi_{st}$ in this form is comparable to $\|H_\psi\|^p_{\mathcal{S}^p}$ with constants depending only on $p \in (0,\infty)$.*

Remark that for $p \in (0,1]$ the series defining $\psi_{st}$ in the theorem above converges absolutely to a bounded function on $\mathbb{R}$, while for $p > 1$ the convergence holds only in the Besov space $\mathbb{B}_p(\mathbb{R})$ (one need to extract constant terms from every summand to get the convergent series, see discussion in [8]). In order to prove an analogous result for Toeplitz operators on the Paley–Wiener space, let us consider the sets

$$\mathcal{U}^\pm_{\eta a,\varepsilon} = \left\{\lambda \in \mathcal{U}_\varepsilon^\pm : |\operatorname{Im}\lambda| > \frac{\varepsilon}{\eta a}\right\}, \qquad \Lambda_{\eta a,\varepsilon} = \mathcal{U}^-_{\eta a,\varepsilon} \cup \mathbb{Z}_{\eta a} \cup \mathcal{U}^+_{\eta a,\varepsilon}.$$

Here $\mathbb{Z}_{\eta a} = \{\frac{2\pi}{\eta a}k, \ k \in \mathbb{Z}\}$. Next, for $a > 0$ and $\lambda \in \mathbb{C}$, denote by $\rho_{a,\lambda}$ the reproducing kernel of the space $\mathrm{PW}_a$ at the point $\lambda$. Recall that

$$\rho_{a,\lambda} : z \mapsto \frac{1}{\pi}\frac{\sin a(z-\bar{\lambda})}{z-\bar{\lambda}}, \qquad z \in \mathbb{C}.$$

We are going to prove the following proposition.

Proposition 3.1. — *Let $a > 0$ and let $\varphi \in L^\infty(\mathbb{R})$. There exist $\varepsilon > 0$, $\eta > 1$ such that $T_\varphi \in \mathcal{S}^p(\mathrm{PW}_a)$ if and only if $\varphi_{st} = \sum_{\lambda \in \Lambda_{\eta a, \varepsilon}} c_\lambda \frac{\rho_{2a,\lambda}}{\|\rho_{a,\lambda}\|^2}$, where $\sum_\lambda |c_\lambda|^p$ is finite and the infimum of $\sum |c_\lambda|^p$ over all possible representations of $\varphi_{st}$ in this form is comparable to $\|T_\varphi\|^p_{\mathcal{S}^p}$ with constants depending only on $p \in (0,1]$.*

We will show how to reduce Proposition 3.1 to the above theorem for Hankel operators using a splitting of the standard symbol into three pieces: analytic, anti-analytic and a piece with "small" Fourier support.

The following two results for $0 < p \leqslant 1$ are consequences of Lemma 1 and Lemma 2 from [5]. The range $1 \leqslant p < \infty$ has been treated earlier in [9], see also Section 2 in [10].

Lemma 3.2. — *Let $a > 0$ and let $\varphi \in L^\infty(\mathbb{R})$. There exist bounded functions $\varphi_\ell$, $\varphi_c$, and $\varphi_r$ such that $T_\varphi = T_{\varphi_\ell} + T_{\varphi_c} + T_{\varphi_r}$ on $\mathrm{PW}_a$,*

$$\operatorname{supp} \mathcal{F}\varphi_\ell \subset [-4a, -\frac{a}{2}], \quad \operatorname{supp} \mathcal{F}\varphi_c \subset [-a, a], \quad \operatorname{supp} \mathcal{F}\varphi_r \subset [\frac{a}{2}, 4a],$$

*and we have $\|T_{\varphi_s}\|_{\mathcal{S}^p} \leqslant c_p \|T_\varphi\|_{\mathcal{S}^p}$ for every $s = \ell, c, r$ for $T_\varphi \in \mathcal{S}^p(\mathrm{PW}_a)$. Here $c_p$ is a constant depending only on $p$.*

Lemma 3.3. — *Let $a > 0$ and let $\varphi \in L^\infty(\mathbb{R})$ be such that $\operatorname{supp} \hat{\varphi} \subset [-a, a]$. Then $T_\varphi \in \mathcal{S}^p(\mathrm{PW}_a)$ if and only if $\varphi \in L^p(\mathbb{R})$, in which case $\|\varphi\|_{L^p(\mathbb{R})}$ is comparable to $\|T_\varphi\|_{\mathcal{S}^p}$ with constants depending only on $p$.*

*Proof of Proposition 3.1.* — Let $\varphi \in L^\infty(\mathbb{R})$ and let $\varphi_{st} = \mathcal{F}^{-1}\chi_{(-2a,2a)}\mathcal{F}\varphi$ be the standard symbol of the operator $T_\varphi$ on $\mathrm{PW}_a$. Then $T_\varphi = T_{\varphi_{st}}$, see Section 2.D in [9]. Suppose that $\varphi_{st} = \sum_{\lambda \in \Lambda_{\eta a, \varepsilon}} c_\lambda \frac{\rho_{2a,\lambda}}{\|\rho_{a,\lambda}\|^2}$ for some $\varepsilon > 0$, $\eta > 0$, and some coefficients $c_\lambda$ such that $\sum_{\lambda \in \Lambda_{\eta a, \varepsilon}} |c_\lambda|^p < \infty$. It follows from the estimate

$$\frac{|\rho_{2a,\lambda}(z)|}{\|\rho_{a,\lambda}\|^2} \leqslant c\, \mathrm{e}^{2a|\operatorname{Im} z|}, \qquad z \in \mathbb{C}, \quad \lambda \in \mathbb{C},$$

that this series converges absolutely to an entire function of exponential type at most $2a$ bounded on the real line $\mathbb{R}$. By triangle inequality (see, e.g., Theorem A1.1 in [6]), we have

$$\|T_\varphi\|^p_{\mathcal{S}^p} = \|T_{\varphi_{st}}\|^p_{\mathcal{S}^p} \leqslant \Big(\sum_{\lambda \in \Lambda_{\eta a, \varepsilon}} |c_\lambda|^p\Big) \sup_{\lambda \in \mathbb{C}} \|T_{\varphi_\lambda}\|^p_{\mathcal{S}^p},$$

where we denoted $\varphi_\lambda = \frac{\rho_{2a,\lambda}}{\|\rho_{a,\lambda}\|^2}$. Take $\lambda \in \mathbb{C}$. For every $f, g \in \mathrm{PW}_a$ we have

$$(T_{\rho_{2a,\lambda}} f, g) = (f\bar{g}, \rho_{2a,\bar{\lambda}}) = f(\bar{\lambda}) \cdot \overline{g(\lambda)} = (f, \rho_{a,\bar{\lambda}})(\rho_{a,\lambda}, g).$$

It follows that the operator $T_{\varphi_\lambda}$ has rank one and $\|T_{\varphi_\lambda}\|_{\mathcal{S}^p} = 1$. Hence $T_\varphi$ belongs to $\mathcal{S}^p(\mathrm{PW}_a)$ and $\|T_\varphi\|^p_{\mathcal{S}^p} \leqslant \sum_\lambda |c_\lambda|^p$.

Now let $\varphi$ be a bounded function on $\mathbb{R}$ such that $T_\varphi \in \mathcal{S}^p(\mathrm{PW}_a)$. We want to show that the standard symbol $\varphi_{st} = \mathcal{F}^{-1}\chi_{(-2a,2a)}\mathcal{F}\varphi$ of $T_\varphi$ can be represented in the form

$$\varphi_{st} = \sum_{\lambda \in \Lambda_{\eta a, \varepsilon}} c_\lambda \frac{\rho_{2a,\lambda}}{\|\rho_{a,\lambda}\|^2}$$

for some positive numbers $\varepsilon$, $\eta$ depending only on $p$ and a sequence $\{c_\lambda\}$ such that $\sum_\lambda |c_\lambda|^p$ is comparable to $\|T_\varphi\|^p_{\mathcal{S}^p}$. By Lemma 3.2, it suffices to consider separately the following three cases:

(1) $\operatorname{supp}\hat{\varphi} \subset (-\infty, 0]$;
(2) $\operatorname{supp}\hat{\varphi} \subset [-a, a]$;
(3) $\operatorname{supp}\hat{\varphi} \subset [0, +\infty)$.

Let us treat the third case first. Denote by $M_{\mathrm{e}^{-\mathrm{i}ax}}$ the operator of multiplication by $\mathrm{e}^{-\mathrm{i}ax}$ on $L^2(\mathbb{R})$. Since $\operatorname{supp}\hat{\varphi} \subset [0, +\infty)$, we have

$$H_{\mathrm{e}^{-2\mathrm{i}ax}\varphi} = M_{\mathrm{e}^{-\mathrm{i}ax}} T_\varphi P_a M_{\mathrm{e}^{-\mathrm{i}ax}},$$

where $H_{\mathrm{e}^{-2\mathrm{i}ax}\varphi} : H^2 \to H^2_-$ is the Hankel operator with symbol $\psi = \mathrm{e}^{-2\mathrm{i}ax}\varphi$. In particular, we have $\|H_\psi\|_{\mathcal{S}^p} \leqslant \|T_\varphi\|_{\mathcal{S}_p}$. By Rochberg's Theorem above, the anti-analytic function $\psi_{st} = \mathcal{F}^{-1}\chi_{(-\infty,0)}\mathcal{F}\,\mathrm{e}^{-2\mathrm{i}ax}\varphi$ admits the following representation:

$$\psi_{st} = \sum_{\lambda \in \mathcal{U}^+_\varepsilon} c_\lambda \frac{\overline{k_\lambda}}{\|k_\lambda\|^2},$$

where $\sum_{\lambda \in \mathcal{U}^+_\varepsilon} |c_\lambda|^p$ is comparable to $\|H_\psi\|^p_{\mathcal{S}^p}$, and $\varepsilon > 0$ does not depend on $\psi$. This gives us decomposition for $\varphi_{st}$:

$$\varphi_{st} = \mathrm{e}^{2\mathrm{i}ax}\psi_{st} = \sum_{\lambda \in \mathcal{U}^+_\varepsilon} c_\lambda \frac{\mathrm{e}^{2\mathrm{i}ax}\,\overline{k_\lambda}}{\|k_\lambda\|^2} = \sum_{\lambda \in \mathcal{U}^+_\varepsilon} c_\lambda \frac{P_{2a}(\mathrm{e}^{2\mathrm{i}ax}\,\overline{k_\lambda})}{\|k_\lambda\|^2},$$

where $P_{2a}$ denotes the orthogonal projection in $L^2(\mathbb{R})$ to $\mathrm{PW}_{2a}$. It is easy to see that $P_{2a}(\mathrm{e}^{2\mathrm{i}ax}\,\overline{k_\lambda}) = \mathrm{e}^{2\mathrm{i}a\bar{\lambda}}\rho_{2a,\bar{\lambda}}$ and $\|\rho_{a,\bar{\lambda}}\|^2 \leqslant 2\,\mathrm{e}^{2a\operatorname{Im}\lambda}\cdot\|k_\lambda\|^2_{L^2(\mathbb{R})}$, hence

$$\varphi_{st} = \sum_{\lambda \in \mathcal{U}^-_\varepsilon} c_{\bar{\lambda}}\beta_\lambda \frac{\rho_{2a,\lambda}}{\|\rho_{a,\lambda}\|^2}$$

for some complex numbers $\beta_\lambda$ such that $\sup_\lambda |\beta_\lambda| \leqslant 2$. Next, in the case where $\operatorname{supp}\varphi \subset (-\infty, 0]$ we can consider the adjoint operator $T^*_\varphi = T_{\varphi^*_{st}}$

with the standard symbol $\varphi_{st}^* : z \mapsto \overline{\varphi_{st}(\bar z)}$ and conclude that in this situation

$$\varphi_{st} = \sum_{\lambda\in\mathcal{U}_\varepsilon^+} \overline{c_\lambda \beta_{\bar\lambda}} \frac{\rho_{2a,\lambda}}{\|\rho_{a,\lambda}\|^2}.$$

Now let $\operatorname{supp}\varphi \subset [-a,a]$. By Lemma 3.3, we have $\varphi \in L^p(\mathbb{R})$. In particular, $\varphi \in \mathrm{PW}_{2a}$ and Plancherel–Polya theorem [7] yields the following decomposition:

$$\varphi = \varphi_{st} = \frac{\pi}{2a} \sum_{\lambda\in\mathbb{Z}_{2a}} f(\lambda)\rho_{2a,\lambda}, \qquad \sum_{\lambda\in\mathbb{Z}_{2a}} |f(\lambda)|^p \leqslant c_p a^p \|\varphi\|^p_{L^p(\mathbb{R})},$$

where the constant $c_p$ depends only on $p$. Put $\Lambda_\varepsilon = \mathcal{U}_\varepsilon^+ \cup \mathbb{Z}_{2a} \cup \mathcal{U}_\varepsilon^-$. To summarize, we have proved that for every bounded function $\varphi$ on $\mathbb{R}$ such that $T_\varphi \in \mathcal{S}^p(\mathrm{PW}_a)$ there are coefficients $c_\lambda$, $\lambda \in \Lambda_\varepsilon$, such that

$$\varphi_{st} = \sum_{\lambda\in\Lambda_\varepsilon} c_\lambda \frac{\rho_{2a,\lambda}}{\|\rho_{a,\lambda}\|^2}, \qquad \sum_{\lambda\in\Lambda_\varepsilon} |c_\lambda|^p \leqslant c_p \|T_\varphi\|^p_{\mathcal{S}^p}. \tag{3.1}$$

It remains to show that the set $\Lambda_\varepsilon$ and coefficients $c_\lambda$ in this decomposition could be replaced by the set $\Lambda_{\eta a,\varepsilon}$ and some new coefficients $c_\lambda$ satisfying the second estimate in (3.1). To this end, for every point $\lambda \in \Lambda_\varepsilon$ denote by $\zeta_\lambda$ the nearest point to $\lambda$ in $\Lambda_{\eta a,\varepsilon} \subset \Lambda_\varepsilon$, where $\eta = 2^k$ and $k \in \mathbb{Z}$ is a positive integer number that will be specified later. Consider the function

$$\tilde\varphi^{(1)} = \sum_{\lambda\in\Lambda_\varepsilon} c_\lambda \frac{\rho_{2a,\zeta_\lambda}}{\|\rho_{a,\zeta_\lambda}\|^2} = \sum_{\lambda\in\Lambda_{\eta a,\varepsilon}} \tilde c_\lambda^{(1)} \frac{\rho_{2a,\lambda}}{\|\rho_{a,\lambda}\|^2}, \qquad \tilde c_\lambda^{(1)} = \sum_{\nu\in\Lambda_\varepsilon,\ \zeta_\nu=\lambda} c_\nu.$$

Note that $\tilde\varphi^{(1)}$ has the required representation and $\sum |\tilde c_\lambda^{(1)}|^p \leqslant \sum |c_\lambda|^p$. Moreover, we have $\|T_\varphi - T_{\tilde\varphi^{(1)}}\|^p_{\mathcal{S}^p} \leqslant \sum_{\lambda\in\Lambda_\varepsilon\setminus\Lambda_{\eta a,\varepsilon}} |c_\lambda|^p \cdot \|T_{\varphi_\lambda} - T_{\varphi_{\zeta_\lambda}}\|^p_{\mathcal{S}^p}$. On the other hand, the quasi-norm in $\mathcal{S}_p$ of the rank two operator

$$T_{\varphi_\lambda} - T_{\varphi_{\zeta_\lambda}} = \frac{\rho_{a,\lambda}}{\|\rho_{a,\lambda}\|} \otimes \frac{\rho_{a,\lambda}}{\|\rho_{a,\bar\lambda}\|} - \frac{\rho_{a,\zeta_\lambda}}{\|\rho_{a,\zeta_\lambda}\|} \otimes \frac{\rho_{a,\zeta_\lambda}}{\|\rho_{a,\zeta_\lambda}\|}$$

does not exceed

$$2^{\frac{1}{p}} \left\| \frac{\rho_{a,\zeta_\lambda}}{\|\rho_{a,\zeta_\lambda}\|} - \frac{\rho_{a,\lambda}}{\|\rho_{a,\lambda}\|} \right\|_{L^2(\mathbb{R})} \leqslant 2^{\frac{1}{p}+\frac{1}{2}} \left(1 - \frac{\operatorname{Re}\rho_{a,\zeta_\lambda}(\lambda)}{\|\rho_{a,\zeta_\lambda}\| \cdot \|\rho_{a,\lambda}\|}\right)^{\frac{1}{2}}.$$

Since $|\zeta_\lambda - \lambda| \leqslant \frac{2\pi}{\eta a}$ for all $\lambda$ by construction, one can choose a large number $\eta = 2^k$ so that $\|T_\varphi - T_{\tilde\varphi}\|^p_{\mathcal{S}^p} \leqslant \frac{1}{2}\|T_\varphi\|^p_{\mathcal{S}^p}$. Clearly, this choice of $\eta$ does not depend on $\varphi$ and $a$. Iterating the process, we see that there are functions

$$\tilde\varphi^{(n)} = \sum_{\lambda\in\Lambda_{\eta a,\varepsilon}} \tilde c_\lambda^{(n)} \frac{\rho_{2a,\lambda}}{\|\rho_{a,\lambda}\|^2}, \quad n = 1, 2, \ldots$$

such that $\|T_\varphi - T_{\tilde\varphi^{(1)}} - \cdots - T_{\tilde\varphi^{(n)}}\|^p_{\mathcal{S}^p} \leqslant \frac{1}{2^n}\|T_\varphi\|^p_{\mathcal{S}^p}$, $\sum_{n,\lambda}|\tilde c^{(n)}_\lambda|^p \leqslant c_p^p\|T_\varphi\|^p_{\mathcal{S}^p}$. Since $\mathcal{S}^p(\mathrm{PW}_a)$ is a complete quasi-normed space and a Toeplitz operator on $\mathrm{PW}_a$ is zero if and only if its standard symbol is zero (see Section 2.D in [9]), this gives us the required decomposition of $\varphi_{st}$ with coefficients $c_\lambda = \sum_{n\geqslant 1}\tilde c^{(n)}_\lambda$, $\lambda \in \Lambda_{\eta a,\varepsilon}$. □

## 4. Interpolation of discrete Besov sequences

Denote by $\mathrm{PW}_{[0,a]}$ the Paley–Wiener space of functions in $L^2(\mathbb{R})$ with Fourier spectrum in the interval $[0,a]$. Recall that the reproducing kernel $k_{a,\lambda}$ of the space $\mathrm{PW}_{[0,a]}$ at a point $\lambda \in \mathbb{C}_+$ has the form

$$k_{a,\lambda}(z) = -\frac{1}{2\pi i}\frac{1-\mathrm{e}^{\mathrm{i}a(z-\bar\lambda)}}{z-\bar\lambda}, \qquad z\in\mathbb{C}.$$

Denote by $\mathcal{C}_0(\mathbb{Z}_a)$ the set of functions on $\mathbb{Z}_a$ tending to zero at infinity. Our aim in this section is to prove the following proposition.

Proposition 4.1. — *Let $0 < p \leqslant 1$, let $\Lambda$ be the set $\Lambda_{\eta a,\varepsilon}$ from Proposition 3.1, and let $F = \sum_{\lambda\in\Lambda} c_\lambda \frac{k_{a,\lambda}}{\|k_{\frac{a}{2},\lambda}\|^2}$ for some $c_\lambda \in \mathbb{C}$ such that $\sum_{\lambda\in\Lambda}|c_\lambda|^p < \infty$. Then the restriction of $F$ to $\mathbb{Z}_a$ belongs to $\mathbb{B}_p(a,\mathrm{osc}) \cap \mathcal{C}_0(\mathbb{Z}_a)$. Conversely, for every function $f \in \mathbb{B}_p(a,\mathrm{osc})$ there exists the unique function $F$ as above and a polynomial $q$ of degree at most $[\frac{1}{p}]$ such that $f = q + F$ on $\mathbb{Z}_a$. Moreover, the infinum of $\sum_{\lambda\in\Lambda}|c_\lambda|^p$ over all possible representations of $F = \sum_{\lambda\in\Lambda} c_\lambda \frac{k_{a,\lambda}}{\|k_{\frac{a}{2},\lambda}\|^2}$ in this form is comparable to $\|f\|^p_{\mathbb{B}_p(\mathrm{osc},a)}$ with constants depending only on $p$.*

The proof of Proposition 4.1 is based on the following lemma.

Lemma 4.2. — *We have $\|k_{a,\lambda}\|_{\mathbb{B}_p(a,\mathrm{osc})} \leqslant c_p\|k_{\frac{a}{2},\lambda}\|^2$ for every $a > 0$, $0 < p \leqslant 1$, and $\lambda \in \mathbb{C}$, where the constant $c_p$ depends only on $p$.*

*Proof.* — At first, consider the points $\lambda$ in the support of $\mu_a$. For $\lambda \in \mathbb{Z}_a$ we have

$$k_{a,\lambda}(x) = \begin{cases} \|k_{a,\lambda}\|^2, & x = \lambda; \\ 0, & x \in \operatorname{supp}\mu_a \setminus \{\lambda\}. \end{cases}$$

Taking $P_I = 0$ for intervals $I \in \mathcal{I}_a$ in the definition of $\operatorname{osc}\big(k_{a,\lambda}, I, \mu_a, \big[\frac{1}{p}\big]\big)$, we obtain the estimate

$$\begin{aligned}\|k_{a,\lambda}\|^p_{\mathbb{B}_p(a,\mathrm{osc})} &\leqslant \sum_{I\in\mathcal{I}_a}\left(\frac{1}{\mu_a(I)}\int_I |k_{a,\lambda}(x)|\,\mathrm{d}\mu_a(x)\right)^p\\ &= \|k_{a,\lambda}\|^{2p}\mu_a(\{\lambda\})^p\sum_{I\in\mathcal{I}_a}\frac{\chi_I(\lambda)}{\mu_a(I)^p}\\ &\leqslant c_p\|k_{\frac{a}{2},\lambda}\|^{2p}.\end{aligned}$$

Now let $\lambda$ be an arbitrary point in $\mathbb{C}\setminus\operatorname{supp}\mu_a$. Then $k_{a,\lambda}(x) = -\frac{1}{2\pi i}\frac{1-\mathrm{e}^{-\mathrm{i}a\bar\lambda}}{x-\bar\lambda}$ for all $x \in \operatorname{supp}\mu_a$. Thus, we need to estimate an oscillation of the function $x \mapsto \frac{1}{x-\bar\lambda}$ on the lattice $\mathbb{Z}_a$. Divide collection $\mathcal{I}_a$ from Section 1 into two parts:

$$\begin{aligned}&\mathcal{I}_{a,1} = \{I \in \mathcal{I}_a\colon I = I_{a,j,k},\ \operatorname{Re}\lambda \notin I_{a,j,k-1}\cup I_{a,j,k}\cup I_{a,j,k+1}\},\\ &\mathcal{I}_{a,2} = \mathcal{I}_a \setminus \mathcal{I}_{a,1}.\end{aligned}$$

For an interval $I \in \mathcal{I}_{a,1}$ with center $x_c$, define the polynomial $P_I$ of degree $[\frac{1}{p}]$ by

$$\frac{1}{x-\bar\lambda} - P_I(x) = \frac{(x-x_c)^{[\frac{1}{p}]+1}}{(x-\bar\lambda)(\bar\lambda-x_c)^{[\frac{1}{p}]+1}}. \tag{4.1}$$

Using this polynomial, we can estimate

$$\operatorname{osc}\left(\frac{1}{x-\bar\lambda}, I, \mu_a, \left[\frac{1}{p}\right]\right) \leqslant \sup_{x\in I}\left|\frac{(x-x_c)^{[\frac{1}{p}]+1}}{(x-\bar\lambda)(\bar\lambda-x_c)^{[\frac{1}{p}]+1}}\right| \leqslant \frac{|I|^{[\frac{1}{p}]+1}}{\operatorname{dist}(\lambda, I)^{[\frac{1}{p}]+2}}, \tag{4.2}$$

where $|I|$ denotes the length of $I$. Since $I \in \mathcal{I}_{a,1}$, we have $\operatorname{dist}(\lambda, I) \geqslant |I|$, hence

$$\sum_{I\in\mathcal{I}_{a,1}}\operatorname{osc}\left(\frac{1}{\bar\lambda - x}, I, \mu_a, \left[\frac{1}{p}\right]\right)^p \leqslant \sum_{I\in\mathcal{I}_{a,1}}\frac{1}{|I|^p} \leqslant c_p\cdot a^p. \tag{4.3}$$

We also will need a more accurate estimate for the left hand side of the inequality above in the case where $|\operatorname{Im}\lambda|$ is large. For every $j \geqslant 0$, let $\mathcal{I}^j_{a,1}$

be the set of intervals $I_{a,j,k}$, $k \in \mathbb{Z}$, belonging to the family $\mathcal{I}_{a,1}$. We have

$$\sum_{I\in\mathcal{I}^j_{a,1}}\left(\frac{|I|^{[\frac{1}{p}]+1}}{\operatorname{dist}(\lambda,I)^{[\frac{1}{p}]+2}}\right)^p=\sum_{I\in\mathcal{I}^j_{a,1}}\left(\frac{|I|^{[\frac{1}{p}]+1}}{\left(|\operatorname{Im}\lambda|^2+\operatorname{dist}(\operatorname{Re}\lambda,I)^2\right)^{([\frac{1}{p}]+2)/2}}\right)^p$$

$$\leqslant c_p\left(\frac{a}{2^j}\right)^p\sum_{m\geqslant 1}\left(\frac{1}{\left(\frac{a}{2^j}\right)^2|\operatorname{Im}\lambda|^2+m^2}\right)^{\frac{1}{2}[\frac{1}{p}]p+p}$$

$$\leqslant c_p\left(\frac{a}{2^j}\right)^p\gamma_j^{1-[\frac{1}{p}]p-2p},$$

where $\gamma_j=\max\bigl(1,\frac{a}{2^j}|\operatorname{Im}\lambda|\bigr)$. Indeed, the last inequality follows from elementary estimates

$$\sum_{m=1}^{\infty}m^{-1-2p}<\infty,\qquad\int_1^\infty\frac{\mathrm{d}x}{(r^2+x^2)^s}\leqslant c_s r^{1-2s},$$

where $r>0$, and the constant $c_s$ depends on $s>1/2$. Put

$$N_\lambda=\begin{cases}\left[\log_2(a|\operatorname{Im}\lambda|)\right], & \text{if } a|\operatorname{Im}\lambda|\geqslant 2,\\ 0, & \text{if } a|\operatorname{Im}\lambda|<2.\end{cases}$$

Note that $\tilde{p}=-1+[\frac{1}{p}]p+p$ is a positive number. It follows

$$\sum_{I\in\mathcal{I}_{a,1}}\operatorname{osc}\left(\frac{1}{\bar\lambda-x},I,\mu_a,\left[\frac{1}{p}\right]\right)^p\leqslant c_p\sum_{j=0}^{\infty}\left(\frac{a}{2^j}\right)^p\gamma_j^{1-[\frac{1}{p}]p-2p}$$

$$\leqslant c_p a^{-\tilde p}|\operatorname{Im}\lambda|^{-\tilde p-p}\sum_{j=0}^{N_\lambda}2^{\tilde p j}+c_p\sum_{j=N_\lambda}^{\infty}\frac{a^p}{2^{pj}}$$

$$\leqslant\frac{c_p}{|\operatorname{Im}\lambda|^p}.$$

Combining the last estimate with (4.3), we get

$$\sum_{I\in\mathcal{I}_{a,1}}\operatorname{osc}\left(\frac{1}{\bar\lambda-x},I,\mu_a,\left[\frac{1}{p}\right]\right)^p\leqslant c_p\min\left(a^p,\frac{1}{|\operatorname{Im}\lambda|^p}\right).$$

Now consider the family $\mathcal{I}_{a,2}=\mathcal{I}_{a,21}\cup\mathcal{I}_{a,22}$,

$$\mathcal{I}_{a,21}=\{I\in\mathcal{I}_{a,2}\colon |I|\leqslant|\operatorname{Im}\lambda|\},\quad\mathcal{I}_{a,22}=\{I\in\mathcal{I}_{a,2}\colon |I|>|\operatorname{Im}\lambda|\}.$$

For an interval $I\in\mathcal{I}_{a,21}$ we use the polynomial $P_I$ defined by (4.1). Then formula (4.2) implies

$$\sum_{I\in\mathcal{I}_{a,21}}\operatorname{osc}\left(\frac{1}{\bar\lambda-x},I,\mu_a,\left[\frac{1}{p}\right]\right)^p\leqslant\sum_{I\in\mathcal{I}_{a,21}}\left(\frac{|I|^{[\frac{1}{p}]+1}}{|\operatorname{Im}\lambda|^{[\frac{1}{p}]+2}}\right)^p\leqslant\frac{c_p}{|\operatorname{Im}\lambda|^p}.$$

Note that if $|\operatorname{Im}\lambda| < \frac{2\pi}{a}$, the set $\mathcal{I}_{a,21}$ is empty. This shows that we can write

$$\sum_{I\in\mathcal{I}_{a,21}} \operatorname{osc}\left(\frac{1}{\bar{\lambda}-x}, I, \mu_a, \left[\frac{1}{p}\right]\right)^p \leqslant c_p \min\left(a^p, \frac{1}{|\operatorname{Im}\lambda|^p}\right).$$

For $I \in \mathcal{I}_{a,22}$ we put $P_I = 0$. Denote by $x_0$ the nearest point to $\lambda$ in $\operatorname{supp}\mu_a$, and set $I' = I \setminus \{x \in \mathbb{R} : |x - \operatorname{Re}\lambda| < \pi/a\}$. We have

$$\begin{aligned}\frac{1}{\mu_a(I)}\int_I \left|\frac{1}{x-\bar{\lambda}}\right| \mathrm{d}\mu_a(x) &\leqslant \frac{\mu_a(\{x_0\})}{\mu_a(I)|x_0-\bar{\lambda}|} + \frac{1}{\mu_a(I)}\int_{I'}\frac{\mathrm{d}x}{|x-\bar{\lambda}|}\\ &\leqslant \frac{c}{a|I||x_0-\bar{\lambda}|} + \frac{c}{|I|}\int_{\pi a^{-1}}^{|I|}\frac{\mathrm{d}x}{\sqrt{x^2+|\operatorname{Im}\lambda|^2}}\\ &\leqslant \frac{c}{a|I||x_0-\bar{\lambda}|} + \frac{c}{|I|}\min\left(\log\frac{a|I|}{\pi}, \log^+\frac{|I|}{|\operatorname{Im}\lambda|}\right).\end{aligned}$$

Using estimates $\sum_{I\in\mathcal{I}_{a,2}} \frac{1}{|I|^p} \leqslant c_p a^p$, $\sum_{I\in\mathcal{I}_{a,2}} \left(\frac{\log a|I|}{|I|}\right)^p \leqslant c_p a^p$, and

$$\sum_{I\in\mathcal{I}_{a,22}} \left(\frac{1}{|I|}\log\frac{|I|}{|\operatorname{Im}\lambda|}\right)^p \leqslant \frac{c_p}{|\operatorname{Im}\lambda|^p},$$

we see that

$$\sum_{I\in\mathcal{I}_{a,22}} \operatorname{osc}\left(\frac{c_p}{\bar{\lambda}-x}, I, \mu_a, \left[\frac{1}{p}\right]\right)^p \leqslant \frac{c_p}{|x_0-\bar{\lambda}|^p} + c_p\min\left(a^p,\ \frac{1}{|\operatorname{Im}\lambda|^p}\right).$$

Eventually, we obtain

$$\left\|\frac{1}{x-\bar{\lambda}}\right\|^p_{\mathbb{B}_p(a,\mathrm{osc})} \leqslant \frac{c_p}{|x_0-\bar{\lambda}|^p} + c_p\min\left(a^p,\ \frac{1}{|\operatorname{Im}\lambda|^p}\right).$$

It follows that

$$\begin{aligned}\|k_{a,\lambda}\|^p_{\mathbb{B}_p(a,\mathrm{osc})} &\leqslant c_p(1+\mathrm{e}^{-a\operatorname{Im}\lambda})^p\min\left(a^p,\ \frac{1}{|\operatorname{Im}\lambda|^p}\right) + c_p\left|\frac{1-\mathrm{e}^{-\mathrm{i}a\bar{\lambda}}}{x_0-\lambda}\right|^p\\ &\leqslant c_p\|k_{\frac{a}{2},\lambda}\|^{2p},\end{aligned}$$

which is the desired estimate. □

Let $\mathcal{C}_0(\mathbb{R})$ denote the set of all continuous functions on $\mathbb{R}$ tending to zero at infinity. For completeness, we include the proof of the following known lemma.

Lemma 4.3. — *Let $0 < p \leqslant 1$, $a > 0$. For every function $f \in \mathbb{B}_p(\mathrm{osc}, a)$ there exists a function $F \in \mathbb{B}_p(\mathbb{R})$ such that $F = f$ on $\mathbb{Z}_a$, and*

$$\|F\|_{\mathbb{B}_p(\mathbb{R})} \leqslant c_p\|f\|_{\mathbb{B}_p(\mathrm{osc},a)},$$

*where the constant $c_p$ depends only $p$.*

*Proof.* — For $k \in \mathbb{Z}$ put $I_k = [\frac{2\pi}{a}[\frac{1}{p}]k, \frac{2\pi}{a}[\frac{1}{p}](k+1)]$. Interiors of intervals $I_k$ are disjoint and every set $I_k \cap \mathbb{Z}_a$ contains $[\frac{1}{p}]+1$ points. On every $I_k$ define the polynomial $P_k$ of degree at most $[\frac{1}{p}]$ such that $P_k(x) = f(x)$ for all $x \in I_k \cap \mathbb{Z}_a$. Next, set $F(x) = P_k(x)$ for $x \in I_k$. We claim that the function $F$ is in $\mathbb{B}_p(\mathbb{R})$. To check this, let us take an interval $J_{j,k} = [\frac{2\pi}{a}[\frac{1}{p}]k\cdot 2^j, \frac{2\pi}{a}[\frac{1}{p}](k+1)\cdot 2^j]$ with $k, j \in \mathbb{Z}$. In the case where $j < 0$ we clearly have $\operatorname{osc}(F, J_{j,k}, m, [\frac{1}{p}]) = 0$ because the function $F$ is a polynomial of degree at most $[\frac{1}{p}]$ on $I$. Hence, we can assume that $J = J_{j,k} = I_\ell \cup \ldots \cup I_{\ell+N}$ for some $\ell \in \mathbb{Z}$ and $N \geqslant 1$. Consider the polynomial $P_J$ of degree at most $[\frac{1}{p}]$ such that

$$\operatorname{osc}\left(f, J, \mu_a, \left[\frac{1}{p}\right]\right) = \frac{1}{\mu_a(J)} \int_J |f(x) - P_J(x)| \,\mathrm{d}\mu_a(x).$$

We have

$$\begin{aligned}\frac{1}{|J|}\int_J |F(x) - P_J(x)|\,\mathrm{d}x &= \frac{1}{|J|}\sum_{s=0}^{N}\int_{I_{\ell+s}} |P_{\ell+s}(x) - P_J(x)|\,\mathrm{d}x \\ &\leqslant \frac{c_p}{|J|}\sum_{s=0}^{N}\int_{I_{\ell+s}} |P_{\ell+s}(x) - P_J(x)|\,\mathrm{d}\mu_a(x) \leqslant c_p \operatorname{osc}\left(f, I, \mu_a, \left[\frac{1}{p}\right]\right),\end{aligned}$$

where we used the fact that

$$\int_{I_\ell} |P(x)|\,\mathrm{d}x \leqslant c_p \int_{I_\ell} |P(x)|\,\mathrm{d}\mu_a(x)$$

for every interval $I_\ell$, $\ell \in \mathbb{Z}$, and every polynomial $P$ of degree at most $[\frac{1}{p}]$. It follows that

$$\|F\|^p_{\mathbb{B}_p(\mathbb{R},m,\mathrm{osc})} \leqslant c_p^p \sum_{j,k} \operatorname{osc}\left(f, J_{j,k}, \mu_a, \left[\frac{1}{p}\right]\right)^p \leqslant c_p^p \|f\|^p_{\mathbb{B}_p(\mathrm{osc},a)},$$

and hence $F$ belongs to the space $\mathbb{B}^{1/p}_{p,p}(\mathbb{R}, dx, \mathrm{osc}) = \mathbb{B}_p(\mathbb{R})$, as required. □

*Proof of Proposition 4.1.* — Consider a function $F$ of the form

$$F = \sum_{\lambda\in\Lambda} c_\lambda \frac{k_{a,\lambda}}{\|k_{\frac{a}{2},\lambda}\|^2}, \qquad \sum_{\lambda\in\Lambda} |c_\lambda|^p < \infty.$$

Since $0 < p \leqslant 1$ and $|k_{a,\lambda}(x)| \leqslant c\|k_{\frac{a}{2},\lambda}\|^2$ for every $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}$, the series above converges absolutely to a function from $\mathcal{C}_0(\mathbb{R})$ by the Lebesgue dominated convergence theorem. By Lemma 4.2, the restriction of $F$ to $\mathbb{Z}_a$ (to be denoted by $f$) is in $\mathbb{B}_p(a, \mathrm{osc})$ and $\|f\|^p_{\mathbb{B}_p(a,\mathrm{osc})} \leqslant c_p \sum_{\lambda\in\Lambda} |c_\lambda|^p$ for a constant $c_p$ depending only on $p$.

Conversely, take $f \in \mathbb{B}_p(a, \mathrm{osc})$ and find a function $\tilde{F} \in \mathbb{B}_p(\mathbb{R})$ such that $\tilde{F} = f$ on $\mathbb{Z}_a$, see Lemma 4.3. Applying Theorem 2.10 from [8] to analytic and anti-analytic parts of $\tilde{F}$, we obtain the representation

$$\tilde{F} = q - \frac{1}{2\pi i} \sum_{\lambda \in \mathcal{U}_\varepsilon} \tilde{c}_\lambda \frac{\operatorname{Im} \lambda}{x - \bar{\lambda}}, \qquad x \in \mathbb{R},$$

where the coefficients $\tilde{c}_k \in \mathbb{C}$ are such that $\sum |\tilde{c}_\lambda|^p \leqslant c_p \|\tilde{F}\|^p_{\mathbb{B}_p(\mathbb{R})}$, and $q$ is a polynomial of degree at most $[\frac{1}{p}]$. Now consider the function

$$F = \sum_{\lambda \in \mathcal{U}_\varepsilon} c_\lambda \frac{k_{\lambda,a}}{\|k_{\frac{a}{2},\lambda}\|^2}, \qquad c_\lambda = \tilde{c}_\lambda \frac{\operatorname{Im} \lambda \cdot \|k_{\frac{a}{2},\lambda}\|^2}{1 - \mathrm{e}^{-\mathrm{i}a\bar{\lambda}}}.$$

Observe that $|c_\lambda| \leqslant |\tilde{c}_\lambda|$ for all $\lambda \in \mathcal{U}_\varepsilon$ and $f = q + F$ on $\mathbb{Z}_a$. We need to replace the set $\mathcal{U}_\varepsilon$ above to the set $\Lambda_{\eta a,\varepsilon}$ from Proposition 3.1. Since $k_{\frac{a}{2},\lambda} = \mathrm{e}^{\frac{\mathrm{i}az}{4}} \mathrm{e}^{-\frac{\mathrm{i}a\bar{\lambda}}{4}} \rho_{\frac{a}{4},\lambda}$, we have $\|k_{\frac{a}{2},\lambda}\|^2 = \mathrm{e}^{-\frac{a \operatorname{Im} \lambda}{2}} \|\rho_{\frac{a}{4},\lambda}\|^2$ and

$$\mathrm{e}^{-\frac{\mathrm{i}ax}{2}} F = \sum_{\lambda \in \mathcal{U}_\varepsilon} c_\lambda \mathrm{e}^{-\frac{\mathrm{i}a\bar{\lambda}}{2}} \frac{\rho_{a/2,\lambda}}{\|k_{a,\lambda}\|^2} = \sum_{\lambda \in \mathcal{U}_\varepsilon} c_\lambda \mathrm{e}^{-\frac{\mathrm{i}a \operatorname{Re} \lambda}{2}} \frac{\rho_{a/2,\lambda}}{\|\rho_{a/4,\lambda}\|^2}.$$

From the beginning of the proof of Proposition 3.1 we see that the Toeplitz operator on $\mathrm{PW}_{a/4}$ with symbol $\mathrm{e}^{-\frac{\mathrm{i}ax}{2}} F$ belongs to the class $\mathcal{S}^p(\mathrm{PW}_{a/4})$. It follows that

$$\mathrm{e}^{-\frac{\mathrm{i}ax}{2}} F = \sum_{\lambda \in \Lambda_{\eta a,\varepsilon}} d_\lambda \frac{\rho_{a/2,\lambda}}{\|\rho_{a/4,\lambda}\|^2}, \qquad \sum_{\lambda \in \Lambda_{\eta a,\varepsilon}} |d_\lambda|^p \leqslant c_p \sum_{\lambda \in \mathcal{U}_\varepsilon} |c_\lambda|^p.$$

This yields the required representation for $F$,

$$F = \sum_{\lambda \in \Lambda_{\eta a,\varepsilon}} c_\lambda \frac{k_{a,\lambda}}{\|k_{\frac{a}{2},\lambda}\|^2}, \qquad \sum_{\lambda \in \Lambda_{\eta a,\varepsilon}} |c_\lambda|^p \leqslant c_p \|f\|_{\mathbb{B}_p(a,\mathrm{osc})},$$

with some new coefficients $c_\lambda$. Since $\sum_\lambda |c_\lambda| < \infty$, the function $G = \mathrm{e}^{\frac{-\mathrm{i}az}{2}} F$ is an entire function of exponential type at most $a/2$ such that $\lim_{x \to \pm\infty} |G(x)| = 0$. In particular, it is uniquely determined by values on $\mathbb{Z}_a$. This proves uniqueness in Proposition 4.1. □

## 5. Proof of Theorem 1.1 for $0 < p \leqslant 1$

*Proof of Theorem 1.1* ($0 < p \leqslant 1$). — Let $\varphi \in L^\infty(\mathbb{R})$ be a function on $\mathbb{R}$ such that the operator $T_\varphi$ is in $\mathcal{S}^p(\mathrm{PW}_a)$, and let $\varphi_{st} = \mathcal{F}^{-1}\chi_{(-2a,2a)}\mathcal{F}\varphi$ be the standard symbol of $T_\varphi$. By Proposition 3.1 and Proposition 4.1, we have $\mathrm{e}^{2\mathrm{i}ax} \varphi_{st} \in \mathbb{B}_p(4a, \mathrm{osc})$ and moreover, $\|\mathrm{e}^{2\mathrm{i}ax} \varphi_{st}\|_{\mathbb{B}_p(4a,\mathrm{osc})} \leqslant c_p \|T_\varphi\|_{\mathcal{S}^p}$ for a constant $c_p$ depending only on $p$.

Conversely, assume that the restriction of the function $\mathrm{e}^{2\mathrm{i}ax}\,\varphi_{st}$ to $\mathbb{Z}_{4a}$ belongs to the space $\mathbb{B}_p(4a,\mathrm{osc})$. By Proposition 4.1, there exists a function $F$ and a polynomial $q$ of degree at most $[\frac{1}{p}]$ such that $q+F=\mathrm{e}^{2\mathrm{i}ax}\,\varphi_{st}$ on $\mathbb{Z}_{4a}$ and

$$F=\sum_{\lambda\in\Lambda_{\eta a,\varepsilon}} c_\lambda \frac{k_{4a,\lambda}}{\|k_{2a,\lambda}\|^2}=\mathrm{e}^{2\mathrm{i}ax}\sum_{\lambda\in\Lambda_{\eta a,\varepsilon}} c_\lambda\,\mathrm{e}^{-2\mathrm{i}a\operatorname{Re}\lambda}\,\frac{\rho_{2a,\lambda}}{\|\rho_{a,\lambda}\|^2} \tag{5.1}$$

for some $c_\lambda\in\mathbb{C}$ such that $\sum|c_\lambda|^p\leqslant c_p\|\,\mathrm{e}^{2\mathrm{i}ax}\,\varphi_{st}\|^p_{\mathbb{B}_p(4a,\mathrm{osc})}$. We claim that $T_{\tilde\varphi}=T_\varphi$ on $\mathrm{PW}_a$, where $\tilde\varphi=\mathrm{e}^{-2\mathrm{i}ax}(q+F)$. Indeed, the entire function $z\mapsto\tilde\varphi-\varphi_{st}$ has exponential type at most $2a$, vanishes on $\mathbb{Z}_{4a}$, and satisfies a polynomial estimate on $\mathbb{R}$. Hence $\tilde\varphi-\varphi_{st}=\tilde q\sin(2az)$ for all $z\in\mathbb{C}$ and a polynomial $\tilde q$. Thus, we have $T_\varphi=T_{\varphi_{st}}=T_{\tilde\varphi}$. It remains to use formula (5.1) and Proposition 3.1. The theorem is proved. □

## 6. Discrete Hilbert transform commutators. Proof of Theorem 1.2

Recall that $\mu_a=\frac{2\pi}{a}\sum_{x\in\mathbb{Z}_a}\delta_x$ is the scalar multiple of the counting measure on the lattice $\mathbb{Z}_a=\left\{\frac{2\pi}{a}k,\ k\in\mathbb{Z}\right\}$. The discrete Hilbert transform $H_{\mu_a}$ on $L^2(\mu_a)$ is defined by

$$H_{\mu_a}: f\mapsto \frac{1}{\pi}\int_{\mathbb{Z}_a}\frac{f(t)}{x-t}\,\mathrm{d}\mu_a(t),$$

and its commutator $C_\psi=M_\psi H_{\mu_a}-H_{\mu_a}M_\psi$ with the multiplication operator $M_\psi: f\mapsto\psi f$ on $L^2(\mu_a)$ by

$$C_\psi: f\mapsto\frac{1}{\pi}\int_{\mathbb{Z}_a}\frac{\psi(x)-\psi(t)}{x-t}f(t)\,\mathrm{d}\mu_a(t),\qquad x\in\operatorname{supp}\mu_a.$$

It is well-known that the operator $H_{\mu_a}$ admits the bounded extension from the dense subset $\mathcal{G}$ of $L^2(\mu_a)$ of finitely supported bounded functions to the whole space $L^2(\mu_a)$. A possible way to define the operator $C_\psi$ on $L^2(\mu_a)$ for any symbol $\psi$ on $\mathbb{Z}_a$ is to consider its bilinear form on elements from the dense subset $\mathcal{G}\times\mathcal{G}$ of $L^2(\mu_a)\times L^2(\mu_a)$. We will also deal with the operators $\tilde C_\psi: L^2(\mu_{\frac{a}{2}})\to L^2(\nu_{\frac{a}{2}})$ defined by

$$\tilde C_\psi: f\mapsto\frac{1}{\pi}\int_{\mathbb{Z}_a}\frac{\psi(x)-\psi(t)}{x-t}f(t)\,\mathrm{d}\mu_{\frac{a}{2}}(t),\qquad x\in\operatorname{supp}\nu_{\frac{a}{2}},$$

where the measure $\nu_{\frac{a}{2}}=\frac{4\pi}{a}\sum_{x\in\mathbb{Z}_{\frac{a}{2}}}\delta_{x+\frac{2\pi}{a}}$ is supported on the lattice $\frac{2\pi}{a}+\mathbb{Z}_{\frac{a}{2}}$. It can be shown that for $1\leqslant p\leqslant\infty$ the operator $C_\psi: L^2(\mu_a)\to L^2(\mu_a)$ is in $\mathcal{S}^p$ if and only if the operator $\tilde C_\psi: L^2(\mu_{\frac{a}{2}})\to L^2(\nu_{\frac{a}{2}})$ is in $\mathcal{S}^p$. As we will see, for $0 < p < 1$ we may have $C_\psi \notin \mathcal{S}^p(L^2(\mu_a))$ for a function $\psi$ on $\mathbb{Z}_a$ such that the operator $\tilde{C}_\psi : L^2(\mu_{\frac{a}{2}}) \to L^2(\nu_{\frac{a}{2}})$ is in $\mathcal{S}^p$.

The discrete Hilbert transform commutators were investigated in details in paper [9]. In particular, it was proved in [9] that $C_\psi$ is bounded on $L^2(\mu_a)$ if and only if its symbol $\psi$ belongs to the discrete $\mathrm{BMO}(\mathbb{Z}_a)$ space of functions $f$ on $\mathbb{Z}_a$ such that $\sup_{I\in\mathcal{I}_a} \operatorname{osc}(f, I, \mu_a, 0) < \infty$, where $\mathcal{I}_a = \{I_{a,j,k},\ j, k \in \mathbb{Z},\ j \geqslant 0\}$ is the collection of intervals defined in Section 1. Another result from [9] says that $C_\psi$ is compact on $L^2(\mu_a)$ if and only if $\psi \in \mathrm{CMO}(\mathbb{Z}_a)$, that is, $\lim_{k\to\pm\infty} \operatorname{osc}(\psi, I_{a,j,k}, \mu_a, 0) = 0$ for every $j \geqslant 0$ and $\lim_{j\to+\infty} \operatorname{osc}(\psi, J_j, \mu_a, 0) = 0$ for any sequence of intervals $J_j \subset \mathbb{R}$ of length $j$ with common center. Finally, the operator $C_\psi$ belongs to $\mathcal{S}^p(L^2(\mathbb{Z}_a))$ for $1 < p < \infty$ if and only if $\psi \in \mathbb{B}_p(a, \mathrm{osc})$, moreover, we have $C_\psi \in \mathcal{S}^1(L^2(\mu_a))$ for every $\psi \in \mathbb{B}_1(a, \mathrm{osc})$. See Theorem 6.2 in [9] and Theorem 4 in [12] for the proof of these results. It was an open question stated in Section 7 of [9] whether $C_\psi \in \mathcal{S}^p(L^2(\mu_a))$ is equivalent to $\psi \in \mathbb{B}_p(a, \mathrm{osc})$ for all positive $p$ (in particular, for $p = 1$). Theorem 1.2 gives the affirmative answer to this question for $p = 1$. On the other hand, for $0 < p < 1$ we show that there exists symbols $\psi \in \mathbb{B}_p(a, \mathrm{osc})$ such that $C_\psi \notin \mathcal{S}^p(L^2(\mu_a))$. In fact, the following modification of Theorem 1.2 holds true.

Theorem 6.2. — *Let $0 < p \leqslant 1$. The operator $\tilde{C}_\psi : L^2(\mu_{\frac{a}{2}}) \to L^2(\nu_{\frac{a}{2}})$ belongs to the class $\mathcal{S}^p$ if and only if $\psi \in \mathbb{B}_p(a, \mathrm{osc}) \cap L^\infty(\mathbb{Z}_a)$. Moreover, the quasi-norms $\|\tilde{C}_\psi\|_{\mathcal{S}^p}$ and $\|\psi\|_{\mathbb{B}_p(a,\mathrm{osc})}$ are comparable with constants depending only on $p$.*

For the proof we need a result on unitary equivalence of discrete Hilbert transform commutators to some truncated Hankel operators. Given a positive number $a > 0$, we denote by $\mathrm{PW}_{[-a,0]}$ the Paley–Wiener space of functions in $L^2(\mathbb{R})$ with Fourier spectrum in the interval $[-a, 0]$. Define the truncated Hankel operator $\Gamma_\psi : \mathrm{PW}_{[0,a]} \to \mathrm{PW}_{[-a,0]}$ with symbol $\psi \in L^\infty(\mathbb{R})$ by

$$\Gamma_\psi : f \mapsto P_{[-a,0]}(\psi f), \qquad f \in \mathrm{PW}_{[0,a]},$$

where $P_{[-a,0]}$ stands for the projection in $L^2(\mathbb{R})$ to the subspace $\mathrm{PW}_{[-a,0]}$. It is easy to see that $\Gamma_\psi$ is completely determined by its standard symbol $\psi_{st,2a} = \mathcal{F}^{-1}\chi_{(-2a,0)}\mathcal{F}\psi$, that is, $\Gamma_\psi f = \Gamma_{\psi_{st,a}} f$ for all functions $f \in \mathrm{PW}_{[0,a]}$ such that $\sup_{x\in\mathbb{R}} |xf(x)| < \infty$. Clearly, such functions form a dense subset in $\mathrm{PW}_{[0,a]}$.

It is known that the embedding operator $V_{\mu_a} : \mathrm{PW}_{[0,a]} \to L^2(\mu_a)$ taking a function $f \in \mathrm{PW}_{[0,a]}$ into its restriction to $\mathbb{Z}_a$ is unitary. The same is true for the embedding operator $\tilde{V}_{\nu_a} : \mathrm{PW}_{[-a,0]} \to L^2(\nu_a)$. A general version of the following result is Lemma 4.2 of [1].

Lemma 6.1. — *Let $a > 0$, $0 < p \leqslant 1$, and let $\psi \in L^\infty(\mathbb{Z}_{2a})$. Then there exists an entire function $\Psi$ such that $\Psi = \psi$ on $\mathbb{Z}_{2a}$, $|F(x)| \leqslant c\log(e + |x|)$ for all $x \in \mathbb{R}$, and the Fourier spectrum of $F$ is contained in the interval $[-2a, 0]$. Moreover, we have*

$$\tilde{V}_{\nu_a}\Gamma_\Psi V_{\mu_a}^{-1} = -i\tilde{C}_\psi. \tag{6.1}$$

*for the operators $\Gamma_\Psi : \mathrm{PW}_{[0,a]} \to \mathrm{PW}_{[-a,0]}$ and $\tilde{C}_\psi : L^2(\mu_a) \to L^2(\nu_a)$.*

*Proof.* — Existence of such a function $\Psi$ follows from a general theory of entire functions, see, e.g., Theorem 1 in Section 21.1 of [2] and Problem 1 after its proof. In order to prove formula (6.1), take a pair of functions $f \in L^2(\mu_a)$, $g \in L^2(\nu_a)$ with finite support. Consider the functions $F, G$ in $\mathrm{PW}_{[0,a]}$ such that $F = V_{\mu_a}^{-1} f$, $\bar{G} = \tilde{V}_{\nu_a}^{-1} g$. It is easy to see that $\int_{\mathbb{R}} |\Psi F G|\,\mathrm{d}x < \infty$ and hence the bilinear form of $\Gamma_\Psi$ is correctly defined on functions $F$, $\bar{G}$. We have

$$\begin{aligned}(\tilde{V}_{\nu_a}\Gamma_\Psi V_{\mu_a}^{-1} f, g)_{L^2(\mathbb{R})} &= (\Gamma_\Psi F, \bar{G})_{L^2(\mathbb{R})} = (FG, \bar{\Psi})_{L^2(\mathbb{R})}\\ &= (V_{\mu_{2a}} FG, V_{\mu_{2a}}\bar{\Psi})_{L^2(\mu_{2a})}\\ &= \frac{1}{2}(Fg, \bar{\psi})_{L^2(\nu_a)} + \frac{1}{2}(fG, \bar{\psi})_{L^2(\mu_a)}.\end{aligned}$$

For every point $x \in \frac{\pi}{a} + \mathbb{Z}_a$ we have

$$F(x) = (V_{\mu_a} F, V_{\mu_a} k_{x,a})_{L^2(\mu_a)} = \frac{2}{\pi i}\int_{\mathbb{R}} \frac{f(t)}{t - x}\,\mathrm{d}\mu_a(t), \qquad x \in \frac{\pi}{a} + \mathbb{Z}_a.$$

Analogously, $G(t) = \frac{2}{\pi i}\int_{\mathbb{R}} \frac{\overline{g(x)}}{x-t}\,\mathrm{d}\nu_a(x)$ for all $t \in \mathbb{Z}_a$. Using these formulas, we get

$$\begin{aligned}(\tilde{V}_{\nu_a}\Gamma_\Psi V_{\mu_a}^{-1} f, g)_{L^2(\mathbb{R})} &= \frac{1}{\pi i}\int_{\mathbb{R}} \frac{\psi(x) - \psi(t)}{x - t} f(t)\overline{g(x)}\,\mathrm{d}\mu_a(t)\,\mathrm{d}\nu_a(x)\\ &= -i(\tilde{C}_\psi f, g)_{L^2(\nu_a)}.\end{aligned}$$

The lemma follows. □

*Proof of Theorem 6.2.* — Let $\psi$ be a function on the lattice $\mathbb{Z}_a$ such that the operator $\tilde{C}_\psi : L^2(\mu_{\frac{a}{2}}) \to L^2(\nu_{\frac{a}{2}})$ belongs to the class $\mathcal{S}^p$. Consider the sequence of points $x_k = \frac{2\pi}{a}k$, $k \in \mathbb{Z}$. Since $0 < p \leqslant 1$, we have

$$\sum_{k\in\mathbb{Z}} |\psi(x_{2k}) - \psi(x_{2k+1})| = \frac{a}{8}\sum_{k\in\mathbb{Z}} |(\tilde{C}_\psi \delta_{x_{2k}}, \delta_{x_{2k+1}})_{L^2(\nu_{\frac{a}{2}})}| < \infty.$$

Hence, the function $\psi$ is bounded on $\mathbb{Z}_a$. Using Lemma 6.1, we can find an entire function $\Psi$ such that $\Psi = \psi$ on $\mathbb{Z}_a$, $|\Psi(x)| \leqslant c\log(e + |x|)$ for all $x \in$

$\mathbb{R}$, the Fourier spectrum of $\Psi$ is contained in $[-a, 0]$, and relation (6.1) holds for the operators $\Gamma_\Psi : \mathrm{PW}_{[0,\frac{a}{2}]} \to \mathrm{PW}_{[-\frac{a}{2},0]}$ and $\tilde{C}_\psi : L^2(\mu_{\frac{a}{2}}) \to L^2(\nu_{\frac{a}{2}})$. In particular, we have $\Gamma_\Psi \in \mathcal{S}^p$. Denote by $M$ the multiplication operator on $L^2(\mathbb{R})$ by the function $\mathrm{e}^{\frac{\mathrm{i}ax}{2}}$. Let $T_{\mathrm{e}^{\frac{\mathrm{i}ax}{2}}\Psi}$ be the Toeplitz operator on $\mathrm{PW}_{\frac{a}{4}}$ with standard symbol $\mathrm{e}^{\frac{\mathrm{i}ax}{2}}\Psi$. Observe that

$$T_{\mathrm{e}^{\frac{\mathrm{i}ax}{2}}\Psi} f = M\Gamma_\Psi M f, \tag{6.2}$$

for every function $f \in \mathrm{PW}_{\frac{a}{4}}$ such that $\sup_{x\in\mathbb{R}} |xf(x)| < \infty$. Since $M$ maps unitarily $\mathrm{PW}_{\frac{a}{4}}$ onto $\mathrm{PW}_{[0,\frac{a}{2}]}$ and $\mathrm{PW}_{[-\frac{a}{2},0]}$ onto $\mathrm{PW}_{\frac{a}{4}}$, the operator $T_{\mathrm{e}^{\frac{\mathrm{i}ax}{2}}\Psi}$ belongs to $\mathcal{S}^p(\mathrm{PW}_{\frac{a}{4}})$. In particular, there exists a function $\varphi \in L^\infty(\mathbb{R})$ such that $T_\varphi = T_{\mathrm{e}^{\frac{\mathrm{i}ax}{2}}\Psi}$ and $\varphi_{st} = \mathrm{e}^{\frac{\mathrm{i}ax}{2}}\Psi + c_1\mathrm{e}^{-\mathrm{i}\frac{a}{2}x} + c_2\mathrm{e}^{\mathrm{i}\frac{a}{2}x}$ for some constants $c_1$, $c_2$. Since $\mathrm{e}^{\frac{\mathrm{i}ax}{2}}\varphi_{st}$ coincides with $\psi + c_1 + c_2$ on $\mathbb{Z}_a$, we have $\psi \in \mathbb{B}_p(a, \mathrm{osc})$ by Theorem 1.1. Moreover, the quasi-norm $\|\tilde{C}_\psi\|_{\mathcal{S}^p}$ is comparable to $\|\psi\|_{\mathbb{B}_p(a,\mathrm{osc})}$ with constants depending only on $p \in (0, 1]$.

Conversely, suppose that $\psi \in \mathbb{B}_p(a, \mathrm{osc}) \cap L^\infty(\mathbb{Z}_a)$. Using Lemma 6.1 again, we find an entire function $\Psi$ such that $\Psi = \psi$ on $\mathbb{Z}_a$, $|\Psi(x)| \leqslant c\log(e+|x|)$ for all $x \in \mathbb{R}$, the Fourier spectrum of $\Psi$ is contained in $[-a, 0]$, and relation (6.1) holds for the operators $\Gamma_\Psi : \mathrm{PW}_{[0,\frac{a}{2}]} \to \mathrm{PW}_{[-\frac{a}{2},0]}$ and $\tilde{C}_\psi : L^2(\mu_{\frac{a}{2}}) \to L^2(\nu_{\frac{a}{2}})$. Since $\psi \in L^\infty(\mathbb{Z}_a)$, the operators $\tilde{C}_\psi$ and $\Gamma_\Psi$ are bounded. Let $\Psi_{st,a}$ be the standard symbol of the operator $\Gamma_\Psi$. Note that $\Psi_{st,a}(x) = \Psi(x) + q(x)$ for all $x \in \mathbb{Z}_a$ and a polynomial $q$ of degree at most one. In particular, we have $\Psi_{st,a} \in \mathbb{B}_p(a, \mathrm{osc})$. By Theorem 1.1, the operator $T_{\mathrm{e}^{\frac{\mathrm{i}ax}{2}}\Psi_{st,a}}$ on $\mathrm{PW}_{\frac{a}{4}}$ is in $\mathcal{S}^p$, hence $\Gamma_\Psi \in \mathcal{S}^p$ by formula (6.2). It follows that the operator $\tilde{C}_\psi$ is in $\mathcal{S}^p$ as well, and, moreover, we have the estimate

$$\|\tilde{C}_\psi\|_{\mathcal{S}^p} = \|\Gamma_\Psi\|_{\mathcal{S}^p} = \left\|T_{\mathrm{e}^{\frac{\mathrm{i}ax}{2}}\Psi_{st,a}}\right\|_{\mathcal{S}^p} \leqslant c_p\|\Psi_{st,a}\|_{\mathbb{B}_p(a,\mathrm{osc})} = c_p\|\psi\|_{\mathbb{B}_p(a,\mathrm{osc})},$$

for a constant $c_p$ depending only on $p$. The theorem is proved. □

*Proof of Theorem 1.2.* — Let $\psi$ be a function on the lattice $\mathbb{Z}_a$ such that we have $C_\psi \in \mathcal{S}^1(L^2(\mu_a))$. Then the operator $\tilde{C}_\psi : L^2(\mu_{\frac{a}{2}}) \to L^2(\nu_{\frac{a}{2}})$ is of trace class as well and $\|\psi\|_{\mathbb{B}_1(a,\mathrm{osc})} \leqslant c_1\|\tilde{C}_\psi\|_{\mathcal{S}^1(L^2(\mu_a))} \leqslant c_1\|C_\psi\|_{\mathcal{S}^1(L^2(\mu_a))}$ by Theorem 6.2.

Conversely, suppose that $\psi \in \mathbb{B}_1(a, \mathrm{osc}) \cap L^\infty(\mathbb{Z}_a)$. By Lemma 4.3, we can find a function $\Psi \in \mathbb{B}_1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that $\Psi = \psi$ on $\mathbb{Z}_a$ and $\|\Psi\|_{\mathbb{B}_1(\mathbb{R})} \leqslant c_1\|\psi\|_{\mathbb{B}_1(\mathrm{osc},a)}$. Denote $\psi_\lambda : t \mapsto \frac{|\operatorname{Im}\lambda|^2}{(t-\bar{\lambda})^2}$ for $\lambda \in \mathbb{C}$. Let us apply Theorem 2.10 in [8] to analytic and anti-analytic parts of $\Psi$: find numbers

$c$, $c_\lambda$ such that $\sum_{\lambda\in\mathcal{U}_\varepsilon}|c_\lambda| \leqslant c_1\|\Psi\|_{\mathbb{B}_1(\mathbb{R})}$ and

$$\psi(x) = \Psi(x) = c + \sum_{\lambda\in\mathcal{U}_\varepsilon} c_\lambda\psi_\lambda(x), \qquad x \in \mathbb{Z}_a.$$

We claim that for every $\lambda \in \mathcal{U}_\varepsilon$ the commutator $C_{\psi_\lambda}$ belongs to the trace class and $\|C_{\psi_\lambda}\|_{\mathcal{S}^1} \leqslant c_1(1+a)$ for a constant $c_1$ do not depending on $\lambda$. Clearly, this will yield the desired estimate $\|C_\psi\|_{\mathcal{S}^1} \leqslant c_1(1+a)\|\psi\|_{\mathbb{B}_1(a,\mathrm{osc})}$. We have

$$\frac{\psi_\lambda(x)-\psi_\lambda(t)}{x-t} = -\frac{|\operatorname{Im}\lambda|^2}{(x-\bar\lambda)^2(t-\bar\lambda)} - \frac{|\operatorname{Im}\lambda|^2}{(x-\bar\lambda)(t-\bar\lambda)^2}.$$

Denote by $K_{\psi_\lambda}$ the integral operator on $L^2(\mu_a)$ with kernel $\frac{\psi_\lambda(x)-\psi_\lambda(t)}{x-t}$:

$$(6.3)\quad (K_{\psi_\lambda}f)(x) = \int_{\mathbb{Z}_a}\frac{\psi_\lambda(x)-\psi_\lambda(t)}{x-t}f(t)\mathrm{d}t = (C_{\psi_\lambda}f)(x) + \frac{2|\operatorname{Im}\lambda|^2}{(x-\bar\lambda)^3}f(x).$$

Observe that the operator $K_{\psi_\lambda}$ has rank 2 and

$$\|K_{\psi_\lambda}\|_{\mathcal{S}^p} \leqslant 2|\operatorname{Im}\lambda|^2\cdot\left\|\frac{1}{(x-\bar\lambda)^2}\right\|_{L^2(\mu_a)}\left\|\frac{1}{x-\bar\lambda}\right\|_{L^2(\mu_a)}.$$

In the case where $\operatorname{dist}(\lambda,\mathbb{Z}_a) \geqslant \frac{\pi}{2a}$, the last expression could be estimated from above by

$$c_1\left(\int_{\mathbb{R}}\frac{|\operatorname{Im}\lambda|\,\mathrm{d}t}{t^2+|\operatorname{Im}\lambda|^2}\int_{\mathbb{R}}\frac{|\operatorname{Im}\lambda|^3\,\mathrm{d}t}{(t^2+|\operatorname{Im}\lambda|^2)^2}\right)^{\frac12} = c_1\left(\int_{\mathbb{R}}\frac{\mathrm{d}t}{t^2+1}\int_{\mathbb{R}}\frac{\mathrm{d}t}{(t^2+1)^2}\right)^{\frac12}.$$

Moreover, the singular numbers of the multiplication operator $f \mapsto \frac{|\operatorname{Im}\lambda|^2}{(x-\bar\lambda)^3}f$ are precisely $\frac{|\operatorname{Im}\lambda|^2}{|x-\bar\lambda|^3}$, $x \in \mathbb{Z}_a$, hence its norm in $\mathcal{S}^1(L^2(\mu_a))$ does not exceed

$$\sum_{x\in\mathbb{Z}_a}\frac{|\operatorname{Im}\lambda|^2}{|x-\bar\lambda|^3} \leqslant \sum_{x\in\mathbb{Z}_a}\frac{|\operatorname{Im}\lambda|^2}{(x^2+|\operatorname{Im}\lambda|^2)^{\frac32}} \leqslant c_1 a$$

for a universal constant $c_1$. This tells us that $\|C_{\psi_\lambda}\|_{\mathcal{S}^p} \leqslant c_1(1+a)$ for all $\lambda \in \mathcal{U}_\varepsilon$ such that $\operatorname{dist}(\lambda,\mathbb{Z}_a) \geqslant \frac{\pi}{2a}$. Now consider the case where $\operatorname{dist}(\lambda,\mathbb{Z}_a) \leqslant \frac{\pi}{2a}$. Let $x_\lambda$ be the nearest point to $\lambda$ in the lattice $\mathbb{Z}_a$. The function $\psi_\lambda$ belongs to $L^1(\mu_a)$ and

$$\begin{aligned}\sum_{x\in\mathbb{Z}_a}|\psi_\lambda(x)| &\leqslant |\psi_\lambda(x_\lambda)| + 2|\operatorname{Im}\lambda|^2\sum_{k=1}^{\infty}\frac{1}{\left(\frac{2\pi}{a}k-\frac{\pi}{2a}\right)^2},\\ &\leqslant \left|\frac{\operatorname{Im}\lambda}{\lambda-x_\lambda}\right|^2 + 2\left(\frac{a|\operatorname{Im}\lambda|}{2\pi}\right)^2\sum_{k=1}^{\infty}\frac{1}{\left(k-\frac14\right)^2} \leqslant c_1,\end{aligned}$$

where the right hand side does not depend on $\lambda$. It follows that the operator $M_{\psi_\lambda}$ lies in $\mathcal{S}^1(L^2(\mu_a))$ and $\|M_{\psi_\lambda}\|_{\mathcal{S}^1} \leqslant c_1$. We also have

$$\left\|C_{\psi_\lambda}\right\|_{\mathcal{S}^p} = \left\|H_{\mu_a} M_{\psi_\lambda} - M_{\psi_\lambda} H_{\mu_a}\right\|_{\mathcal{S}^1} \leqslant c_1,$$

for another constant $c_1$, because the discrete Hilbert transform $H_{\mu_a}$ is bounded on $L^2(\mu_a)$. This completes the proof. ☐

Remark that the second part of the proof of Theorem 1.2 is almost literal repetition of the corresponding part of the proof of Theorem 6.2 in [9]. However, the original argument in [9] has a gap: it does not involve the estimate of the $\mathcal{S}^1$-norm of the multiplication operator $f \mapsto \frac{|\operatorname{Im}\lambda|^2}{(x-\bar\lambda)^3} f$ from formula (6.3). This technical place turns out to be crucial in the case $0 < p < 1$. More precisely, we have the following result.

Proposition 6.2. — *Let $0 < p < 1$ and let $a > 0$. There exists a function $\psi \in \mathbb{B}_p(\mathbb{Z}_a)$ such that $C_\psi \notin \mathcal{S}^p(L^2(\mu_a))$.*

*Proof.* — Suppose that $C_\psi \in \mathcal{S}^p(L^2(\mu_a))$ for every $\psi \in \mathbb{B}_p(a,\mathrm{osc})$. Then it is easy to see from the closed graph theorem that there exists a constant $c_{p,a}$ such that $\|C_\psi\|_{\mathcal{S}^p} \leqslant c_{p,a}\|\psi\|_{\mathbb{B}_p(a,\mathrm{osc})}$ for all $\psi \in \mathbb{B}_p(a,\mathrm{osc})$. Take $\lambda \in \mathbb{C}^+$ such that $\operatorname{Im}\lambda \geqslant \frac{2\pi}{a}$ and consider the function $\psi_\lambda : t \mapsto \frac{\operatorname{Im}\lambda}{t-\bar\lambda}$. Analogously to (6.3), we have $K_{\psi_\lambda} = C_{\psi_\lambda} + M_\lambda$, where $K_{\psi_\lambda}$ is the integral operator with kernel

$$\frac{\psi_\lambda(x) - \psi_\lambda(t)}{x-t} = -\frac{\operatorname{Im}\lambda}{(x-\bar\lambda)(t-\bar\lambda)},$$

and $M_\lambda : f \mapsto \frac{\operatorname{Im}\lambda}{(x-\bar\lambda)^2} f$ is the multiplication operator on $L^2(\mu_a)$ by $\frac{\operatorname{Im}\lambda}{(x-\bar\lambda)^2}$. Observe that $K_{\psi_\lambda}$ is the rank-one operator whose norm does not exceed

$$\operatorname{Im}\lambda \cdot \left\|\frac{1}{x-\bar\lambda}\right\|^2_{L^2(\mu_a)} \leqslant c_p \int_{\mathbb{R}} \frac{\operatorname{Im}\lambda \,\mathrm{d}t}{t^2 + (\operatorname{Im}\lambda)^2} = c_p \int_{\mathbb{R}} \frac{\mathrm{d}t}{t^2+1}.$$

It follows from our assumption and Lemma 4.2 that $\|M_\lambda\|_{\mathcal{S}^p} \leqslant c_{p,a}$ for all $\lambda \in \mathbb{C}$ with $\operatorname{Im}\lambda \geqslant \frac{2\pi}{a}$ and a universal constant $c_p$. On the other hand, we have

$$\|M_\lambda\|^p_{\mathcal{S}^p} = \sum_{x\in\mathbb{Z}_a} \frac{(\operatorname{Im}\lambda)^p}{|x-\bar\lambda|^{2p}} \geqslant a c_p \int_{\mathbb{R}} \frac{(\operatorname{Im}\lambda)^p \,\mathrm{d}x}{(x^2 + (\operatorname{Im}\lambda)^2)^p} \leqslant a\tilde{c}_p (\operatorname{Im}\lambda)^{1-p}.$$

Since the right hand side is unbounded in $\lambda$, we get the contradiction. ☐

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