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LOCAL GALOIS GROUP OF IRREGULAR q-DIFFERENCE EQUATIONS

by Virginie BUGEAUD

Abstract. — Relying on the normal forms of pure isoclinic modules with non integral slopes, due to van der Put and Reversat, we extend the isoformal analytic classification of Ramis, Sauloy and Zang made for the case of integral slopes. We obtain an analogue of Birkhoff–Guenther normal forms in the case of two slopes which are not integral. Computing Stokes operators in the case of two slopes, we prove a theorem of classification by the H^1. Moreover, we describe in a matricial form the formal Galois group when the denominator of the slopes is fixed. Finally, we prove a density theorem similar to that of Ramis and Sauloy to describe the Galois group.

Résumé. — Sur la base des travaux de van der Put et Reversat sur les formes normales des modules purs isoclines à pentes non entières, nous poursuivons la classification analytique locale des modules aux q-différences réalisée pour le cas des modules à pentes entières par Ramis, Sauloy et Zang. Nous obtenons un analoque des formes normales de Birkhoff–Guenther dans le cas à deux pentes non entières. En calculant les opérateurs de Stokes dans le cas à deux pentes, nous démontrons un théorème de classification par le H^1. De plus, nous décrivons le groupe de Galois sous forme matricielle dans le cas où le dénominateur des pentes est fixé. Enfin, nous démontrons un théorème de densité similaire à celui de Ramis et Sauloy pour décrire le groupe de Galois.

Introduction

This article deals with linear q-difference equations. These are functional equations defined with an operator denoted by σ_q which operates on complex functions by $\sigma_q(f(z)) = f(qz)$. As in differential theory, a Newton polygon can be associated to a q-difference equation, and it gives rational slopes.

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Studying analytic linear q-difference equations is the same as studying the systems:

$$Y(qz) = A(z)Y(z),$$

where $A(z)$ is an invertible matrix with analytic coefficients, that is, in $K = \mathbb{C}(\{z\})$. In this article, we study pairs (K^n, Φ_A) which represent q-difference modules and where $\Phi_A : K^n \to K^n$ is a σ_q-linear automorphism of K^n defined by $\Phi_A(X) = A^{-1}\sigma_q(X)$.

Birkhoff and Guenther in [1] said that any q-difference system is equivalent to a polynomial system, which means that the matrix A has polynomial coefficients. It is easy to prove that a q-difference module with only one integral slope μ is equivalent to a module with matrix A of the form $z^\mu C$ where $C \in \mathrm{GL}_n(\mathbb{C})$. This form, under some additional conditions, is called normal form.

van der Put and Reversat gave a normal form to any q-difference module with only one non integral slope, considering irreducible and indecomposable modules. As for q-difference module with several slopes, Ramis, Sauloy and Zhang found an explicit method to obtain a matrix with coefficients in $\mathbb{C}[z, z^{-1}]$. Nevertheless, it was only done when the slopes are integral. This form is called the Birkhoff–Guenther normal form. Local analytic classification, in the case of integral slopes, is partially built thanks to these normal forms.

Another aspect of q-difference equations is Galois theory. Many approaches exist to define the Galois group associated with a q-difference module (Picard–Vessiot theory by van der Put and Singer in [8], tannakian theory...). The main difficulty compared to differential equations, lies in the fact that the constant field is the field of elliptic functions, so it is rather big: yet, one looks for classification and Galois theory over the complex numbers.

The approach of Sauloy in [14] is analytic and uses tannakian theory. In his article, Sauloy obtained a description of the local Galois group of q-difference equations with null slope. Later, Ramis and Sauloy deduced an explicit matricial description of the formal Galois group of q-difference equations with integral slopes.

Local analytic classification by Ramis, Sauloy and Zhang and the construction of Stokes operators enabled Ramis and Sauloy to study the local Galois group of q-difference equations with integral slopes. They obtained a Zariski dense subgroup of the local Galois group generated by the local formal Galois group and by Stokes operators associated to q-alien derivations.

In this article, we have tried to complete local analytic classification of q-difference modules and to compute Stokes operators in the case of non integral slopes. The case of two slopes is totally done. We also go on understanding local Galois theory: we obtain an explicit description of the local formal Galois group and a density theorem to describe the local Galois group in the case of non integral slopes. The case of three slopes or more is incomplete here, we comment on this at the beginning of Section 2.

Contents

The complete description of normal forms associated with pure isoclinic modules with non integral slopes, by van der Put and Reversat in [7], is our starting point to extend a part of the results of [10], [11] and [12] about classification and Galois group. Note that van der Put and Reversat use Picard–Vessiot theory, while we follow a tannakian approach, so that our results in Galois theory are mostly independent of theirs.

In the first section, we introduce notations and definitions about q-difference modules. We define the Newton polygon and the slopes. We give a new, effective proof of the theorems of [7] which describe the normal forms of q-difference modules with a unique non integral slope. We also study the tensor product of two irreducible modules: we prove with an explicit isomorphism that a tensor product of two irreducible modules is a direct sum of irreducible modules. This result is one of the keys of the isoformal analytic classification in the case of two slopes not necessarily integral.

The second section deals with the analytic classification, it consists, as in [12], in describing the equivalence classes of modules which have the same graded module. We are looking for a Birkhoff–Guenther normal form for the case of two slopes. It exists when the slopes are integral for any number of slopes (see [12]). In the case of two slopes no necessarily integral, we prove that any class admits a representative in polynomial form and we obtain an isomorphism between the set of isoformal analytic classes and an explicit quotient of a polynomial space (Theorem 2.8).

In the third section, we want to compute Stokes operators. Like in the case of integral slopes, we start by establishing a classification by the H^1 of the vector bundle associated with a q-difference module (Theorem 3.5). The cocycles obtained to prove Theorem 3.5 enable us to construct Stokes operators associated with a module with slopes non necessarily integral.

The last section is devoted to Galois theory. Our approach is tannakian. In Theorem 4.1, we give a matricial description of the formal Galois group. The case of two slopes leads us to study also extensions of representations of the Galois group (Theorem 4.12). Finally, proving that our Stokes operators are Galoisian shows us the way to the density theorem. The whole Galois group is generated by the Stokes operators obtained by iteration of the operator σ_q and by the formal Galois group (Theorem 4.30).

1. Definitions

1.1. Notations

Let $K := \mathbb{C}(\{z\}) = \mathbb{C}\{z\}[z^{-1}]$ be the field of convergent Laurent series, namely meromorphic germs at 0, it is the field of fractions of $\mathbb{C}\{z\}$. Let $\hat{K} := \mathbb{C}[\![z]\!][z^{-1}]$ be the field of fractions of $\mathbb{C}[\![z]\!]$, the ring of formal power series.

We fix $q \in \mathbb{C}^*$ such that $|q| > 1$, and we define the operator σ_q by $\sigma_q(f(z)) = f(qz)$, it is an automorphism of the field K (and of $\hat{K}$). We define also $\mathcal{D}_{K,\sigma_q} := K\langle \sigma_q, \sigma_q^{-1}\rangle$ the Ore ring of q-difference operators $\sum_{finite} a_i \sigma_q^i$. It has an algebra structure, characterized by a commutation relation: for all $x \in K$, for all $k \in \mathbb{Z}$, $\sigma_q^k.x = \sigma_q^k(x).\sigma_q^k$. The ring $\mathcal{D}_{K,\sigma_q} := K\langle \sigma_q, \sigma_q^{-1}\rangle$ is left euclidean (cf. [12]).

We denote by E_q the elliptic curve $E_q := \mathbb{C}^*/q^{\mathbb{Z}}$, the natural projection $\mathbb{C}^* \to E_q$ induces a bijection between the fundamental annulus $C_q := \{z \in \mathbb{C}^* \mid 1 \leqslant |z| < |q|\}$ and E_q.

For all $r \in \mathbb{N}^*$, let $\xi_r = e^{\frac{2i\pi}{r}}$ be a primitive rth root of unity in $\mathbb{C}$. We choose once and for all $\tau \in \mathbb{C}^*$ such that $q = e^{2i\pi\tau}$, $\operatorname{Im}\tau < 0$. Let $q_r = e^{\frac{2i\pi\tau}{r}}$ be a rth root of q, so that compatibility relations are satisfied: $q_r^r = q$ and $q_{rs}^s = q_r$.

1.2. q-difference modules, morphisms

General references for this subsection and the following (*e.g.* cyclic vector lemma, Newton polygon, fuchsian equations…) are [13, 16].

Definition 1.1. — *A q-difference module M over (K, σ_q) is a pair (V, Φ) where V is a K-vector space with finite dimension and Φ a σ_q-linear automorphism on V, which means, for all $a \in K$, and for all $X \in V$, $\Phi(aX) = \sigma_q(a)\Phi(X)$.*

A morphism *of q-difference modules $M = (V_1, \Phi_1) \to N = (V_2, \Phi_2)$ is a K-linear application $F : V_1 \to V_2$ such that $\Phi_2 \circ F = F \circ \Phi_1$.*

By choosing a basis of the K-vector space V, a q-difference module M is isomorphic to a pair (K^n, Φ_A) where:

$$\Phi_A(X) = A^{-1}\sigma_q(X), \quad A \in \mathrm{GL}_n(K).$$

A morphism from (K^n, Φ_A) to (K^p, Φ_B) is a matrix $F \in M_{p,n}(K)$ satisfying $\sigma_q(F)A = BF$. An isomorphism from (K^n, Φ_A) to (K^n, Φ_B) is a matrix $F \in \mathrm{GL}_n(K)$ such that $F[A] = B$, where we define $F[A] := \sigma_q(F)AF^{-1}$.

Definition 1.2. — *Let $M = (V, \Phi)$ be a q-difference module of rank n and let e be a vector of the K-vector space V. We say that e is a* cyclic vector *if the family $\underline{e} = (e, \Phi(e), \dots, \Phi^{n-1}(e))$ is a basis of V.*

By the cyclic vector lemma, every q-difference module $M = (V, \Phi)$ admits a cyclic vector e. By defining for all $x \in V$, $\left(\sum_{i=0}^n a_i\sigma_q^i\right).x := \sum_{i=0}^n a_i\Phi^i(x)$, then, every q-difference module is isomorphic to a module $\mathcal{D}_{K,\sigma_q}/\mathcal{D}_{K,\sigma_q}P$ where $P \in \mathcal{D}_{K,\sigma_q}$ is entire unitary; it is obtained by expressing $\Phi^n(e)$ as a combination of the $\Phi^i(e)$ for $i = 0, \dots, n-1$.

Let us define the Newton polygon associated with a q-difference module. The z-adic valuation of K or $\hat{K}$ is denoted by v and defined by: $v\left(\sum a_n z^n\right) = \min_{a_n \neq 0} n$ and $v(0) = -\infty$.

Definition 1.3. — *The* Newton polygon *associated with the operator $P = \sum a_i\sigma_q^i \in \mathcal{D}_{K,\sigma_q}$ is the convex hull of $\{(i, j) \in \mathbb{Z}\times\mathbb{R} \mid j \geqslant v(a_i)\}$ in $\mathbb{R}^2$.*

The lower boundary of the Newton polygon is made of k vectors $(r_i, d_i) \in \mathbb{N}^* \times \mathbb{Z}$ ordered from left to right, the $\mu_i = d_i/r_i$ are the slopes of these vectors. Necessarily $\mu_i \in \mathbb{Q}$ and $\mu_1 < \cdots < \mu_k$. The Newton function associated with P is defined by $r_P(\mu_i) = r_i$ and $r_P(\mu) = 0$ otherwise.

By [12, Theorem 2.2.1], all unitary entire P such that $M \cong \mathcal{D}_{K,\sigma_q}/\mathcal{D}_{K,\sigma_q}P$ have the same Newton function denoted by r_M.

As a consequence, we can associate with a q-difference module a set of slopes $S(M) = \{\mu_1, \dots, \mu_k\}$.

A module with only one slope is said to be *pure isoclinic* and a module which is a direct sum of pure isoclinic modules is said to be *pure*. More particularly, a pure isoclinic module of slope 0 is called a *fuchsian* module. The equivalence of this definition with the classical one by Birkhoff is proved in [13, 16].

In the formal case (over $\hat{K}$), every q-difference module M is isomorphic to $M_0 = P_1 \oplus \cdots \oplus P_k$ where for all $i = 1, \dots, k$, P_i is pure isoclinic of slope μ_i and rank $r_M(\mu_i)$.

In the analytic case, according to the canonical filtration by slopes in [16], for all q-difference module M such that $S(M) = \{\mu_1, \dots, \mu_k\}$ and $\mu_1 < \dots < \mu_k$, there exists unique submodules such that $\{0\} \subset M_1 \subset \dots \subset M_k = M$, where for all $i = 1, \dots, k$, $P_i = M_i/M_{i-1}$ is pure isoclinic of slope μ_i and rank $r_M(\mu_i)$. We denoted by $\operatorname{gr} M$, the graded module $P_1 \oplus \dots \oplus P_k$, this module is pure by definition. In fact, gr is a functor from the category of q-differences modules to the category of pure q-differences modules. This functor is exact, faithful and tensor compatible (cf. [16]).

1.3. Normal forms

According to [12, 2.2.2], a pure isoclinic module with integral slope μ is isomorphic to a module of the form $(K^n, \Phi_{z^\mu A})$, where $A \in \mathrm{GL}_n(\mathbb{C})$ and $Sp(A) \subset C_q$ ($Sp(A)$ is the spectrum of the matrix A). The matrix A is unique up to conjugation in $\mathrm{GL}_n(\mathbb{C})$. It is the *normal form* of the module M. In this section, we give normal forms associated with q-difference modules with non integral slopes. These results are due to van der Put and Reversat in [7], but, except for Theorem 1.10 which we copy directly, we give different proofs and a concrete description that will be needed afterwards.

1.3.1. Irreducible modules

Definition 1.4. — *An* irreducible *q-difference module is a non trivial q-difference module which has no submodules except $\{0\}$ and itself. As a $\mathcal{D}_{K,\sigma_q}$-module, an irreducible module is simple.*

Theorem 1.5 ([7, Proposition 1.3]). — *For all irreducible q-difference module M over K (or $\hat{K}$), there are unique integers $d \in \mathbb{Z}$, $r \geqslant 1$ with $\gcd(d,r) = 1$, and a unique $c \in \mathbb{C}^*$, $1 \leqslant |c| < |q|$, such that:*

$$M \cong E(r,d,c) := \mathcal{D}_{K,\sigma_q}/\mathcal{D}_{K,\sigma_q}\left(\sigma_q^r - q^{\frac{-d(r-1)}{2}} c^{-1} z^{-d}\right).$$

Remark 1.6. — From this theorem, an irreducible module of rank $r > 1$ is pure isoclinic with slope $\mu = d/r$ such that $\gcd(r,d) = 1$. Conversely, a module of rank $r > 1$ and of slope $\mu = d/r$ such that $\gcd(r,d) = 1$, is irreducible. So, it is isomorphic to some $E(r,d,c)$ which can be written (K^r, Φ_B), where B is the following invertible matrix:

$$B := \begin{pmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & & & 1 \\ q^{\frac{d(r-1)}{2}} c z^d & 0 & \dots & 0 \end{pmatrix}.$$

For convenience, we will often denote c by b^r. $E(r,d,b^r)$ depends on b^r and not on the choice of b. By convention, if $r=1$, $E(1,d,c)=(K,\Phi_{cz^d})$.

More generally, every module of rank r and slope d/r ($\gcd(d,r)=1$) will be denoted by $E(r,d,c)=(K^r,\Phi_B)$ with $c\in\mathbb{C}^*$ but no necessarily in C_q.

Lemma 1.7. — *Let $d\in\mathbb{Z}^*$, $r\in\mathbb{N}^*$ such that $\gcd(d,r)=1$ and let $c\in\mathbb{C}^*$, there is an isomorphism of q-difference modules:*

$$E(r,d,c)\cong E(r,d,qc)\,.$$

Proof. — We identify $E(r,d,c)$ with (K^r,Φ_B) and we denote by $e={}^t(1,0,\dots,0)$ the cyclic vector associated with $E(r,d,c)=(K^r,\Phi)$. We have $\Phi^r(e)=q^{\frac{-d(r-1)}{2}}c^{-1}z^{-d}\,e$. We take $z^{-u}\Phi^v(e)$ as a new cyclic vector where $u,v\in\mathbb{Z}$ are Bezout's coefficients such that $ur+vd=1$. Indeed,

$$\Phi^r\left(z^{-u}\Phi^v(e)\right)=q^{-(ur+vd)}q^{\frac{-d(r-1)}{2}}c^{-1}z^{-d}\,z^{-u}\Phi^v(e)\,.\qquad\square$$

Remark 1.8. — We set $K_r:=K(z^{1/r})$, it is an extension of the field K, and the automorphism σ_{q_r} extends σ_q: $\sigma_{q_r}(f(z'))=f(q_rz')$ where $z'=z^{1/r}$. Then (K_r,σ_{q_r}) is q_r-difference field and the slopes are calculated with the z'-adic valuation associated with K_r. The operation of ramification on M consists in defining the q_r-difference module $M'=K_r\otimes M$, where if $M=(V,\Phi)$ then $M'=(V',\Phi')$, $V'=K_r\otimes_K V$ and $\Phi'=\sigma_{q_r}\otimes\Phi$. If μ_i are the slopes of M, the slopes of M' are $r\mu_i$. If M is irreducible and $M\cong E(r,d,c)$, we choose $b\in\mathbb{C}^*$ such that $c=b^r$, a matrix A such that $(M',\Phi_{M'})\cong(K_r^r,\Phi_{z^{d/r}A})$ is:

$$A:=\begin{pmatrix} b & & & 0\\ & \xi_r b & & \\ & & \ddots & \\ & & & \xi_r^{(r-1)}b\end{pmatrix}.$$

The matrix of the isomorphism G over K_r such that $\sigma_{q_r}(G)\,B=z^{d/r}A\,G$ is:

$$G=\begin{pmatrix} 1 & 1 & \dots & & 1\\ \xi_r^{-1} & \xi_r^{-2} & \dots & & 1\\ \xi_r^{-2} & \xi_r^{-4} & & \dots & 1\\ \vdots & & & & \vdots\\ \xi_r^{-(r-1)} & \xi_r^{-2(r-1)} & \dots & & 1\end{pmatrix}\begin{pmatrix} g_1 & & & \\ & g_2 & & 0\\ & & \ddots & \\ & 0 & & \ddots & \\ & & & & g_r\end{pmatrix}$$

where $g_i=q^{\frac{d(r-1)}{2}}b^{r-i}q_r^{\frac{d(i-1)(2r-i)}{2}}q_r^{-\,\mathrm{id}(r-1)}z^{\frac{-d(i-1)}{r}}$. In fact, these equations stay valid if d and r are not coprime.

1.3.2. Indecomposable modules

Definition 1.9. — *An* indecomposable *q-difference module over $\hat{K}$ or K, is a non trivial q-difference module which can not split into the direct sum of two q-difference submodules.*

Theorem 1.10 ([7, Corollary 1.6]). — *Let M be a pure isoclinic indecomposable q-difference module over K (or $\hat{K}$). There are unique integers $d \in \mathbb{Z}^*$, $r \geqslant 1$ and $m \geqslant 1$ such that $\gcd(d, r) = 1$ and unique $c \in \mathbb{C}^*$ satisfying $1 \leqslant |c| < |q|$ such that:*

$$M \cong E(r, d, c) \otimes_K U_m \, .$$

U_m is the unipotent q-difference module (K^m, Φ_{W_m}) (or over $\hat{K}$), where $W_m \in \mathrm{GL}_m(\mathbb{C})$ is the Jordan block of rank m:

$$W_m := \begin{pmatrix} 1 & 1 & 0 \\ & \ddots & 1 \\ & 0 & 1 \end{pmatrix}.$$

(For the definition of the tensor product, see 1.4).

Remark 1.11. — If M is indecomposable over $\hat{K}$ then M is automatically pure isoclinic because in the formal case, M is a direct sum of its maximal pure isoclinic submodules.

Theorem 1.10 tells nothing about an indecomposable q-difference module with several slopes. The study of modules with several slopes will be done in Section 2. Nevertheless, a pure isoclinic module is a direct sum of indecomposable pure isoclinic modules, therefore we have a normal form for every pure isoclinic q-difference module.

1.4. Tensor product

The tensor product of two q-difference modules $M = (V, \Phi)$ and $N = (W, \Psi)$ is $M \otimes N = (V \otimes_K W, \Phi \otimes \Psi)$ where $\Phi \otimes \Psi$ is the unique σ_q-linear automorphism of $V \otimes_K W$ such that $x \otimes y \mapsto \Phi(x) \otimes \Psi(y)$.

Convention. — Let us write $e_1, \dots, e_m$ for the canonical basis of K^m and $f_1, \dots, f_n$ the one of K^n. The family $(e_i \otimes f_j)_{i,j}$ form a basis of $K^m \otimes K^n$. We will use convention "$\hat{\otimes}$" to identify $K^m \otimes K^n$ with K^{mn}.

It consists in taking basis $\widehat{\mathcal{C}} = (e_1 \otimes f_1, e_2 \otimes f_1, \dots, e_m \otimes f_1, \dots, e_1 \otimes f_n, \dots, e_m \otimes f_n)$. Let $A = (a_{i,j}) \in M_m(K)$ and $B = (b_{i,j}) \in M_n(K)$ two matrices. With this convention, tensor product of A and B is denoted by $A \hat{\otimes} B$ and we have:

$$A \hat{\otimes} B = \begin{pmatrix} Ab_{1,1} & Ab_{1,2} & & \\ Ab_{2,1} & Ab_{2,2} & \ddots & \\ \vdots & \vdots & & \\ Ab_{n,1} & & & Ab_{n,n} \end{pmatrix}.$$

Lemma 1.12. — *If F, G are matrices in $M_{m,n}(K)$ then $AFB = G$ if, and only if, $(A \hat{\otimes} {}^t B)\hat{F} = \hat{G}$, where $\hat{F}$ is the vector ${}^t(f_{1,1}, f_{2,1}, \dots, f_{m,1}, \dots, f_{1,n}, \dots, f_{m,n})$.*

Proof. — Straightforward computation left to the reader. □

Let us have look at the tensor product of two irreducible modules. An irreducible q-difference module with an integral slope $\mu \in \mathbb{Z}$ is necessarily of rank one and of the form (K, Φ_{az^μ}). We saw that an irreducible module with non integral slope is of the form $E(r, d, c)$. We will give explicit formula for the tensor product of two irreducible modules. It is mentioned in [7, p. 681], we will give more details in 1.13 and 1.14.

Proposition 1.13. — *The tensor product of an irreducible q-difference module with non integral slope $\frac{d}{r}$ and an irreducible module with integral slope $\mu \in \mathbb{Z}$, of rank one, is isomorphic to an irreducible q-difference module of slope $\frac{d+\mu r}{r}$:*

$$E(r, d, c) \otimes (K, \Phi_{az^\mu}) \cong E(r, d + \mu r, ca^r)\,;$$

this isomorphism is explicit.

Proof. — The matrix $G = \mathrm{diag}(1, az^\mu, az^\mu \sigma_q(az^\mu), \dots)$ realizes this isomorphism. □

When the two irreducible modules have non integral slopes, we obtain a direct sum of irreducible modules with the same slope.

Proposition 1.14. — *Let $E(r_1, d_1, b_1^{r_1})$ and $E(r_2, d_2, b_2^{r_2})$ be two irreducible q-difference modules. Let $p = \gcd(r_1, r_2)$, $m = \mathrm{lcm}(r_1, r_2)$ and $d \in \mathbb{Z}$, $r \geqslant 1$ such that $\gcd(d, r) = 1$ and $\frac{d_1}{r_1} + \frac{d_2}{r_2} = \frac{d}{r}$ and $k := \frac{m}{r}$. We have an isomorphism of q-difference modules:*

$$E(r_1, d_1, b_1^{r_1}) \otimes E(r_2, d_2, b_2^{r_2}) \cong \bigoplus_{r=1}^{p/k} \bigoplus_{i=0}^{k-1} \bigoplus_{j=0}^{k-1} E(r, d, q_k^i\, \xi_k^j\, (b_1 b_2)^r)\,.$$

Proof. — It is a consequence of the two following lemmas. □

Lemma 1.15. — *Let $E(r_1,d_1,c_1)$ and $E(r_2,d_2,c_2)$ two irreducible q-difference modules. Let $c_1=b_1^{r_1}$ and $c_2=b_2^{r_2}$. We have an explicit isomorphism of q-difference modules which does not depend on the choice of b_1 and b_2:*

$$E(r_1,d_1,b_1^{r_1})\otimes E(r_2,d_2,b_2^{r_2})\cong\bigoplus_{i=0}^{p-1}\tilde{E}(m,t,q^{iu_2d_1}(b_1b_2)^m)$$

where $p=\gcd(r_1,r_2)$, $m=\operatorname{lcm}(r_1,r_2)$, $r_1=pu_1$, $r_2=pu_2$, $t=u_2d_1+u_1d_2$.

$\tilde{E}(m,t,a_i^m)$ represents a q-difference module of rank m, of slope $\frac{t}{m}=\frac{d_1}{r_1}+\frac{d_2}{r_2}$ and is equal to (K^m,Φ_{A_i}) where A_i has the following form:

$$A_i=\begin{pmatrix} 0 & 1 & & 0\\ \vdots & & \ddots & \\ 0 & & & 1\\ a_i^m q^{\frac{t(m-1)}{2}}z^t & 0 & \dots & 0\end{pmatrix}\quad \textit{where}\quad a_i^m=q^{iu_2d_1}(b_1b_2)^m.$$

Remark 1.16. — The modules $\tilde{E}$ are not a priori irreducible and t and m are not coprime. This lemma introduces an asymmetry that we correct in the Proposition 1.14.

Proof. — Let $M_i:=E(r_i,d_i,b_i^{r_i})=(K^{r_i},\Phi_i)$ where $\Phi_i:=\Phi_{B_i}$. The vector $e={}^t(1,0,\dots,0)$ of rank r_1 is a cyclic vector associated with M_1, indeed, $\underline{e}=\big(e,\Phi_1(e),\dots,\Phi_1^{r_1-1}(e)\big)$ is a basis of K^r and $\Phi_1^{r_1}(e)=q^{\frac{-d_1(r_1-1)}{2}}b_1^{-r_1}z^{-d_1}\ e$. Thus $\Phi_1(\underline{e})=\underline{e}B_1^{-1}$. If we denote by f the cyclic vector associated with M_2, we have $\Phi_2^{r_2}(f)=q^{\frac{-d_2(r_2-1)}{2}}b_2^{-r_2}z^{-d_2}\ f$.

We are looking for a basis of $K^{r_1}\otimes K^{r_2}$ formed with cyclic vectors for $\Phi=\Phi_1\otimes\Phi_2$. The vector $e\otimes f$ is not cyclic, indeed if $m=\operatorname{lcm}(r_1,r_2)\leqslant r_1r_2$, the family $(e\otimes f,\Phi(e\otimes f),\dots,\Phi^{m-1}(e\otimes f))$ is free over K but if $m\neq r_1r_2$, it is not a basis. Let $p=\gcd(r_1,r_2)$, the family

$$\begin{gathered} e\otimes f,\ \Phi_1(e)\otimes\Phi_2(f),\dots,\Phi_1^{m-1}(e)\otimes\Phi_2^{m-1}(f),\\ \Phi_1(e)\otimes f,\ \Phi_1^2(e)\otimes\Phi_2(f),\dots,\Phi_1^{m}(e)\otimes\Phi_2^{m-1}(f),\\ \vdots\\ \Phi_1^{p-1}(e)\otimes f,\ \Phi_1^{p}(e)\otimes\Phi_2(f),\dots,\Phi_1^{p-1+m-1}(e)\otimes\Phi_2^{m-1}(f)\end{gathered}$$

is a basis $\mathcal{B}$ of $K^{r_1}\otimes K^{r_2}$ which is then identified by $K^{r_1r_2}$, and the matrix of Φ has the wished form. □

We have something better thanks to the following lemma which allows to see the modules $\tilde{E}$ as a direct sum of irreducible modules:

Lemma 1.17. — *We have an explicit isomorphism of q-difference modules:*

$$\tilde{E}(m,t,a^m) \cong \bigoplus_{j=0}^{k-1} E(r,d,\xi_k^j a^r)$$

where $\gcd(d,r)=1$, $m=kr$ *and* $\frac{t}{m}=\frac{d}{r}$.

Proof. — It is clear that the vector $e = {}^t(1,0,\ldots,0)$ is cyclic for the module $\tilde{E}(m,t,a^m) =: (K^m,\Phi)$ that is $\{e,\Phi(e),\ldots,\Phi^{m-1}(e)\}$ is a basis of K^m and $\Phi^m(e) = a^{-m}q^{\frac{-t(m-1)}{2}}z^{-t}\,e$.

We are looking for vectors $f_0, f_1, \ldots, f_{k-1}$ such that the family

$$\{f_0, \Phi(f_0), \ldots, \Phi(f_0^{r-1}), \ldots, f_{k-1}, \ldots, \Phi^{r-1}(f_{k-1})\}$$

is a basis of K^m and $\Phi^r(f_j) = \xi_k^{-j}a^{-r}q^{\frac{-d(r-1)}{2}}z^{-d}\,f_j$. We write

$$f_j = a_{1(j)}e + a_{2(j)}\Phi^r(e) + \cdots + a_{k(j)}\Phi^{m-r}(e)$$

and we verify that $a_{i(j)} = \xi_k^{ij}c_i$ with $c_i = a^{-m+ir}q^{\frac{-(k-i)d}{2}r(k+i-2)-1}z^{(i-1)d}$, is suitable. The transition matrix from the basis $\{e,\Phi(e),\ldots,\Phi^{m-1}(e)\}$ to $\{f_0,\Phi(f_0),\ldots,\Phi^{r-1}(f_0),\ldots,f_{k-1},\ldots,\Phi^{r-1}(f_{k-1})\}$ is the matrix of the isomorphism. $\square$

2. Isoformal analytic classification

In this section, we want to study the analytic classes of q-difference modules over K whose graded module is fixed (we will define this notion later).

When slopes are integral, according to [12], it leads to the normal form of Birkhoff–Guenther[(1)] and it gives an explicit matricial normal form for the representatives of the classes. Here, we choose the same approach allowing non integral slopes. We will use the same tool: an adapted q-Borel–Ramis transform and we obtain normal forms with polynomial coefficients. However, we are only able to treat the case of two slopes, when one or two slopes are not integral. Our method does not work with three slopes or more. Indeed, as shown by the computations in [15], upper levels involve nonlinear formulas for which I have found no equivalent in the case of non integral slopes.

(1) These authors also use a cohomological description inspired by the theorem of Birkhoff–Malgrange–Sibuya, we shall follow that method in Section 3 for two arbitrary slopes.

2.1. The space of isoformal analytic classes

We fix a class $M_0 = P_1 \oplus \cdots \oplus P_k$ where P_i is pure isoclinic of slope $\mu_i \in \mathbb{Q}$ and of rank $r_i \in \mathbb{N}^*$. Moreover, we suppose $\mu_1 < \cdots < \mu_k$. As M_0 is a direct sum of pure isoclinic q-difference module, the class M_0 can be identified with a formal class.

Definition 2.1 ([12]). — *$\mathcal{F}(P_1, \ldots, P_k)$ is the set of equivalence classes of pairs (M, u) with M a q-difference module over K and $u : \operatorname{gr} M \to M_0$ an isomorphism, where $(M, u) \sim (M', u')$ if there exists a morphism $f : M \to M'$ such that $u = u' \circ \operatorname{gr} f$ and f is automatically an isomorphism.*

According to [12, Theorem 3.1.4], the set $\mathcal{F}(P_1, \ldots, P_k)$ is an affine space of dimension over $\mathbb{C}$ $\sum_{i<j} r_i r_j (\mu_j - \mu_i)$.

In term of matrices, the fixed formal class is $M_0 = (K^n, \Phi_{A_0})$ and every class of $\mathcal{F}(P_1, \ldots, P_k)$ can be represented by $M_U = (K^n, \Phi_{A_U})$ where

$$A_0 = \begin{pmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_k \end{pmatrix}, \quad A_U = \begin{pmatrix} B_1 & & U_{i,j} \\ & \ddots & \\ 0 & & B_k \end{pmatrix} \tag{2.1}$$
$$\forall\, 1 \leqslant i < j \leqslant k, \quad U_{i,j} \in M_{r_i, r_j}(K)$$

each matrice $B_i \in \mathrm{GL}_{r_i}(K)$ represents the pure isoclinic module $P_i = (K^{r_i}, \Phi_{B_i})$ of slope μ_i and rank r_i (we generally denote $M_{r,s}(K)$ the space of $r \times s$ matrices over K).

We will denote by $[M_U]$ the class of the module M_U in $\mathcal{F}(P_1, \ldots, P_k)$. A morphism respecting the graduation from M_U to M_V is a matrix:

$$F = \begin{pmatrix} I_{r_1} & & F_{i,j} \\ & \ddots & \\ 0 & & I_{r_k} \end{pmatrix} \qquad \forall\, 1 \leqslant i < j \leqslant k, \quad F_{i,j} \in M_{r_i, r_j}(K) \tag{2.2}$$

such that $\sigma_q(F) A_U = A_V F$. The set of matrices of the form (2.2) forms a unipotent algebraic subgroup of GL_n that we denote by $\mathfrak{G}_{r_1, \ldots, r_k}$.

Two modules M_U and M_V are equivalent if there exists an isomorphism $F \in \mathfrak{G}_{r_1, \ldots, r_k}(K)$ such that $F[A_U] = A_V$, in this case, we denote $M_U \sim M_V$ and $A_U \sim A_V$.

When the slopes are integral, $\mu_i \in \mathbb{Z}$, the matrices associated with P_i have the form $B_i = z^{\mu_i} A_i$, $A_i \in \mathrm{GL}_{r_i}(\mathbb{C})$. According to Proposition 3.3.4 p. 40 of [12], a class of $\mathcal{F}(P_1, \ldots, P_k)$ admits a unique representative with the associated matrix:

$$A_U = \begin{pmatrix} z^{\mu_1} A_1 & & U_{i,j} \\ & \ddots & \\ 0 & & z^{\mu_k} A_k \end{pmatrix} \ \forall\, 1 \leqslant i < j \leqslant k,\ \ U_{i,j} \in M_{r_i,r_j}\left(\bigoplus_{l=\mu_i}^{\mu_j - 1} \mathbb{C} z^l\right).$$

This is called the normal form of Birkhoff–Guenther.

2.2. The case of two slopes: extensions

As we shall see, the space of isoformal analytic classes with two slopes is special because it is a vector space over $\mathbb{C}$. Moreover, thanks to the normal forms of pure isoclinic modules with integral slopes or not, we finally have to study only one case: $\mathcal{F}(E, \underline{1})$ in the above E is an irreducible module.

Let P_1 and P_2 be two pure isoclinic q-difference modules with slopes $\mu_1 < \mu_2$. The space $\mathcal{F}(P_1, P_2)$ is $\mathbb{C}$-vector space which can be identify with $\mathrm{Ext}(P_2, P_1)$, which is the $\mathbb{C}$-vector space of the extensions of the q-difference module P_2 by the q-difference module P_1 (cf. [12, Proposition 2.3.9 p. 25]; our Ext can be identified with the Ext^1 space of homological algebra for $\mathcal{D}_{K,\sigma_q}$-modules, this allows us to use [3]).

Let $P_1 = (V_1, \Phi_1)$ and $P_2 = (V_2, \Phi_2)$ be two pure isoclinic q-differences modules of slopes $\mu_1 < \mu_2$. An extension of P_2 by P_1 is an exact sequence of q-differences modules:

$$0 \to P_1 \to M \to P_2 \to 0$$

giving an exact sequence of K-vector spaces: $0 \to V_1 \to V \to V_2 \to 0$ where $M = (V, \Phi)$. The following diagram is commutative:

$$\begin{tikzcd} 0 \arrow[r] & V_1 \arrow[r] \arrow[d, "\Phi_1"] & V \arrow[r] \arrow[d, "\Phi"] & V_2 \arrow[r] \arrow[d, "\Phi_2"] & 0 \\ 0 \arrow[r] & V_1 \arrow[r] & V \arrow[r] & V_2 \arrow[r] & 0\,. \end{tikzcd}$$

For $i = 1, 2$, P_i is isomorphic to (K^{n_i}, Φ_{A_i}), $A_i \in \mathrm{GL}_{n_i}(K)$, an extension of P_2 by P_1 is a module of the form $M_U = (K^{n_1+n_2}, \Phi_{A_U})$ where:

$$A_U = \begin{pmatrix} A_1 & U \\ 0 & A_2 \end{pmatrix}, \quad U \in M_{n_1,n_2}(K)\,.$$

We can easily see that $A_U \sim A_V$ if, and only if, $A_0 \sim A_{V-U}$:

$$\begin{aligned} A_U \sim A_V &\Leftrightarrow \exists\, F \in M_{n_1,n_2}(K), \sigma_q(F) A_2 + U = A_1 F + V \\ &\Leftrightarrow \exists\, F \in M_{n_1,n_2}(K), \sigma_q(F) A_2 - A_1 F = V - U\,. \end{aligned}$$

We notice that $\mathcal{F}(P_1, P_2)$ can be identified with the cokernel of $X \mapsto \sigma_q(X)A_2 - A_1X$ in $M_{n_1,n_2}(K)$ which is isomorphic to the cokernel of $X \mapsto \sigma_q(X) - A_1XA_2^{-1}$ in $M_{n_1,n_2}(K)$.

We have $\mathrm{Ext}(P_2, P_1 \oplus P_3) \cong \mathrm{Ext}(P_2, P_1) \oplus \mathrm{Ext}(P_2, P_3)$ (cf. [12]) that gives the following isomorphism: $\mathcal{F}(P_1 \oplus P_3, P_2) \cong \mathcal{F}(P_1, P_2) \oplus \mathcal{F}(P_3, P_2)$ which is defined by:

$$[M_U] = [(K^{n_1+n_3+n_2}, \Phi_{A_U})] \mapsto ([M_{U_1}], [M_{U_2}]) \tag{2.3}$$

where $M_{U_1} = (K^{n_1+n_2}, \Phi_{A_{U_1}})$, $M_{U_2} = (K^{n_3+n_2}, \Phi_{A_{U_2}})$ and by identifying P_1 with (K^{n_1}, Φ_{B_1}) and P_3 with (K^{n_3}, Φ_{B_3}),

$$A_U = \begin{pmatrix} B_1 & 0 & U_1 \\ 0 & B_3 & U_2 \\ 0 & 0 & B_2 \end{pmatrix},\ A_{U_1} = \begin{pmatrix} B_1 & U_1 \\ 0 & B_2 \end{pmatrix}\ A_{U_2} = \begin{pmatrix} B_3 & U_2 \\ 0 & B_2 \end{pmatrix},$$
$$U_1 \in M_{n_1,n_2}(K),\ U_2 \in M_{n_3,n_2}(K).$$

Moreover, $\mathrm{Ext}(P_2, P_1) \cong \mathrm{Ext}(\underline{1}, P_2^\vee \otimes P_1)$ (cf. [12]), then we have an isomorphism: $\mathcal{F}(P_1, P_2) \cong \mathcal{F}(P_1 \otimes P_2^\vee, \underline{1})$ where $P_2^\vee$ is the dual of P_2. This isomorphism is defined by:

$$[M_U] = [(K^{n_1+n_2}, \Phi_{A_U})] \mapsto [M'_{U'}] = [(K^{n_1n_2+1}, \Phi_{A'_{U'}})] \tag{2.4}$$

where,

$$A_U = \begin{pmatrix} B_1 & U \\ 0 & B_2 \end{pmatrix},\ U \in M_{n_1,n_2}(K)$$
$$\text{and } A'_{U'} = \begin{pmatrix} B_1 \hat{\otimes} B_2^\vee & U' \\ 0 & 1 \end{pmatrix},\ U' = \widehat{UB_2^{-1}} \in M_{n_1n_2,1}(K).$$

It is well defined, indeed M_0 is equivalent to M_U if, and only if, there exists $F \in M_{n_1,n_2}(K)$ such that:

$$\sigma_q(F)B_2 = B_1F + U \Leftrightarrow \sigma_q(F) = B_1FB_2^{-1} + UB_2^{-1}$$
$$\Leftrightarrow \sigma_q\hat{F} = B_1 \hat{\otimes} B_2^\vee \hat{F} + \widehat{UB_2^{-1}}$$

where $B_2^\vee = {}^tB_2^{-1}$ is the matrix associated with $P_2^\vee$. The last equation means that M'_0 is equivalent to $M'_{U'}$.

Proposition 2.2. — *The module M' is a pullback:*

$$M' = (M \otimes P_2^\vee) \times_{P_2 \otimes P_2^\vee} \underline{1}.$$

Proof. — We have the following exact sequence:

$$0 \to P_1 \to M \to P_2 \to 0.$$

By tensoring by $P_2^\vee$, it becomes (since tensoring is exact in this tannakian category):

$$0 \to P_1 \otimes P_2^\vee \to M \otimes P_2^\vee \to P_2 \otimes P_2^\vee \to 0. \tag{2.5}$$

Let $M'' = (M \otimes P_2^\vee) \times_{P_2 \otimes P_2^\vee} \underline{1}$ (using the usual operator $\underline{1} \to P_2 \otimes P_2^\vee$ of rigid categories); we have the following lemma:

Lemma 2.3. — *The following diagram of exact sequences of q-difference modules is commutative:*

$$\begin{tikzcd}
0 \arrow[r] & P_1 \otimes P_2^\vee \arrow[r] & M \otimes P_2^\vee \arrow[r] & P_2 \otimes P_2^\vee \arrow[r] & 0 \\
0 \arrow[r] & P_1 \otimes P_2^\vee \arrow[r] \arrow[u, "id"] & M'' \arrow[r] \arrow[u, hook] & \underline{1} \arrow[r] \arrow[u, hook] & 0.
\end{tikzcd}$$

Remark 2.4. — This a general lemma (see for example [12] or [3]). We give here an explicit matricial proof.

Proof of the lemma. — For convenience, we denote $P_1 = (V_1, \Phi_1)$, $P_2 = (V_2, \Phi_2)$. We have $M = (V_1 \times V_2, \Phi_u)$ such that

$$\Phi_u(x_1, x_2) = (\Phi_1(x_1) + u(x_2), \Phi_2(x_2)),$$

$u \in \mathcal{L}_{\sigma_q}(V_2, V_1)$ (the set of the σ_q-linear maps from V_2 to V_1), with the point of view of matrices, u corresponds to $-B_1 U B_2^{-1} \sigma_q$ (since $\Phi_U = (A_U)^{-1}\sigma_q$). From now, we identify $N \otimes P_2^\vee = \underline{\mathrm{Hom}}(P_2, N)$, where by definition, the internal Hom of $M = (V, \Phi)$ and $N = (W, \Psi)$, denoted by $\underline{\mathrm{Hom}}(M, N)$, is the q-difference module $(\mathcal{L}(V, W), T_{\Phi,\Psi})$ where $T_{\Phi,\Psi}$ is the σ_q-linear automorphism defined by: $f \mapsto \Psi \circ f \circ \Phi^{-1}$.

The exact sequence of q-difference modules (2.5) becomes:

$$0 \to \underline{\mathrm{Hom}}(P_2, P_1) \to \underline{\mathrm{Hom}}(P_2, M) \to \underline{\mathrm{Hom}}(P_2, P_2) \to 0.$$

Indeed, we have an exact sequence of the underlying K-vector spaces:

$$0 \to \mathcal{L}(V_2, V_1) \to \mathcal{L}(V_2, V_1) \times \mathcal{L}(V_2, V_2) \to \mathcal{L}(V_2, V_2) \to 0$$

with $\mathcal{L}(V_2, V_1)$ provided with the σ_q-linear automorphism $f \mapsto \Phi_1 \circ f \circ \Phi_2^{-1}$, $\mathcal{L}(V_2, V_2)$ provided with the σ_q-linear automorphism $g \mapsto \Phi_2 \circ g \circ \Phi_2^{-1}$ and $\mathcal{L}(V_2, V_1) \times \mathcal{L}(V_2, V_2)$ provided with the σ_q-linear automorphism

$$(f, g) \mapsto \Psi_u(f, g) = \Phi_u \circ (f, g) \circ \Phi_2^{-1} = (\Phi_1 \circ f \circ \Phi_2^{-1} + u \circ g \circ \Phi_2^{-1}, \Phi_2 \circ g \circ \Phi_2^{-1}).$$

Therefore, the module $\underline{\mathrm{Hom}}(P_2, M)$ is an extension of q-difference modules $\underline{\mathrm{Hom}}(P_2, P_2)$ by $\underline{\mathrm{Hom}}(P_2, P_1)$ which corresponds to the element $g \mapsto u \circ g \circ \Phi_2^{-1}$ of $\mathcal{L}_{\sigma_q}(\mathcal{L}(V_2, V_2), \mathcal{L}(V_2, V_1))$.

According to [3] (Algebra, chapter X on homological algebra, p. 113), we have a morphism $\mathrm{Hom}(N,\tilde{M})\times Ext(\tilde{M},\tilde{M}')\to Ext(N,\tilde{M}')$. Replacing N by $\underline{1}$, $\tilde{M}$ by $P_2\otimes P_2^\vee$ and $\tilde{M}'$ by $P_1\otimes P_2^\vee$, we make the pullback with the morphism $\underline{1}\to\underline{\mathrm{Hom}}(P_2,P_2)$ and we obtain (keeping only the degree 0 terms of the Ext sequence):

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
 & \underline{\mathrm{Hom}}(P_2,P_1) \arrow[r, "\mathrm{id}"] \arrow[d] & \underline{\mathrm{Hom}}(P_2,P_1) \arrow[d] \\
(2.6) & \underline{\mathrm{Hom}}(P_2,M)\times_{\underline{\mathrm{Hom}}(P_2,P_2)}\underline{1} \arrow[r] \arrow[d] & \underline{\mathrm{Hom}}(P_2,M) \arrow[d] \\
 & \underline{1} \arrow[r, hook] \arrow[d] & \underline{\mathrm{Hom}}(P_2,P_2) \arrow[d] \\
 & 0 & 0.
\end{tikzcd}$$

As far as the underlying K-vector spaces are concerned, the second exact sequence of the diagram gives the exact sequence of K-vector spaces:

$$0\to\mathcal{L}(V_2,V_1)\to(\mathcal{L}(V_2,V_1)\times\mathcal{L}(V_2,V_2))\times_{\mathcal{L}(V_2,V_2)}K\to K\to 0$$

with $\mathcal{L}(V_2,V_1)$ provided with the σ_q-linear automorphism $f\mapsto\Phi_1\circ f\circ\Phi_2^{-1}$, K provided with the σ_q-linear automorphism σ_q.

And yet, $(\mathcal{L}(V_2,V_1)\times\mathcal{L}(V_2,V_2))\times_{\mathcal{L}(V_2,V_2)}K=\{(f,g,\lambda)\mid g=\lambda\,\mathrm{Id}\}=\{(f,\lambda)\}$, so it is provided with the σ_q-linear automorphism $\Psi_u\times\sigma_q|_{\{(f,\lambda)\}}$:

$$(f,\lambda)\mapsto\Big(\Phi_1\circ f\circ\Phi_2^{-1}+\sigma_q(\lambda)u\circ\Phi_2^{-1}),\sigma_q(\lambda)\Big).$$

The module M'' corresponds to the extension of $\underline{1}$ by $(\mathcal{L}(V_2,V_1),\Phi_1\otimes\Phi_2^\vee)$ defined by the element of $\mathcal{L}_{\sigma_q}(K,\mathcal{L}(V_2,V_1))$, $\lambda\mapsto\sigma_q(\lambda)(u\circ\Phi_2^{-1})$. Notice that $u\circ\Phi_2^{-1}$ is associated with the matrix $-B_1UB_2^{-1}B_2=-B_1U$. This ends the proof of the lemma. □

With the same notation as the lemma, M' is identified with the module $(\mathcal{L}(V_2,V_1)\times K,\Phi_v)$ where $\Phi_v(f,\lambda)=(\Phi_1\circ f\circ\Phi_2^{-1}+\sigma_q(\lambda)v,\sigma_q(\lambda))$, $v\in\mathcal{L}_{\sigma_q}(K,\mathcal{L}(V_2,V_1))$, with a matricial point of view v corresponds to $-B_1U\sigma_q$. It is exactly the module M' defined, as a consequence, there are the same modules. This ends the proof of the proposition. □

A pure isoclinic module of slope $\mu \in \mathbb{Z}$ is isomorphic to a module of the form $(K^n, \Phi_{z^\mu A})$, by Jordan's decomposition of A, it is isomorphic to a module of the form $\bigoplus_i (K, \Phi_{a_i z^\mu}) \otimes U_{m_i}$, where $a_i \in C_q$ is an eigenvalue of A.

A pure isoclinic module with non integral slope is a direct sum of indecomposable modules, so it is isomorphic to $\bigoplus_i E(r,d,c_i) \otimes U_{m_i}$. As a consequence, the study of $\mathcal{F}(P_1,P_2)$ amounts to the study of $\mathcal{F}(E_1 \otimes U_{m_1}, E_2 \otimes U_{m_2})$ where E_1, E_2 are irreducible with an integral slope or not.

According to the explicit isomorphism (2.4), $\mathcal{F}(E_1 \otimes U_{m_1}, E_2 \otimes U_{m_2})$ is isomorphic to $\mathcal{F}(E_1 \otimes E_2^\vee \otimes U_{m_1} \otimes U_{m_2}^\vee, \underline{1})$. Moreover, $U_{m_1} \otimes U_{m_2}^\vee$ is isomorphic to a direct sum of unipotent modules (for details, see for instance [14]).

Thus, we have to study $\mathcal{F}(E_1 \otimes E_2^\vee \otimes U_m, \underline{1})$. And yet, in the Proposition 1.14, we have seen that the tensor product of two irreducible modules is isomorphic to a direct sum of irreducible modules. Finally, the only one case to study is $\mathcal{F}(E \otimes U_m, \underline{1})$, where E is an irreducible module with non integral slope.

2.3. $\mathcal{F}(E,1)$ and $\mathcal{F}(E \otimes U_m, 1)$

We begin to study $\mathcal{F}(P_1,P_2)$ when the slope of P_1 is negative and non integral and the slope of P_2 is zero. According to the last paragraph, we just have to deal with $\mathcal{F}(E,\underline{1})$ where $E = E(r,-d,c) = (K^r, \Phi_B)$ is an irreducible module with non integral negative slope $\frac{-d}{r}$.

So, let $E = E(r,-d,c)$ be is an irreducible module with non integral slope $\mu = -d/r < 0$ and of rank r, such that $E = (K^r, \Phi_B)$ we set $c' = q^{\frac{-d(r-1)}{2}} c \in \mathbb{C}^*$.

Remember that a class of $\mathcal{F}(E,\underline{1})$ admits a representative whose associated matrix has the following form (cf. (2.1)):

$$A_U = \begin{pmatrix} B & U \\ 0 & 1 \end{pmatrix} \text{ where } U = {}^t(u_1, \dots, u_r) \in K^r.$$

In [16, 15], it is proved following Birkhoff and Guenther [1], that a class of $\mathcal{F}(E,\underline{1})$ admits at least a polynomial representative (that is to say U has polynomial coefficients), in the following lemma, we give an effective way to find one. This representative is non unique contrary to the case of integral slopes. Indeed, the simplest non trivial example (E having rank 2 and slope $-1/2$) yields a space of classes of dimension 1 to be evenly distributed in $\mathbb{C}^2$, explaining the impossibility of a straight generalisation of Birkhoff–Guenther normal form.

Lemma 2.5 (cf. [12]). — *Let L' be an operator of $\mathbb{C}(\{z\})$ with values in itself of the form $L' = az^d\sigma_q^r - 1$, $a \in \mathbb{C}^*$ and $d, r > 0$. Let $k \in \mathbb{Z}$ fixed, then we have:*

$$\mathbb{C}(\{z\}) = \bigoplus_{l=k}^{k+d-1} \mathbb{C}z^l \oplus \operatorname{Im} L'$$

Remark 2.6. — Let us fix $k \in \mathbb{Z}$, we denote by $\pi_{r,d,k}$ the projection of $\mathbb{C}(\{z\})$ on $\bigoplus_{l=k}^{k+d-1} \mathbb{C}z^l$ parallel to $\operatorname{Im} L'$. Thus, the equation $az^d\sigma_q^r(f) - f = \alpha$ has a solution in $\mathbb{C}(\{z\})$ if, and only if, $\pi_{r,d,k}(\alpha) = 0$.

Let $V_{r,d}$ be the $\mathbb{C}$-vector space $(\sum_{l=0}^{d-1} \mathbb{C}z^l)^r$ of dimension rd.

Lemma 2.7. — *The linear map from $V_{r,d}$ to $\mathcal{F}(E, \underline{1})$ defined by:*

$$U \in V_{r,d} \mapsto [M_U]$$

is onto. So, every class of $\mathcal{F}(E, \underline{1})$ admits a representative (non unique) M_U with $U \in V_{r,d}$. Such a representative can be given by the $\mathbb{C}$-linear map $\Pi_{r,d} : K^r \to V_{r,d}$ defined by:

$$U = (u_1, \ldots, u_r) \mapsto (v_1, \ldots, v_r) \ \text{ such that } \ z^d\sigma_q^{r-i}(v_i) = \pi_{r,d,d}\big(z^d\sigma_q^{r-i}(u_i)\big),$$

($\pi_{r,d,d}$ is the $\mathbb{C}$-linear defined in the Remark 2.6, and we choose here $k = d$).

Proof. — Let M_U be a representative of a class of $\mathcal{F}(E, \underline{1})$ with matrix A_U, we write $U = {}^t(u_1, \ldots, u_r)$, then

$$(2.7) \quad A_0 \sim A_U \Leftrightarrow \exists\, F = {}^t(f_1, \ldots, f_n) \in K^r,\ \sigma_q F = BF + U$$

$$(2.8) \qquad \Leftrightarrow \begin{cases} f_2 = \sigma_q(f_1) - u_1 \\ f_3 = \sigma_q^2(f_1) - (\sigma_q(u_1) + u_2) \\ \quad\vdots \\ f_r = \sigma_q^{r-1}(f_1) - (\sigma_q^{r-2}(u_1) + \cdots + \sigma_q(u_{r-2}) + u_{r-1}) \\ \sigma_q^r(f_1) = c' z^{-d} f_1 + \sigma_q^{r-1}(u_1) + \cdots + \sigma_q(u_{r-1}) + u_r\,. \end{cases}$$

We have to solve an equation of the type: $az^d\sigma_q^r(f) - f = \alpha$, with $a = c'^{-1} \in \mathbb{C}^*$ and

$$\alpha = az^d\left(\sigma_q^{r-1}(u_1) + \cdots + \sigma_q(u_{r-1}) + u_r\right).$$

According to Lemma 2.5, for a fixed $k = d \in \mathbb{Z}$, this equation have a solution in K if, and only if, $\pi_{r,d,d}(\alpha) \in \bigoplus_{l=d}^{d+d-1} \mathbb{C}z^l = 0$. Let

$$z^d\sigma_q^{(r-i)}(v_i) = \big(\pi_{r,d,d}(z^d\sigma_q^{r-i}(u_i))\big),$$

we have $\pi_{r,d,d}\left(z^d\sigma_q^{r-i}(v_i) - z^d\sigma_q^{r-i}(u_i)\right) = 0$. Thus, the equations $az^d\sigma_q^r(g_i) - g_i = z^d\sigma_q^{r-i}(v_i) - z^d\sigma_q^{r-i}(u_i)$ admit a solution g_i in K. Let $f = g_1 + \cdots + g_r$.

With the same system, replacing U by $V - U$, then $A_0 \sim A_{V-U}$ so $A_U \sim A_V$ with $V \in V_{r,d}$. □

Theorem 2.8. — *The $\mathbb{C}$-linear map φ from $V_{r,d}$ to $\sum_{l=0}^{d-1}\mathbb{C}z^l$ defined by*

$$\varphi((u_1, \ldots, u_r)) = \sum_{j=1}^{r} \sigma^{r-j}(u_j)$$

induces an isomorphism of $\mathbb{C}$-vector spaces:

$$\mathcal{F}(E, \underline{1}) \cong \frac{V_{r,d}}{Ker\varphi} \cong \operatorname{Im}\varphi \cong \sum_{l=0}^{d-1}\mathbb{C}z^l.$$

The dimension of $\mathcal{F}(E, \underline{1})$ is known to be equal to d, we find it again in this theorem, which can be summarized by the following commutative diagram:

$$\begin{tikzcd} V_{r,d} \arrow[d, "\phi"'] \arrow[dr, "\text{onto}"] & \\ V_{1,d} \arrow[r, "\simeq"] & \mathcal{F}(E, \underline{1}) \end{tikzcd}$$

Proof. — According to the Lemma 2.7, we can suppose that a class of $\mathcal{F}(E, \underline{1})$ is represented by M_U with $U \in V_{r,d}$.

Thanks to system (2.8) of the previous proof, we now have $\alpha = az^d\left(\sigma_q^{r-1}(u_1) + \cdots + \sigma_q(u_{r-1}) + u_r\right)$ which is in $\bigoplus_{l=d}^{d+d-1}\mathbb{C}z^l$, so the equation $az^d\sigma_q^r(f) - f = \alpha$ admits a convergent solution if, and only if, $\alpha = 0$, it is equivalent to $\varphi(U) = \sigma_q^{r-1}(u_1) + \cdots + \sigma_q(u_{r-1}) + u_r = 0$. As a consequence, the map from $\mathcal{F}(E, \underline{1})$ to$\sum_{l=0}^{d-1}\mathbb{C}z^l$ is injective. To prove the bijectivity, we can use the dimension or see that the antecedent of $u \in \sum_{l=0}^{d-1}\mathbb{C}z^l$ is the class $[M_U]$ where $U = {}^t(0, \ldots, 0, u)$, because $[M_U] \mapsto \varphi(U) = u$. □

Remark 2.9. — According to the proof, the coefficients of a morphism F verifying $\sigma_q(F) - BF = U$ with $U \in V_{r,d}$ are in $V_{r,d}$, because we have $f_1 = 0$ and for all $i > 1$,

$$f_i = -(\sigma_q^{i-2}(u_1) + \cdots + \sigma_q(u_{i-2}) + u_{i-1}).$$

Thanks to this theorem, we can choose the matrix of a representative of a class of $\mathcal{F}(E, \underline{1})$, of the form A_U where $U = {}^t(u_1, \ldots, u_r)$ has polynomial coefficients in $\mathbb{C}[z]_{d-1}$. In other word, such a q-difference module is in the same class as A_0 if, and only if, $\varphi((u_1, \ldots, u_r)) = 0$.

From this result, we can achieve the study of $\mathcal{F}(E_1, E_2)$ when E_1 and E_2 are irreducible. It remains now to study $\mathcal{F}(E \otimes U_m, \underline{1})$.

Lemma 2.10. — *Every class of $\mathcal{F}(E \otimes U_m, \underline{1})$ admits a representative M_U such that $U \in V_{r,d}^m$.*

Proof. — With the convention $\hat{\otimes}$ defined in 1.4, a module M_U representing a class of $\mathcal{F}(E \otimes U_m, \underline{1})$ has a matrix of the form:

$$A_U = \begin{pmatrix} B & B & & 0 & U_1 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & B & B & U_{m-1} \\ 0 & \dots & 0 & B & U_m \\ 0 & & & 0 & 1 \end{pmatrix}$$

Thus, $A_0 \sim A_U$ if, and only if, there exists $F = (F_1, \dots, F_m) \in (K^r)^m$, such that

$$\begin{pmatrix} \sigma_q(F_1) \\ \vdots \\ \sigma_q(F_{m-1}) \\ \sigma_q(F_m) \end{pmatrix} = \begin{pmatrix} B & B & & 0 \\ 0 & B & \ddots & \\ & & \ddots & B \\ 0 & & 0 & B \end{pmatrix} \begin{pmatrix} F_1 \\ \vdots \\ F_{m-1} \\ F_m \end{pmatrix} + \begin{pmatrix} U_1 \\ \vdots \\ U_{m-1} \\ U_m \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} \sigma_q(F_1) = BF_1 + BF_2 + U_1 \\ \sigma_q(F_2) = BF_2 + BF_3 + U_2 \\ \qquad \vdots \\ \sigma_q(F_{m-1}) = BF_{m-1} + BF_m + U_{m-1} \\ \sigma_q(F_m) = BF_m + U_m \,. \end{cases}$$

With the notations of Lemma 2.7, we put:

- $V_m = \Pi_{r,d}(U_m)$ then there exists $F_m \in K^r$ such that $\sigma_q(F_m) = BF_m + V_m - U_m$;
- $V_{m-1} = \Pi_{r,d}(BF_m + U_{m-1})$ then there exists $F_{m-1} \in K^r$ such that $\sigma_q(F_{m-1}) = BF_{m-1} + V_{m-1} - (BF_m + U_{m-1}) \dots$

Therefore by induction, we obtain the lemma. □

Let us suppose now that $U \in V_{r,d}^m$. We have $A_0 \sim A_U$ that implies $\varphi(U_m) = 0$ and according to the Remark 2.9, the morphism F_m such that $\sigma_q(F_m) = BF_m + U_m$ is explicitly known and depends on U_m, it has also his coefficients in $V_{r,d}$, and BF_m too. Indeed, $BF_m = -\Psi(U_m)$ where Ψ is the following map:

$$\Psi(U_m) = \begin{pmatrix} u_1 \\ \sigma_q(u_1) + u_2 \\ \vdots \\ \sigma_q^{r-2}(u_1) + \cdots + \sigma_q(u_{r-2}) + u_{r-1} \\ \varphi(U_m) \end{pmatrix} \text{ and } U_m = {}^t(u_1, \dots, u_r).$$

So $A_0 \sim A_U$ implies $\varphi(U_m) = 0$ and $\varphi(-\Psi(U_m) + U_{m-1})$. By induction, we prove that $A_0 \sim A_U$ implies $F_1, \dots, F_m \in V_{r,d}$ and $\varphi(U_m) = 0$, for all $i = 1, \dots m-1$, $\varphi(BF_{m-i+1} + U_{m-i}) = 0$. We obtain the following theorem:

Theorem 2.11. — *The $\mathbb{C}$-linear map ψ from $(K^r)^m$ to $(\sum_{l=0}^{d-1} \mathbb{C}z^l)^m$ defined by:*

$$U = (U_1, \dots, U_m) \in (V_{r,d})^m \mapsto \psi(U) = (\varphi(S_1), \dots, \varphi(S_m))$$

where

$$S_i = (-1)^i \Psi^{m-i}(U_m) + (-1)^{i-1} \Psi^{m-i-1}(U_{m-1}) + \cdots - \Psi(U_{m-i+1}) + U_{m-i}$$

induces an isomorphism between $\mathcal{F}(E \otimes U_m, \underline{1})$ and $(\sum_{l=0}^{d-1} \mathbb{C}z^l)^m$, which, to a class $[M_U]$ with $U \in V_{r,d}^m$, associates $\psi(U)$.

Proof. — From the previous paragraph, it is injective and because of the dimension, it is bijective. □

This theorem is summarized by the following commutative diagram:

$$\begin{tikzcd} (K^r)^m \arrow[d, "U \mapsto [M_U]"'] \arrow[dr, "\psi"] & \\ \mathcal{F}(E \otimes U_m, \underline{1}) \arrow[r, "\simeq"] & (\sum_{l=0}^{d-1} \mathbb{C}z^l)^m \end{tikzcd}$$

3. H^1 and Stokes operators

In [12], the space of isoformal classes is shown (in the case of an arbitrary number of integral slopes) to be isomorphic to the first cohomology group of some sheaf of unipotent groups over the elliptic curve E_q (q-analogue of the theorem of Birkhoff–Malgrange–Sibuya). We do the same here for arbitrary slopes (not necessarily integral) but restricting to the case of two slopes. Then, the sheaf of groups is a vector bundle $\mathcal{F}_E$ (it will be defined precisely just before Theorem 3.5).

3.1. Some notations

The discrete q-spiral of $a \in \mathbb{C}^*$ is $[a;q] := aq^{\mathbb{Z}}$, every point of $[a;q]$ has the same image by the canonical projection $\pi : \mathbb{C}^* \to E_q$. Similarly, for all $r \in \mathbb{N}^*$, the q^r-spiral will be $[a;q^r] := aq^{r\mathbb{Z}}$.

We define the Theta function $\theta_{q^r,c}$, $c \in \mathbb{C}^*$. It is an analytic solution on $\mathbb{C}^*$ of the equation $\sigma_q^r(\theta) = \frac{z}{c}\theta$ and is equal to:

$$\theta_{q^r,c}(z) = \sum_{n \in \mathbb{Z}} q^{r\frac{-n(n+1)}{2}} \left(\frac{z}{c}\right)^n .$$

The Jacobi's triple product formula gives:

$$\theta_{q^r,c}(z) = \prod_{n \geqslant 1} (1 - q^{-rn}) \prod_{n \geqslant 1} \left(1 + q^{-rn}\frac{z}{c}\right) \prod_{n \geqslant 0} \left(1 + q^{-rn}\left(\frac{z}{c}\right)^{-1}\right).$$

The function $\theta_{q^r,c}$ has simple zeroes all located on the q-spiral $[-c;q^r]$.

Moreover, for all $d \in \mathbb{N}^*$, $\theta^d_{q^r,c}$ is solution of $\sigma_q^r(\theta) = \left(\frac{z}{c}\right)^d \theta$ and its zeros are located on $[-c;q^r]$ and are of multiplicity d.

3.2. Stokes operators in the case of two slopes

Let us take $E = E(r,-d,b^r)$ an irreducible q-difference module of slope $-d/r < 0$, we have $E = (K^r, \Phi_B)$ where

$$B = \begin{pmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & & & 1 \\ b' z^{-d} & 0 & \dots & 0 \end{pmatrix} \quad \text{where} \quad b' = q^{\frac{-d(r-1)}{2}} b^r \in \mathbb{C}^*.$$

In this section, we want to prove that $\mathcal{F}(E,\underline{1})$ and $H^1(E_q, \mathcal{F}_E)$ (the sheaf $\mathcal{F}_E$ is defined just before Theorem 3.5) are isomorphic. Therefore, we are going to construct privileged cocycles with poles on E_{q^r} but not on E_q as for the case of integral slopes in [15]. Here $M_0 = (K^{r+1}, \Phi_{A_0})$ and a class of $\mathcal{F}(E,\underline{1})$ is represented by a module $M_U = (K^{r+1}, \Phi_{A_U})$ where:

$$A_0 = \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A_U = \begin{pmatrix} B & U \\ 0 & 1 \end{pmatrix} \; U \in (\mathbb{C}[z]_{d-1})^r.$$

Indeed, we may and will suppose $U \in (\mathbb{C}[z]_{d-1})^r$ thanks to Lemma 2.7 in all this part.

We are looking for an isomorphism, more precisely a matrix, between the q-difference modules (K^{r+1}, Φ_{A_0}) and (K^{r+1}, Φ_{A_U}). We want that isomorphism to have a matrix of the form $\mathfrak{S}_{r,1}$:

$$\begin{pmatrix} I_r & F \\ 0 & 1 \end{pmatrix}$$

with $\sigma_q(F) - BF = U$.

In the general case, there exists a unique formal morphism $\hat{F}$ satisfying this equation but, a priori, the modules associated to the matrices A_0 and A_U are not in the same analytic class, therefore $\hat{F}$ has not its coefficients in K. We shall be looking for F meromorphic on $\mathbb{C}^*$.

We denote by $\bar{\bar{c}}$ the class of $c \in \mathbb{C}^*$ in E_{q^r} and write $\mathcal{O}(\mathbb{C}^*)$ the set of holomorphic functions over $\mathbb{C}^*$.

Proposition 3.1. — *Let* $\Sigma(A_0) = \left\{ c \in \mathbb{C}^* \,\middle|\, \exists\, n \in \mathbb{Z},\ c^d = b^r q^{\frac{-d(r-1)}{2}} q^{rn} \right\}$ *modulo* q^r, *the set* $\Sigma(A_0)$ *is finite.*

For all $U \in (\mathbb{C}[z]_{d-1})^r$, *for all* $\bar{\bar{c}} \in E_{q^r} \setminus \Sigma(A_0)$ *(the notation is defined hereafter), there exists a unique vector of meromorphic functions* $\tilde{F}_{\bar{\bar{c}}}$ *over* $\mathbb{C}^*$, $\tilde{F}_{\bar{\bar{c}}} = {}^t(f_1, \ldots, f_r)$, *such that the poles of* f_i *are the* q^r*-spiral* $[-cq^{-i+1}; q^r]$, *of multiplicity* $\leqslant d$ *and such that* $\sigma_q(\tilde{F}_{\bar{\bar{c}}}) - B\tilde{F}_{\bar{\bar{c}}} = U$.

Proof. — We look for F with meromorphic coefficients on $\mathbb{C}^*$ such that $\sigma_q(F) - BF = U$. By setting conditions on poles using Theta functions, we would like to obtain uniqueness of a such F.

Let $F = T^{-1}G$ such that $G \in (\mathcal{O}(\mathbb{C}^*))^r$ and $T \in \mathrm{GL}_r(\mathcal{O}(\mathbb{C}^*))$ is the diagonal matrix:

$$T = \begin{pmatrix} \theta_1 & & 0 \\ & \ddots & \\ 0 & & \theta_r \end{pmatrix}, F = \begin{pmatrix} g_1/\theta_1 \\ \vdots \\ g_r/\theta_r \end{pmatrix}.$$

We have $\sigma_q(F) - BF = U \Leftrightarrow \sigma_q(G) - T[B]G = \sigma_q(T)U$.

Let us denote $A = T[B]$ and let $G = \sum_{n \in \mathbb{Z}} G_n z^n$, and $V = \sigma_q(T)U = \sum_{n \in \mathbb{Z}} V_n z^n$ then:

$$\begin{aligned} \sigma_q(G) - AG = V &\Leftrightarrow \forall\, n \in \mathbb{Z},\ q^n G_n - AG_n = V_n \\ &\Leftrightarrow \forall\, n \in \mathbb{Z},\ G_n = (q^n I_r - A)^{-1} V_n\,. \end{aligned}$$

and,

$$A = T[B] = \sigma_q(T)BT^{-1} = \begin{pmatrix} 0 & \frac{\sigma_q(\theta_1)}{\theta_2} & 0 & \\ & & \ddots & 0 \\ 0 & & & \frac{\sigma_q(\theta_{r-1})}{\theta_r} \\ b'z^{-d}\frac{\sigma_q(\theta_r)}{\theta_1} & 0 & \dots & 0 \end{pmatrix}.$$

We choose A of the following form:

$$\begin{pmatrix} 0 & 1 & 0 & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ c' & 0 & \dots & 0 \end{pmatrix}, \quad c' \in \mathbb{C}^*.$$

Its eigenvalues are the rth roots of c'. Therefore, G is unique if, and only if, for all $n \in \mathbb{Z}$, $c' \neq q^{rn}$. Now, we have to determine θ_i satisfying:

$$\begin{cases} \theta_2 = \sigma_q(\theta_1) \\ \theta_3 = \sigma_q^2(\theta_1) \\ \vdots \\ \sigma_q^r(\theta_1) = c'b'^{-1}z^d\theta_1 \,. \end{cases}$$

Let $c \in \mathbb{C}^*$, we put $c' = b'c^{-d}$ and $\theta_1 = \left(\theta_{q^r,c}\right)^d$. The zeroes of θ_1 have multiplicity d and are the q^r-spiral $[-c; q^r]$. The Jacobi's triple product formula gives

$$\begin{aligned} \theta_{i+1} &= \sigma_q^i\left(\theta_{q^r,c}^d(z)\right) \\ &= \prod_{n\geqslant 1}(1-q^{-rn})^d \prod_{n\geqslant 1}\left(1+q^{-rn}q^i\frac{z}{c}\right)^d \prod_{n\geqslant 0}\left(1+q^{-rn}q^{-i}\left(\frac{z}{c}\right)^{-1}\right)^d, \end{aligned}$$

so that the zeroes of θ_i have multiplicity d and are the q^r-spiral $[-cq^{-i+1}; q^r]$.

Then, we put $\tilde{F}_c = T^{-1}G$. For the chosen matrix T, F exists and is unique on condition that for all $n \in \mathbb{Z}$, $c^d \neq b'q^{rn}$.

In order to finish, we need to verify that our computation of $\tilde{F}_c$ depends only on $\bar{\bar{c}}$, that is $\tilde{F}_c = \tilde{F}_{cq^r}$. Writing $G = {}^t(g_1, \dots, g_r)$, $U = {}^t(u_1, \dots, u_r)$, we have:

$$\sigma_q(G) - AG = \sigma_q(T)U \Leftrightarrow \begin{cases} \sigma_q(g_1) - g_2 = \sigma_q(\theta_1)u_1 \\ \sigma_q(g_2) - g_3 = \sigma_q(\theta_2)u_2 \\ \vdots \\ \sigma_q(g_r) - c'g_1 = \sigma_q(\theta_r)u_r \end{cases} \tag{3.1}$$

$$\Leftrightarrow \begin{cases} g_2 = \sigma_q(g_1) - \sigma_q(\theta_1)u_1 \\ g_3 = \sigma_q^2(g_1) - \sigma_q^2(\theta_1)(\sigma_q(u_1) + u_2) \\ \vdots \\ \sigma_q^r(g_1) = c'g_1 + \sigma_q^r(\theta_1)\varphi(U), \end{cases} \tag{3.2}$$

where $\varphi(U) = \sigma_q^{r-1}(u_1) + \cdots + \sigma_q(u_{r-1}) + u_r$ is the map defined in Theorem 2.8.

We replace c by cq^r,let $\theta_1' = (\theta_{q^r,cq^r})^d$ and $\theta_i' = \sigma_q^{i-1}(\theta_1')$. We remark thanks to the Jacobi's triple product formula that:

$$\theta_1' = (\theta_{q^r,cq^r})^d = \left(q^r \left(\frac{z}{c}\right)^{-1} \theta_{q_r,c}\right)^d = q^{rd} \left(\frac{z}{c}\right)^{-d} \theta_1 .$$

As c is replaced by cq^r, c' is replaced by $c'q^{-rd}$. If $\tilde{F}_{cq^r} = T'^{-1}G'$, the equation verified by g_1' is:

$$\begin{aligned} \sigma_q^r(g_1') &= c'q^{-rd}g_1' + \sigma_q^r(\theta_1')\varphi(U) \\ &\Leftrightarrow \sigma_q^r(g_1') = c'q^{-rd}g_1' + c^d z^{-d}\sigma_q^r(\theta_1)\varphi(U) \\ &\Leftrightarrow \sigma_q^r\left(q^{-rd}\left(\frac{z}{c}\right)^d g_1'\right) = c'q^{-rd}\left(\frac{z}{c}\right)^d g_1' + \sigma_q^r(\theta_1)\varphi(U). \end{aligned}$$

And yet, g_1 is the unique solution on $\mathbb{C}^*$ of $\sigma_q^r(g_1) = c'g_1 + \sigma_q^r(\theta_1)\varphi(U)$, so $g_1 = q^{-rd}\left(\frac{z}{c}\right)^d g_1'$. We easily verify that $\frac{g_i}{\theta_i} = \frac{g_i'}{\theta_i'}$, so $\tilde{F}_c = \tilde{F}_{cq^r}$. □

Remark 3.2. — We notice the apparition of the elliptic curve E_{q^r} instead of E_q. Nevertheless, $\tilde{F}_{\overline{\overline{c}}}$ does not have a unique pole $\overline{\overline{-c}}$, every coordinate of $\tilde{F}_{\overline{\overline{c}}}$ has poles $\overline{\overline{-cq^{-i+1}}}$.

Remark 3.3. — The hypothesis $U \in (\mathbb{C}[z]_{d-1})^r$, can be weakened in this proposition. Indeed, it is sufficient that $U \in K^r$ to obtain existence and uniqueness of F.

Remark 3.4. — In the proof of Proposition 3.1, the last equality $\tilde{F}_c = \tilde{F}_{cq^r}$ can also be justified without any calculation because the two members satisfy the same constraint of uniqueness.

Let $\tilde{F}_{\bar{\bar{c}},\bar{\bar{c}}'}(A_U) = \tilde{F}_{\bar{\bar{c}}'} - \tilde{F}_{\bar{\bar{c}}} = {}^t(f_{\bar{\bar{c}},\bar{\bar{c}}'}, \sigma_q(f_{\bar{\bar{c}},\bar{\bar{c}}'}), \ldots, \sigma_q^{r-1}(f_{\bar{\bar{c}},\bar{\bar{c}}'}))$ where $f_{\bar{\bar{c}},\bar{\bar{c}}'} = f_{\bar{\bar{c}}'} - f_{\bar{\bar{c}}}$ satisfy this equation:

$$\sigma_q^r(f_{\bar{\bar{c}}}) - b^r q^{-d(r-1)/2} z^{-d} f_{\bar{\bar{c}}} = \varphi(U),$$

φ is the map defined in Theorem 2.8. The poles of $\tilde{F}_{\bar{\bar{c}},\bar{\bar{c}}'}(A_U)$ are located on E_q and correspond to $\overline{-c}, \overline{-c'}$. The function $\tilde{F}_{\bar{\bar{c}},\bar{\bar{c}}'}(A_U)$ satisfies $\sigma_q(\tilde{F}_{\bar{\bar{c}},\bar{\bar{c}}'}) - B\tilde{F}_{\bar{\bar{c}},\bar{\bar{c}}'} = 0$.

Let $M = (K^n, \Phi_A)$ be a q-difference module in normal form, the holomorphic vector bundle on E_q associated with M is defined by:

$$\mathcal{F}_M = \frac{\mathbb{C}^* \times \mathbb{C}^n}{(z, X) \sim (qz, A(z)X)} \longrightarrow E_q.$$

As a sheaf, it associates to every open set $V \subset E_q$ the space of holomorphic solutions over $\pi^{-1}(V) \subset \mathbb{C}^*$ of the q-difference system $Y(qz) = A(z)Y(z)$, or rather their germs near 0 (cf. [17]).

The functions $\tilde{F}_{\bar{\bar{c}},\bar{\bar{c}}'}(A_U)$ are sections of the vector bundle $\mathcal{F}_E$ on the open sets $V_{\bar{c},\bar{c}'} = E_q \setminus \{\overline{-c}, \overline{-c'}\}$.

Theorem 3.5. — *There is an isomorphism of $\mathbb{C}$-vector spaces:*

$$\mathcal{F}(E, \underline{1}) \cong H^1(E_q, \mathcal{F}_E)$$

induced by:

$$M_U = (K^{r+1}, \Phi_{A_U}) \mapsto (\tilde{F}_{\bar{\bar{c}},\bar{\bar{c}}'}(A_U)).$$

Proof. — According to the previous proposition, for all $\bar{\bar{c}}, \bar{\bar{c}}' \in E_{q^r} \setminus \Sigma(A_0)$, we can associate with each matrix A_U, $U \in (\mathbb{C}[z]_{d-1})^r$, a unique vector of holomorphic functions on $V_{\bar{c},\bar{c}'}$, denoted by $\tilde{F}_{\bar{\bar{c}},\bar{\bar{c}}'}(A_U)$. First, let us prove that the $\tilde{F}_{\bar{\bar{c}},\bar{\bar{c}}'}(A_U)$ depends only on the class of M_U in $\mathcal{F}(E, \underline{1})$. Let $U' \in \mathbb{C}[z]_{d-1}$ such that $[M_U] = [M_{U'}]$, that is, thanks to the Theorem 2.8, $\varphi(U) = \varphi(U')$ and there exists an isomorphism $H \in \mathfrak{G}_{r,1}(K)$ between M_U and $M_{U'}$. By Proposition 3.1, the uniqueness of $\tilde{F}_{\bar{\bar{c}}}(A_U)$ and $\tilde{F}_{\bar{\bar{c}}'}(A_{U'})$ gives us the following commutative diagram:

$$\begin{tikzcd}
M_0 \arrow[r, "\tilde{S}_{\bar{\bar{c}}}\hat{F}(A_U)"] \arrow[d, "\mathrm{id}"'] & M_U \arrow[d, "H"] \\
M_0 \arrow[r, "\tilde{S}_{\bar{\bar{c}}}\hat{F}(A_{U'})"'] & M_{U'}
\end{tikzcd}$$

where

$$\tilde{S}_{\bar{\bar{c}}}\hat{F}(A_U) = \begin{pmatrix} I_r & \tilde{F}_{\bar{\bar{c}}}(A_U) \\ 0 & 1 \end{pmatrix} \text{ and } \tilde{S}_{\bar{\bar{c}}}\hat{F}(A_{U'}) = \begin{pmatrix} I_r & \tilde{F}_{\bar{\bar{c}}}(A_{U'}) \\ 0 & 1 \end{pmatrix}.$$

As a consequence, we easily see that $\tilde{F}_{\bar{\bar{c}},\bar{\bar{c}}'}(A_U) = \tilde{F}_{\bar{\bar{c}},\bar{\bar{c}}'}(A_{U'})$.

The map $[M_U] \mapsto \tilde{F}_{\bar{\bar{c}},\bar{\bar{c}}'}(A_U)$ is well defined and induces a map:

$$\alpha : \mathcal{F}(E, \underline{1}) \to Z^1(\mathcal{U}', \mathcal{F}_E)$$

where $\mathcal{U}'$ is a covering of E_q that we are going to describe. We have $q^{r\mathbb{Z}} \subset q^{\mathbb{Z}}$ hence an onto map $p : E_{q^r} \to E_q$ such that the following diagram is commutative:

$$\begin{tikzcd} \mathbb{C}^* \arrow[r, "\pi_r"] \arrow[dr, "\pi_1"'] & E_{q^r} = \mathbb{C}^*/q^{r\mathbb{Z}} \arrow[d, "p"] \\ & E_q = \mathbb{C}^*/q^{\mathbb{Z}}, \end{tikzcd}$$

π_1 and π_r being the canonical projections. We set:

$$W_{\bar{\bar{c}}} = E_q \setminus p\left(\{\bar{\bar{c}}\}\right).$$

It is an open set of E_q and in fact $W_{\bar{\bar{c}}} = V_{\bar{c}}$. Thus $\tilde{F}_{\bar{\bar{c}}}$ is holomorphic on $W_{\bar{\bar{c}}}$. Putting:

$$\mathcal{U}' = \bigcup_{\bar{\bar{c}} \in E_{q^r} \setminus \Sigma(A_0)} W_{\bar{\bar{c}}}\,,$$

we obtain an open covering of E_q. The map $\alpha : \mathcal{F}(E, \underline{1}) \to Z^1(\mathcal{U}', \mathcal{F}_E)$ is well defined and $\mathbb{C}$-linear, It is easy to see that $\tilde{F}_{\bar{\bar{c}},\bar{\bar{c}}'}(A_{U+U'}) = \tilde{F}_{\bar{\bar{c}},\bar{\bar{c}}'}(A_U) + \tilde{F}_{\bar{\bar{c}},\bar{\bar{c}}'}(A_{U'})$ and $\tilde{F}_{\bar{\bar{c}},\bar{\bar{c}}'}(\lambda U) = \lambda \tilde{F}_{\bar{\bar{c}},\bar{\bar{c}}'}(U)$ for all $\lambda \in \mathbb{C}$.

The map α is injective. Indeed, let us suppose that $\tilde{F}_{\bar{\bar{c}},\bar{\bar{c}}'}(U) = 0$ and show that $[M_U] = [M_0]$. We have $\tilde{F}_{\bar{\bar{c}},\bar{\bar{c}}'}(U) = 0 \Leftrightarrow \tilde{F}_{\bar{\bar{c}}} = \tilde{F}_{\bar{\bar{c}}'}$. So, if $\bar{\bar{c}} \neq \bar{\bar{c}}'$, $\tilde{F}_{\bar{\bar{c}}}$ and $\tilde{F}_{\bar{\bar{c}}'}$ have no poles on $\mathbb{C}^*$. Thus, we have an holomorphic function F on $\mathbb{C}^*$ such that $\sigma_q(F) - BF = U$. The following lemma shows that necessarily if F is meromorphic at 0 so that the morphism $\left(\begin{smallmatrix} I_r & F \\ 0 & 1 \end{smallmatrix}\right)$ is an isomorphism over K from M_0 to M_U, hence $[M_U] = [M_0]$.

Lemma 3.6. — *A holomorphic solution on $\mathbb{C}^*$ of the equation $\sigma_q(F) - BF = U$ is automatically meromorphic on $\mathbb{C}$.*

Proof of the lemma. — We write $F = {}^t(f_1, \ldots, f_r)$ then

$$\sigma_q(F) - BF = U$$

$$\Leftrightarrow \begin{cases} \qquad f_2 = \sigma_q(f_1) - u_1 \\ \qquad\quad \vdots \\ \qquad f_r = \sigma_q^{r-1}(f_1) - (\sigma_q^{r-2}(u_1) + \cdots + \sigma_q(u_{r-2}) + u_{r-1}) \\ \sigma_q^r(f_1) = b' z^{-d} f_1 + \sigma_q^{r-1}(u_1) + \cdots + \sigma_q(u_{r-1}) + u_r\,. \end{cases}$$

If $f_1 = \sum_{n\in\mathbb{Z}} a_n z^n$ and $\varphi(U) = \sigma_q^{r-1}(u_1)+\cdots+\sigma_q(u_{r-1})+u_r = \sum_{k=0}^{d-1} v_n z^n$, the last equation is equivalent to $q^{rn}a_n = b'a_{n+d} + v_n$ for all $n \in \mathbb{Z}$.

When $n < 0$ the equation becomes $q^{rn}a_n = b'a_{n+d}$. We denote by s_n the rest of the euclidean division of n by d, $n = -k_n d + s_n$ ($k_n \geqslant 0$). Then

$$a_n = b'q^{-rn}a_{n+d} = b'^{k_n} q^{-r(k_n n + \frac{k_n(k_n-1)d}{2})} a_{s_n}.$$

When n tends to $-\infty$, a_n is of order q^{rn^2} so there is divergence. Necessarily, $a_n = 0$ when $n << 0$. □

In the same way, we see that the map $\bar{\alpha} : \mathcal{F}(E, \underline{1}) \to H^1(\mathcal{U}', \mathcal{F}_E)$ is $\mathbb{C}$-linear and injective. Indeed, the cohomology class of $\tilde{F}_{\bar{\bar{c}},\bar{\bar{c}}'}(A_U)$ is zero if $\tilde{F}_{\bar{\bar{c}},\bar{\bar{c}}'}(A_U) = X_{\bar{\bar{c}}'} - X_{\bar{\bar{c}}}$ with $X_{\bar{\bar{c}}}$ and $X_{\bar{\bar{c}}'}$ holomorphic sections on $W_{\bar{\bar{c}}}$ and $W_{\bar{\bar{c}}'}$; the argument is the same as the one to prove that α is injective.

Since in Cech cohomology, $H^1(\mathcal{U}', \mathcal{F}_E)$ embeds into the direct limit $H^1(E_q, \mathcal{F}_E)$. $\bar{\alpha}$ induces a map from $\mathcal{F}(E, \underline{1})$ to $H^1(E_q, \mathcal{F}_E)$ that is $\mathbb{C}$-linear and injective. Because of the dimension, it is an isomorphism. Indeed, we know by the general theory in [12] that $\dim_{\mathbb{C}} \mathcal{F}(E, \underline{1}) = d$ and $\dim_{\mathbb{C}} H^1(E_q, \mathcal{F}_E) = d$; this uses the results on holomorphic vector bundles from [5, p. 64]. □

These results given in Proposition 3.1 and Theorem 3.5 can be generalized for an indecomposable module of the form $M = E \otimes U_m$. With the convention $\hat{\otimes}$, M has an associated matrix C of rank mr:

$$C = \begin{pmatrix} B & B & & 0 \\ & \ddots & \ddots & \\ & & B & B \\ 0 & & & B \end{pmatrix}.$$

Putting

$$A_U = \begin{pmatrix} C & U \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} U_1 \\ \vdots \\ U_m \end{pmatrix}, \quad U_i \in (\mathbb{C}[z]_{d-1})^r$$

and writing A_0 the matrix associated with the graded module (i.e. $U = 0$), we have the following proposition.

Proposition 3.7. — *For all $U \in (\mathbb{C}[z]_{d-1})^n$, for all $\bar{\bar{c}} \in E_{q^r} \setminus \Sigma(A_0)$, there exists a unique vector of meromorphic functions $\tilde{F}_{\bar{\bar{c}}} = (F_i)_{i=1\ldots m}$ on $\mathbb{C}^*$, $F_i = {}^t(f_{1,i}, \ldots, f_{r,i})$, such that the poles of $f_{k,i}$ are the q^r-spiral $[-cq^{-i+1}; q^r]$, of multiplicity $\leqslant d$ and $\tilde{F}_{\bar{\bar{c}}}$ satisfies $\sigma_q(\tilde{F}_{\bar{\bar{c}}}) - B\tilde{F}_{\bar{\bar{c}}} = U$. The set $\Sigma(A_0)$ is equal to $\left\{c \in \mathbb{C}^* \,\middle|\, \exists\, n \in \mathbb{Z},\ c^d = b^r q^{\frac{-d(r-1)}{2}} q^{rn}\right\}$ modulo q^r and is finite.*

Proof. — The proof is the same as the case $\mathcal{F}(E,\underline{1})$ by putting $F = T'^{-1}G$ where

$$T' = \begin{pmatrix} T & & 0 \\ & \ddots & \\ 0 & & T \end{pmatrix} \text{ and } T'[C] = \begin{pmatrix} T[B] & T[B] & & 0 \\ & \ddots & \ddots & \\ & & T[B] & T[B] \\ 0 & & & T[B] \end{pmatrix},$$

This matrix is fuchsian and

$$\sigma_q(F) - CF = U \Leftrightarrow \sigma_q(G) - T'[C]G = \sigma_q(T')U\,. \qquad \square$$

In the next paragraph, we generalize the previous calculations to a q-difference module with two slopes non necessarily integral. We are guided in the same way as the study of $\mathcal{F}(P_1,P_2)$ by the isomorphism $\mathcal{F}(P_1,P_2) \cong \mathcal{F}(P_1 \otimes P_2^{\vee},\underline{1})$ given by the formula (2.4).

We suppose that E_1 and E_2 are two irreducible modules of slopes $\mu_1 < \mu_2$ and their associated matrices are B_1 and B_2, one of them can have an integral slope. We look for a meromorphic isomorphism on $\mathbb{C}^*$ between the q-difference modules $(K^{r_1+r_2},\Phi_{A_0})$ and $(K^{r_1+r_2},\Phi_{A_U})$ where:

$$A_0 = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \text{ and } A_U = \begin{pmatrix} B_1 & U \\ 0 & B_2 \end{pmatrix},\ U \in M_{r_1,r_2}(K)$$

and the matrix of the isomorphism has the following form: $\left(\begin{smallmatrix} I_{r_1} & F \\ 0 & I_{r_2} \end{smallmatrix}\right) \in \mathfrak{G}_{r_1,r_2}$ satisfying $\sigma_q(F)B_2 - B_1F = U$. But,

$$\sigma_q(F)B_2 - B_1F = U \Leftrightarrow \sigma_q(\hat{F}) - B_1 \hat{\otimes} B_2^{\vee}\hat{F} = \widehat{UB_2^{-1}}.$$

So, it is equivalent to look for F such that $\sigma_q(F) - B_1 \hat{\otimes} B_2^{\vee} F = U'$, $U' = \widehat{UB_2^{-1}}$. Let $M' = (K^{r_1r_2+1},\Phi_{A'_{U'}})$ where:

$$A'_{U'} = \begin{pmatrix} B_1 \hat{\otimes} B_2^{\vee} & U' \\ 0 & 1 \end{pmatrix}, \quad U' = \widehat{UB_2^{-1}}.$$

$B_1 \hat{\otimes} B_2^{\vee}$ is associated with the module $E_1 \otimes E_2^{\vee}$, according to Proposition 1.14, this module is isomorphic to a direct sum of irreducible q-difference modules of rank r and slope $\frac{-d}{r}$. Then, there exists an isomorphism of q-difference modules P (given by the proposition) such that $E_1 \otimes E_2^{\vee}$ is isomorphic to a module with a matrix which is diagonal by blocks written B. The blocks are matrices $B_{i,j,l}$, $i,j = 0,\dots,k-1$ and $l = 1,\dots,\frac{p}{k}$ associated with irreducible modules $E(r,-d,b_{i,j})$, $b_{i,j} = q_k^i\xi_r^j(b_1b_2^{-1})^r$, so, for all i,j, there are p/k identical blocks $B_{i,j,l}$. We have $\sigma_q(P)B_1\hat{\otimes}B_2^{\vee} = BP$.

Let $M'' = (K^{r_1 r_2+1}, \Phi_{A''_{U''}})$ where $\sigma_q(P')A'_{U'} = A''_{U''}P'$,

$$P' = \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A''_{U''} = \begin{pmatrix} B & U'' \\ 0 & 1 \end{pmatrix}.$$

We denote by $U'' = (U''_{i,j,l})_{i,j,l}$ and A''_0 is the matrix of the graded module M'' corresponding to $U'' = 0$. We apply Proposition 3.1, to each block $B_{i,j,l}$ associated with the vector $U''_{i,j,l} \in K^r$ (cf. Remark 3.3), for all $\bar{\bar{c}} \in E_{q^r} \setminus \Sigma(A''_0)$, there exists a unique vector of meromorphic functions on $\mathbb{C}^*$, $\tilde{F}_{\bar{\bar{c}},(i,j,l)}$ satisfying the polar conditions of Proposition 3.1 and such that $\sigma_q(\tilde{F}_{\bar{\bar{c}},(i,j,l)}) - B_{i,j,l}\tilde{F}_{\bar{\bar{c}},(i,j,l)} = U''_{i,j,l}$. The set $\Sigma(A''_0)$ is equal to $\left\{c \in \mathbb{C}^* \,\middle|\, \exists\, n \in \mathbb{Z}, \exists\, i,j \in \{0,\dots,k-1\},\ c^d = b_{i,j} q^{\frac{-d(r-1)}{2}} q^{rn}\right\}$ modulo q^r and it is finite.

We denote by $\tilde{F}_{\bar{\bar{c}}}(A''_{U''}) = (\tilde{F}_{\bar{\bar{c}},(i,j,l)})$ the vector we obtain by concatenating the vectors corresponding to the blocks (i,j,l) and satisfying the polar conditions of Proposition 3.1. So, let $\tilde{F}_{\bar{\bar{c}}}(A'_{U'}) = P^{-1}\tilde{F}_{\bar{\bar{c}}}(A''_{U''})$. Thus, we define $\tilde{F}_{\bar{\bar{c}}}(A_U)$ by:

$$\widehat{\tilde{F}_{\bar{\bar{c}}}(A_U)} = \tilde{F}_{\bar{\bar{c}}}(A'_{U'}) = P^{-1}\tilde{F}_{\bar{\bar{c}}}(A''_{U''}).$$

This latter verifies $\sigma_q(\tilde{F}_{\bar{\bar{c}}}(A_U))B_2 = B_1\tilde{F}_{\bar{\bar{c}}}(A_U) + U$ and is meromorphic on $\mathbb{C}^*$, it satisfies the polar conditions denoted by $(*)$ imposed by those of $\tilde{F}_{\bar{\bar{c}},(i,j,l)}$. Putting $\Sigma(A_0) := \Sigma(A''_0)$, we have the following proposition:

Proposition 3.8. — *For all $U \in M_{r_1,r_2}(K)$, for all $\bar{\bar{c}} \in E_{q^r} \setminus \Sigma(A_0)$ there exists a unique matrix $\tilde{F}_{\bar{\bar{c}}}(A_U)$ of rank $r_1 \times r_2$ with meromorphic coefficients on $\mathbb{C}^*$ and satisfying the polar conditions $(*)$, such that $\sigma_q(F)B_2 = B_1F + U$.*

Then, we obtain a meromorphic isomorphism on $\mathbb{C}^*$ from the module $(K^{r_1+r_2}, \Phi_{A_0})$ to the module $(K^{r_1+r_2}, \Phi_{A_U})$, that we denote $\tilde{S}_{\bar{\bar{c}}}\hat{F}(A_U) = \left(\begin{smallmatrix} I_{r_1} & \tilde{F}_{\bar{\bar{c}}}(A_U) \\ 0 & I_{r_2} \end{smallmatrix}\right)$.

We denote also $\tilde{F}_{\bar{\bar{c}},\bar{\bar{c}}'}(A_U) = \tilde{F}_{\bar{\bar{c}}'}(A_U) - \tilde{F}_{\bar{\bar{c}}}(A_U)$, so the Stokes operators associated with the module $(K^{r_1+r_2}, \Phi_{A_U})$ are the meromorphic automorphisms of $(K^{r_1+r_2}, \Phi_{A_0})$:

$$\tilde{S}_{\bar{\bar{c}},\bar{\bar{c}}'}\hat{F}(A_U) = \begin{pmatrix} I_{r_1} & \tilde{F}_{\bar{\bar{c}},\bar{\bar{c}}'}(A_U) \\ 0 & I_{r_2} \end{pmatrix}, \quad \bar{\bar{c}}, \bar{\bar{c}}' \in E_{q^r} \setminus \Sigma(A_0).$$

This proposition can be generalized for all modules with two slopes where at least one is non integral but we don't give the details because the calculations become unreadable but are explicit.

We are able to compute the Stokes operators associated with a q-difference module with two slopes. These Stokes operators don't have galoisian properties yet, they only express an ambiguity of summation. Nevertheless, we will prove in the next section that they are galoisian.

4. Local Galois theory

In this section, we shall give a more concrete description of the results of van der Put and Reversat on the formal Galois group [7], then apply them as well as our results on Stokes operators to the description of the local analytic Galois group in the case of two slopes.

4.1. Formal Galois group

Let $q' \in \mathbb{C}^*$, $|q'| > 1$, $(K', \sigma_{q'})$ is a q'-difference field such that $(K', \sigma_{q'}) = (K, \sigma_q)$ or (K_r, σ_{q_r}) or (K, σ_{q^r}). We will denote by:

- $\mathcal{E}_p(K', q')$: the category of pure q'-difference modules over K'.
- $\mathcal{E}_f(K', q')$: the category of pure q'-difference modules of slope equal to 0 over K' (i-e fuchsian modules)
- $\mathcal{E}_{p,r}(K', q')$, $r \in \mathbb{N}^*$: the category of pure q'-difference modules over K' of slopes $\frac{k}{r}$ for some $k \in \mathbb{Z}$.

If $r = 1$, it is the category of pure q'-difference modules with integral slopes. When it is not specified, $\mathcal{E}_p$, respectively $\mathcal{E}_{p,r}$ denote the category $\mathcal{E}_p(K, q)$, respectively $\mathcal{E}_{p,r}(K, q)$. These categories are $\mathbb{C}$-linear rigid abelian tensor categories (see [4]).

From now on, the pure isoclinic q-difference modules are supposed to be in normal form, in particular, their associated matrices have their coefficients in $\mathbb{C}[z, z^{-1}]$ (see paragraph 1.3).

Let $(K', \sigma_{q'})$ be (K, σ_q). Let $z_0 \in \mathbb{C}^*$, and ω_{z_0} be the fiber functor from the category $\mathcal{E}_p$ to the category of the $\mathbb{C}$-vector spaces of finite dimension, defined in [9] by:

$$\begin{array}{cccc} \omega_{z_0} : & \mathcal{E}_p & \to & \mathrm{Vect}_{\mathbb{C}}^f \\ & (K^n, \Phi_A) & \rightsquigarrow & \mathbb{C}^n \\ & (K^n, \Phi_A) \xrightarrow{F} (K^p, \Phi_B) & \rightsquigarrow & F(z_0) \end{array}$$

This fiber functor turns $\mathcal{E}_p$ into a neutral tannakian category and allows one to define the tannakian Galois groups associated with the previous categories over (K, σ_q).

The Galois group associated with the category $\mathcal{E}_p$ is the group of $\otimes$-compatible automorphisms of the fiber functor ω_{z_0} and we denote it by $G_p := \mathrm{Aut}^{\otimes}(\omega_{z_0})$.

The categories, $\mathcal{E}_f$, respectively $\mathcal{E}_{p,r}$, $r > 0$, are full rigid abelian tensor subcategories of $\mathcal{E}_p$, and the fiber functor ω_{z_0} can be restricted to those tannakian categories. Their Galois group are defined by $G_f = \mathrm{Aut}^{\otimes}(\omega_{z_0}|_{\mathcal{E}_f})$ respectively, $G_{p,r} = \mathrm{Aut}^{\otimes}(\omega_{z_0}|_{\mathcal{E}_{p,r}})$. If we have to precise, we will denote $G_p(K,q)$, $G_f(K,q)$ and $G_{p,r}(K,q)$. As $\mathcal{E}_f$ and $\mathcal{E}_{p,r}$ are full subcategories of $\mathcal{E}_p$ and $\mathcal{E}_f$ is a full subcategory of $\mathcal{E}_{p,r}$, according to [4, Proposition 2.21] the following group morphisms are onto (for details, see [15, 12]):

$$G_p \to G_f\,, \quad G_p \to G_{p,r} \quad \text{and} \quad G_{p,r} \to G_f\,.$$

From now on, a pure q-difference module will always be a pair (K^n, Φ_A), where A is a matrix in $\mathrm{GL}_n(\mathbb{C}[z, z^{-1}])$ in normal form. Instead of writing $\varphi(M)$, we make reference to the matrix in normal form writing $\varphi(A)$. Thus, an element φ of the Galois group is determined by the $\varphi(A)$, A being the matrix associated with the module (cf. [14, 1.1.2] for more details).

Our goal is to describe explicitly the matrices $\varphi(A)$ for all matrices A in normal form.

As far as integral slopes are concerned, the formal Galois group is described explicitly in [14] and [9]:

$$G_{p,1} = \mathbb{C}^* \times \overbrace{E_q^{\vee} \times \mathbb{C}}^{G_f}$$

where $E_q^{\vee} := \mathrm{Hom}_{\mathrm{gr}}(E_q, \mathbb{C}^*)$ denote the morphisms of "abstract" group from E_q to $\mathbb{C}^*$.

For a module $M = (K^n, \Phi_{z^{\mu}A})$ where $A \in \mathrm{GL}_n(\mathbb{C})$, an element $\varphi = (t, \gamma, \lambda) \in G_{p,1}$ acts in the following way:

$$\varphi(A) = t^{\mu}\gamma(A_s)A_u^{\lambda}$$

where $t \in \mathbb{C}^*$, $\mu \in \mathbb{Z}$ is the slope of M, $\lambda \in \mathbb{C}$, $A = A_sA_u$ is the multiplicative Dunford decomposition, A_s is the semisimple part and A_u the unipotent part. Let us define $\gamma(A_s)$: if $A_s = P\,\mathrm{diag}(a_1, \dots, a_n)\,P^{-1}, P \in \mathrm{GL}_n(\mathbb{C}), a_i \in \mathbb{C}^*$ then $\gamma(A_s) = P\ \mathrm{diag}(\gamma(a_1), \dots, \gamma(a_n))\,P^{-1}$ (it does not depend on diagonalisation).

As A_u is a unipotent matrix $A_u^{\lambda} = \sum_{k \geqslant 0} \binom{\lambda}{k} (A_u - I_n)^k$ is well-defined.

Now, we want to describe the formal Galois group of q-difference modules with non integral slopes. Let us fix the denominator of the slopes $r \in \mathbb{N}^*$, let $z_{0,r}$ be a rth root of z_0. We want to describe the Galois group $G_{p,r}$

associated with the category $\mathcal{E}_{p,r}$ of pure q-difference modules of slopes $\frac{k}{r}$, $k \in \mathbb{Z}$.

After Theorem 1.10, an element of the category $\mathcal{E}_{p,r}$ is a direct sum of indecomposable modules $E(s,t,c) \otimes U_m$ of slope $\frac{k}{r} = \frac{tm}{sm}$ and modules with integral slopes. For all $\varphi \in G_{p,r}$, we have $\varphi(E(s,t,c) \otimes U_m) = \varphi(E(s,t,c)) \otimes \varphi(U_m)$.

And if s divides r, then the inclusion $i_{r,s}$ of the categories $\mathcal{E}_{p,s} \subset \mathcal{E}_{p,r}$ induces an onto morphism $i^*_{r,s}$ of the associated Galois group: $G_{p,r} \to G_{p,s}$, so that for all $\varphi \in G_{p,r}$, $\varphi(E(s,t,c)) = i^*_{r,s}(\varphi)(E(s,t,c))$.

As a consequence, to study the image of the matrix associated with a module of $\mathcal{E}_{p,r}$ by an element of the Galois group, it is sufficient to study the image $\varphi(A)$, $\varphi \in G_{p,r}$, where A is associated with $E(r,d,a^r)$, an irreducible q-difference module of slope $\frac{d}{r}$ ($a \in \mathbb{C}^*$, $\gcd(d,r) = 1$):

$$A = \begin{pmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ a^r q^{\frac{d(r-1)}{2}} z^d & 0 & \dots & 0 \end{pmatrix} \in \mathrm{GL}_r(\mathbb{C})$$

The category $\mathcal{E}_{p,1}$ is a full subcategory of $\mathcal{E}_{p,r}$, we denote by i this inclusion. There is a morphism of groups $i^* : G_{p,r} \to G_{p,1}$ (restriction to a subcategory) such that $i^*(\varphi)(B) = \varphi(B)$ for all matrices B with integral slope (these are at the same time in $E_{p,r}$ and $E_{p,1}$). As we know how to describe the action of the Galois group on a module with integral slopes, we try to be in this case to describe $\varphi(A)$. Then, we obtain the following theorem.

We recall from Section 1, Subsection 1.1 that $\xi_r = e^{\frac{2i\pi}{r}}$, a primitive rth root of unity in $\mathbb{C}$, and that $q_r = e^{\frac{2i\pi\tau}{r}}$, a rth root of $q = e^{2i\pi\tau}$.

Theorem 4.1. — *Let $\varphi \in G_{p,r}$. Then there exists $t \in \mathbb{C}^*$, $\gamma \in E_q^\vee$ and $\lambda \in \mathbb{C}$ such that $\varphi = (t, \gamma, \lambda)$ and $i^*(\varphi) = (t^r, \gamma, \lambda)$. Moreover, if A is the matrix in normal form which is associated with an irreducible module $E(r,d,a^r)$, there exists a matrix $G_{a,d}(z_{0,r}) \in \mathrm{GL}_r(\mathbb{C})$ such that:*

$$\varphi(A) = G_{a,d}^{-1}(z_{0,r}) \;\; t^d \gamma(a)\, \gamma(T_r) \;\; G_{a,d}(z_{0,r}) \begin{pmatrix} 1 & & & 0 \\ & \gamma(q_r) & & \\ & & \ddots & \\ 0 & & & \gamma(q_r)^{(r-1)} \end{pmatrix}^d$$

and

$$G_{a,d}(z_{0,r}) = \mathrm{diag}(1, \alpha_0, \alpha_0\alpha_1, \dots, \alpha_0 \dots \alpha_{r-2}), \quad \alpha_j = a^{-1} q_r^{-jd} z_{0,r}^{-d},$$

$$T_r = \begin{pmatrix} 0 & 1 & \\ 0 & \ddots & 1 \\ 1 & \dots & 0 \end{pmatrix} \in \mathrm{GL}_r(\mathbb{C}).$$

Remark 4.2. — The action of λ is on the unipotent part, it is described in the course of the proof.

Remark 4.3. — It is easy to see that t depends only on φ, we have $\varphi(z) = t$, and $\varphi(A)$ depends only on a^r. The matrix $G_{a,d}(z_{0,r})$ depends only on A.

Remark 4.4.

(1) The matrix T_r is conjugate to the diagonal matrix $\mathrm{diag}(1, \xi_r, \dots, \xi_r^{r-1})$, we have $T_r = V^{-1} \mathrm{diag}(1, \xi_r, \dots, \xi_r^{r-1}) V$ where V is the Vandermonde matrix:

$$V = \begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ \xi_r^{-1} & \xi_r^{-2} & & \xi_r^{-(r-1)} & \vdots \\ \xi_r^{-2} & \xi_r^{-4} & & \xi_r^{-2(r-1)} & \vdots \\ \vdots & \vdots & & \vdots & \vdots \\ \xi_r^{-(r-1)} & \xi_r^{-2(r-1)} & \dots & \xi_r^{-(r-1)^2} & 1 \end{pmatrix}.$$

(2) For all $\gamma \in E_q^\vee$, $\gamma(q_r)^r = 1$ and $\gamma(\xi_r)^r = 1$, so $\gamma(q_r)$ and $\gamma(\xi_r)$ are rth roots of unity. In particular, if $\gamma(\xi_r) = \xi_r^k$, then $\gamma(T_r) = T_r^k$.

Proof. — The category $\mathcal{E}_f$ is a full subcategory of $\mathcal{E}_{p,r}$, we denote by j this inclusion. Then, we have a morphism of groups $j^* : G_{p,r} \to G_f$ (restriction to a subcategory) such that $j^*(\varphi)(B) = \varphi(B)$ for all fuchsian matrices B.

We consider another irreducible module $E(r, r-d, b^r)$ of slope $1 - \frac{d}{r}$ and associated matrix B. The tensor product adds the slopes so, $E(r, d, a^r) \otimes E(r, r-d, b^r)$ has its slope equal to 1 and we can notice that

$$\underbrace{E(r, d, a^r) \otimes \dots \otimes E(r, d, a^r)}_{r \text{ times}}$$

is of slope d, that is to say, we have two ways to obtain integral slopes.

We use the convention $\hat{\otimes}$ for the tensor product of matrices and to simplify notations, it will be denoted by $\otimes$.

Study of the non fuchsian part. — Let $\varphi \in \mathrm{Ker}(j^*)$ be an element of the Galois group $G_{p,r}$ which is trivial on the fuchsian part. As $E(r,d,a^r) \otimes E(r,r-d,b^r)$ is of slope 1, it is isomorphic as a q-difference module over K (and $\hat{K}$) to a module (K^{r^2}, Φ_{zC}), $C \in \mathrm{GL}_{r^2}(\mathbb{C})$. Hence, $\varphi(A) \otimes \varphi(B)$ is similar to $\varphi(zC)$. Putting $\varphi(z) = t^r, t \in \mathbb{C}^*$, we have:

$$\varphi(A) \otimes \varphi(B) = t^r I_{r^2}$$

if we identify $K^r \otimes K^r$ with K^{r^2} taking for basis $\{e \otimes f, \Phi_A(e) \otimes f, \ldots, \Phi_A^{r-1}(e) \otimes f, e \otimes \Phi_B(f), \Phi_A(e) \otimes \Phi_B(f), \ldots\}$ (basis of convention $\hat{\otimes}$) ; e being the cyclic vector of A and f the cyclic vector of B, here e and f are the first vector of the canonical basis of K^r.

Necessarily $\varphi(A)$ and $\varphi(B)$ have to be diagonal.

Writing $\varphi(A) = \mathrm{diag}(a_1, \ldots, a_r)$ and $\varphi(B) = \mathrm{diag}(b_1, \ldots, b_r)$ then,

$$\begin{aligned}\varphi(A) \otimes \varphi(B) &= \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_r \end{pmatrix} \otimes \begin{pmatrix} b_1 & & 0 \\ & \ddots & \\ 0 & & b_r \end{pmatrix} \\ &= \begin{pmatrix} a_1 b_1 & & & 0 \\ & a_2 b_1 & & \\ & & \ddots & \\ 0 & & & a_r b_r \end{pmatrix} = t^r I_{r^2}\end{aligned}$$

Consequently $\varphi(A) \otimes \varphi(B) = t^r(\alpha I_r) \otimes (\frac{1}{\alpha} I_r)$, $\alpha \in \mathbb{C}^*$. So $\varphi(A) = a_1 I_r$ and $\varphi(B) = b_1 I_r$ such that $a_1 b_1 = t^r$.

As

$$\overbrace{\varphi(A) \otimes \cdots \otimes \varphi(A)}^{r \text{ times}} = t^{rd} I_{r^r}$$

then $a_1^r I_{r^r} = t^{rd} I_{r^r}$, so $a_1 = (\xi_r^l t)^d$, $l \in \{0, \ldots, r-1\}$ and $b_1 = (\xi_r^{l'} t)^{r-d}$.

As $a_1 b_1 = t^r$, we have $l = l'$ so: $\varphi(A) = (\alpha_d t)^d I_r$ and $\varphi(B) = (\alpha_d t)^{r-d} I_r$ where α_d is a rth root of 1. The root α_d might depend on d and φ. Let us prove that α_d depends only on φ. Let d and d' be two integers coprime to r (so $\gcd(dd', r) = 1$), let A_d be the matrix of $E(r,d,a^r)$ and $A_{d'}$ the matrix of $E(r,d',a'^r)$:

$$\begin{aligned}&\underbrace{\varphi(A_d) \otimes \cdots \otimes \varphi(A_d)}_{r-d' \text{ times}} \otimes \underbrace{\varphi(A_{d'}) \otimes \cdots \otimes \varphi(A_{d'})}_{d \text{ times}} \\ &\qquad = (\alpha_d t)^{d(r-d')} \times (\alpha_{d'} t)^{dd'} I = \alpha_d^{-dd'} \times \alpha_{d'}^{dd'} t^{rd} I \\ &\qquad\qquad \Rightarrow (\alpha_d^{-1} \alpha_{d'})^{dd'} = 1 \Rightarrow \alpha_d = \alpha_{d'}\end{aligned}$$

Even if we have to change t in αt, we have:

$$\text{if } \varphi \in \mathrm{Ker} j^*, \text{ then } \varphi(A) = t^d I_r\,.$$

Study of the fuchsian part. — In the previous part, we studied the effect of the slopes on an element of the Galois group. We used the tensor product of two irreducible module to have an integral slope. With the same method, we study the effect of the fuchsian part. Let us recall some facts about the tensor product.

According to Lemma 1.15, there is an explicit matrix P such that:

$$A\otimes B=\sigma_q(P)\begin{pmatrix}C_1 & & 0\\ & \ddots & \\ 0 & & C_r\end{pmatrix}P^{-1},\ \text{and } C_j=\begin{pmatrix}0 & 1 & \\ & \ddots & 1\\ \tilde{c_j} & & 0\end{pmatrix},$$
$$\text{where } \tilde{c_j}=q^{\frac{r(r-1)}{2}}(ab)^r q^{(j-1)d}z^r$$

and P is equal to:

$$\left(\begin{array}{cccc|cccc|cccc|c|cccc}
1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & & 0 & 0 & \dots & 0\\
0 & \vdots & & \vdots & 1 & \vdots & & \vdots & 0 & \vdots & & \vdots & & \vdots & \vdots & & \vdots\\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & 1 & \vdots & & \vdots & & 0 & \vdots & & \vdots\\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \vdots & 0 & \dots & 0 & & 1 & 0 & \dots & 0\\
\hline
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & & & & & & 0 & a' & \dots & 0\\
\vdots & 1 & & \vdots & \vdots & 0 & & \vdots & & & & & & \vdots & 0 & & \vdots\\
\vdots & \vdots & & \vdots & \vdots & 1 & & \vdots & & & & & & \vdots & \vdots & & \vdots\\
0 & 0 & \dots & 0 & 0 & \vdots & & & & & & & & 0 & 0 & \dots & 0\\
\hline
 & & & & & & & & & & & & & & & & \\
\hline
0 & \dots & 0 & 0 & 0 & \dots & 0 & a' & 0 & \dots & 0 & 0 & & 0 & \dots & 0 & 0\\
\vdots & & \vdots & \vdots & \vdots & & \vdots & 0 & \vdots & & \vdots & \sigma_q(a') & & \vdots & & \vdots & \vdots\\
\vdots & & \vdots & 0 & \vdots & & \vdots & \vdots & \vdots & & \vdots & 0 & & \vdots & & \vdots & \sigma_q^{r-2}(a')\\
0 & \dots & 0 & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \vdots & & 0 & \dots & 0 & 0
\end{array}\right)$$

where $a'=\left(a^r q^{\frac{d(r-1)}{2}}z^d\right)^{-1}$ and the blocks have size r.

Let $\varphi\in G_{p,r}$, it is sufficient to study the image by $\varphi\in G_{p,r}$ of the blocks C_j. The matrix C_j represents a module of slope 1 so we have to find an isomorphism with a matrix of the form $zA,\ A\in\mathrm{GL}_r(\mathbb{C})$. In fact, we have:

$$C_j=\begin{pmatrix}0 & 1 & \\ & \ddots & 1\\ \tilde{c_j} & & 0\end{pmatrix}=\sigma_q(Q_j)^{-1}z\begin{pmatrix}c_j & 0 & & 0\\ 0 & c_j\xi_r & \ddots & \\ & \ddots & \ddots & 0\\ 0 & & 0 & c_j\xi_r^{r-1}\end{pmatrix}Q_j$$

where

$$Q_j = \underbrace{\begin{pmatrix} 1 & 1 & \dots & 1 & 1 \\ \xi_r^{-1} & \xi_r^{-2} & & \xi_r^{-(r-1)} & \vdots \\ \xi_r^{-2} & \xi_r^{-4} & & \xi_r^{-2(r-1)} & \vdots \\ \vdots & \vdots & & \vdots & \vdots \\ \xi_r^{-(r-1)} & \xi_r^{-2(r-1)} & \dots & \xi_r^{-(r-1)^2} & 1 \end{pmatrix}}_{V} \underbrace{\begin{pmatrix} g_1^{(j)} & 0 & & 0 \\ 0 & g_2^{(j)} & \ddots & \\ & \ddots & \ddots & 0 \\ 0 & & 0 & g_r^{(j)} \end{pmatrix}}_{G_j}$$

and $g_i^{(j)} = c_j^{r-i} q^{\frac{r(r-1)}{2}} q^{-i(r-1)} q^{r-1} \dots q^{r-i+1} z^{-(i-1)}$, cf. Remark 1.8. By restriction to $\mathcal{E}_{p,1}$, $i^*(\varphi) = (t^r, \gamma, \lambda)$, $t^r \in \mathbb{C}^*, \gamma \in \mathrm{Hom}_{\mathrm{gr}}(E_q, \mathbb{C}^*), \lambda \in \mathbb{C}$. So:

$$\varphi(C_j) = Q_j(z_0)^{-1} \quad t^r \gamma(c_j) \begin{pmatrix} 1 & 0 & & 0 \\ 0 & \gamma(\xi_r) & \ddots & \\ & \ddots & \ddots & 0 \\ 0 & & 0 & \gamma(\xi_r)^{r-1} \end{pmatrix} Q_j(z_0)\,.$$

To continue, we need to explicit Q_j^{-1} and thanks to the following lemma, the inverse of the matrix of Vandermonde V is:

$$V^{-1} = \frac{1}{r} \begin{pmatrix} 1 & \xi_r & \xi_r^2 & \dots & \xi_r^{(r-1)} \\ 1 & \xi_r^2 & \xi_r^4 & \dots & \xi_r^{2(r-1)} \\ \vdots & & & & \vdots \\ 1 & \xi_r^{(r-1)} & \xi_r^{2(r-1)} & \dots & \xi_r^{(r-1)^2} \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix}.$$

Lemma 4.5. — *Let ξ_r be a primitive rth root of unity, $\xi_r = e^{2i\pi/r}$, and let $k \in \mathbb{Z}$ then:*

$$\sum_{j=1}^{r} (\xi_r^k)^j = \{\begin{cases} 0 & si\ k \neq 0 \mod r \\ r & otherwise. \end{cases}$$

Proof. — For all $s \in \mathbb{N}^*$, the sum of sth roots of unity is equal to zero. □

Thus,

$$\varphi(C_j) = t^r \gamma(c_j)\ G_j(z_0)^{-1}\ V^{-1} \begin{pmatrix} 1 & 0 & & 0 \\ 0 & \gamma(\xi_r) & \ddots & \\ & \ddots & \ddots & 0 \\ 0 & & 0 & \gamma(\xi_r)^{r-1} \end{pmatrix} V\ G_j(z_0)$$

and, $V^{-1}\operatorname{diag}(1,\gamma(\xi_r),\dots,\gamma(\xi_r)^{r-1})V = \left(\frac{1}{r}\sum_{l=0}^{r-1}\left(\xi_r^{-(j-i)}\gamma(\xi_r)\right)^l\right)_{1\leqslant i,j\leqslant r}$.

We have $\gamma(\xi_r)^r = 1$ so $\gamma(\xi_r)$ is a rth root of unity, there exists $k \in \{0,\dots,r-1\}$ such that $\gamma(\xi_r) = \xi_r^k$. According to Lemma 4.5, we can easily describe this product: $\varphi(C_j) = t^r\gamma(c_j)\, G_j(z_0)^{-1}\, T_r^k\, G_j(z_0)$ where

$$T_r = \begin{pmatrix} 0 & 1 & & \\ & & \ddots & 1 \\ 1 & & & 0 \end{pmatrix} = V^{-1}\begin{pmatrix} 1 & 0 & & 0 \\ 0 & \xi_r & \ddots & \\ & \ddots & \ddots & 0 \\ 0 & & 0 & \xi_r^{r-1} \end{pmatrix} V\,.$$

Moreover $\gamma(T_r) = T_r^k$ so $\varphi(C_j) = t^r\gamma(c_j)\, G_j(z_0)^{-1}\, \gamma(T_r)\, G_j(z_0)$.

Now, we have:

$$\begin{aligned}\varphi(A\otimes B) &= P(z_0)\begin{pmatrix} \varphi(C_1) & & \\ & \ddots & \\ & & \varphi(C_r)\end{pmatrix}P(z_0)^{-1} \\ &= t^r\gamma(ab)P(z_0)G(z_0)^{-1}\begin{pmatrix}\gamma(T_r) & & & 0 \\ & \gamma(q_r)^d\gamma(T_r) & & \\ & & \ddots & \\ 0 & & & \gamma(q_r)^{d(r-1)}\gamma(T_r)\end{pmatrix}G(z_0)P(z_0)^{-1}.\end{aligned}$$

$G(z_0)$ is the diagonal matrix with diagonal blocks $G_j(z_0)$ $j = 1,\dots,r$. We set $G'(z_0) =$

$$\left(\begin{array}{ccc|ccc|cccc|c}
{g_1^{(1)}}^{-1} & & 0 & & & & & & & & \\
 & \ddots & & & & & & & & & \\
0 & & {g_1^{(r)}}^{-1} & & & & & & & & \\
\hline
 & & & {g_2^{(r)}}^{-1}a' & & 0 & & & & & \\
 & & & & {g_2^{(1)}}^{-1} & & & & & & \\
 & & & & \ddots & & & & & & \\
 & & & 0 & & {g_2^{(r-1)}}^{-1} & & & & & \\
\hline
 & & & & & & {g_3^{(r-1)}}^{-1}a' & & & 0 & \\
 & & & & & & & {g_3^{(r)}}^{-1}\sigma_q(a') & & & \\
 & & & & & & & & {g_3^{(1)}}^{-1} & & \\
 & & & & & & 0 & & & \ddots &
\end{array}\right)$$

where $g_i^{(j)} := g_i^{(j)}(z_0)$, $a' := a'(z_0)$, $\sigma_q^l(a') := a'(q^l z_0)$ by abusing the notation,

$$g_i^{(j)} = (ab)^{r-i}q_r^{(j-1)d(r-i)}q^{\frac{r(r-1)}{2}}q^{-i(r-1)}q^{r-1}\dots q^{r-i+1}\; z_0^{-(i-1)}.$$

Writing $\gamma(T_r) = (j_{i,j})_{1\leqslant i,j\leqslant r}$, we obtain

$$\varphi(A\otimes B) = t^r\gamma(a)\gamma(b)G'(z_0)HG'(z_0)^{-1},$$

where H is described in Figure 4.1.

A tensor product appears in the structure of this matrix:

$$H = \gamma(T_r)\begin{pmatrix}1 & & & 0\\ & \gamma(q_r) & & \\ & & \ddots & \\ 0 & & & \gamma(q_r)^{(r-1)}\end{pmatrix}^d \otimes \gamma(T_r)\begin{pmatrix}1 & & & 0\\ & \gamma(q_r) & & \\ & & \ddots & \\ 0 & & & \gamma(q_r)^{(r-1)}\end{pmatrix}^{r-d}.$$

Thus,

$$\begin{aligned}\varphi(A\otimes B) = {} & t^r\gamma(a)\gamma(b)\; G'(z_0)\;\gamma(T_r)\otimes\gamma(T_r)\; G'(z_0)^{-1}\\ & \times\begin{pmatrix}1 & & & 0\\ & \gamma(q_r) & & \\ & & \ddots & \\ 0 & & & \gamma(q_r)^{(r-1)}\end{pmatrix}^d \otimes \begin{pmatrix}1 & & & 0\\ & \gamma(q_r) & & \\ & & \ddots & \\ 0 & & & \gamma(q_r)^{(r-1)}\end{pmatrix}^{r-d}\end{aligned}$$

($G'(z_0)$ being diagonal, it commutes with the other diagonal matrices).

We have to find two matrices U and V such that:

$$\begin{pmatrix}0 & u_2 & \\ & \ddots & u_r\\ u_1 & & 0\end{pmatrix} \otimes \begin{pmatrix}0 & v_2 & \\ & \ddots & v_r\\ v_1 & & 0\end{pmatrix} = G'(z_0)\; T_r\otimes T_r\; G'(z_0)^{-1}.$$

We notice that

$$\frac{g_i^{(j)}}{g_k^{(j)}} = (ab)^{k-i}q_r^{(j-1)(k-i)}q^{\frac{(k-i)(k+i-3)}{2}}\; z_0^{k-i}.$$

Then we have:

$$\begin{aligned}
u_1v_1 &= (ab)^{r-1}q^{\frac{(r-1)(r-2)}{2}}z_0^{r-1},\\
u_2v_1 &= a^{-1}b^{r-1}q_r^{d(r-1)}q^{\frac{(r-1)(r-2)}{2}}q^{\frac{-d(r-1)}{2}}z_0^{r-d-1},\\
u_jv_1 &= a^{-1}b^{r-1}q_r^{d(j-1)(r-1)}q^{\frac{(r-1)(r-2)}{2}}q^{\frac{-d(r-1)}{2}}q^{-(j-2)d}z_0^{r-d-1},\\
u_rv_1 &= a^{-1}b^{r-1}q_r^{d(r-1)^2}q^{\frac{(r-1)(r-2)}{2}}q^{\frac{-d(r-1)}{2}}q^{-(r-2)d}z_0^{r-d-1},\\
u_1v_2 &= a^{r-1}b^{-1}q_r^{-d(r-1)}q^{\frac{d(r-1)}{2}}z_0^{d-1}\\
u_2v_2 &= (ab)^{-1}z_0^{-1}\\
u_jv_2 &= (ab)^{-1}q_r^{-d(j-2)}z_0^{-1}\\
u_rv_2 &= (ab)^{-1}q_r^{-d(r-2)}z_0^{-1}\;\;\ldots
\end{aligned}$$

$$
\left(\begin{array}{cccc|cccc|c|cccc}
j_{11} & & & & 0 & j_{12} & & & & 0 & & & j_{1r} \\
& j_{11}\gamma(q_r)^{d} & & & \vdots & & \ddots & & & j_{1r}\gamma(q_r)^{d} & & & 0 \\
& & \ddots & & 0 & & & j_{12}\gamma(q_r)^{d(r-2)} & & & & \ddots & \vdots \\
& & & j_{11}\gamma(q_r)^{(r-1)d} & j_{12}\gamma(q_r)^{d(r-1)} & 0 & \dots & 0 & & & & & 0 \\
\hline
0 & & & j_{21}\gamma(q_r)^{(r-1)d} & j_{22}\gamma(q_r)^{d(r-1)} & & & & & & & & \\
j_{21} & & & 0 & & j_{22} & & & & & & & \\
& j_{21}\gamma(q_r)^{d} & & \vdots & & & \ddots & & & & & & \\
& & \ddots & 0 & & & & j_{22}\gamma(q_r)^{d(r-2)} & & & & & \\
\hline
& & & & & & & & & & & & \\
\hline
0 & j_{r1}\gamma(q_r)^{d} & & & & & & & & j_{rr}\gamma(q_r)^{d} & & & \\
\vdots & & \ddots & & & & & & & & j_{rr}\gamma(q_r)^{2d} & & \\
0 & & & j_{r1}\gamma(q_r)^{d(r-1)} & & & & & & & & \ddots & \\
j_{r1} & 0 & \dots & 0 & & & & & & & & & j_{rr}
\end{array}\right)
$$

Figure 4.1.

Let $z_{0,r}$ be the rth root of z_0 that we choose to begin, there exists $\alpha_{(d)} \in \mathbb{C}^*$ such that:

$$\begin{aligned}
u_1 &= \alpha_{(d)} a^{r-1} q_r^{\frac{d(r-1)(r-2)}{2}} z_{0,r}^{d(r-1)}, & v_1 &= \alpha_{(d)}^{-1} b^{r-1} q_r^{\frac{(r-d)(r-1)(r-2)}{2}} z_{0,r}^{(r-d)(r-1)},\\
u_2 &= \alpha_{(d)} a^{-1} z_{0,r}^{-d}, & v_2 &= \alpha_{(d)}^{-1} b^{-1} z_{0,r}^{-(r-d)},\\
u_j &= \alpha_{(d)} a^{-1} q_r^{-d(j-2)} z_{0,r}^{-d}, & v_j &= \alpha_{(d)}^{-1} b^{-1} q_r^{-(r-d)(j-2)} z_{0,r}^{-(r-d)},\\
u_r &= \alpha_{(d)} a^{-1} q_r^{-d(r-2)} z_{0,r}^{-d}, & v_r &= \alpha_{(d)}^{-1} b^{-1} q_r^{-(r-d)(r-2)} z_{0,r}^{-(r-d)}.
\end{aligned}$$

In fact, we have:

$$\begin{aligned}
\varphi(A) = t^d \alpha_{(d)} \gamma(a) &\begin{pmatrix} 0 & \alpha_0^{-1} & & \\ & & \ddots & \\ 0 & & & \alpha_{r-2}^{-1} \\ \alpha_0\alpha_1\ldots\alpha_{r-2} & 0 & \ldots & 0 \end{pmatrix}^k \\
&\times \begin{pmatrix} 1 & & & 0 \\ & \gamma(q_r) & & \\ & & \ddots & \\ 0 & & & \gamma(q_r)^{(r-1)} \end{pmatrix}^d
\end{aligned}$$

where $\alpha_j = a q_r^{jd} z_{0,r}^d$ and $\gamma(T_r) = T_r^l$. In the same way as the first part, we prove that $\alpha_{(d)}$ depends only on d and φ, and that $\alpha_{(d)} = \alpha_{(1)}^d$ is rth root of 1. Even if we change t in $\alpha_{(1)} t$,we can consider $\alpha_{(d)} = 1$.

And if we set:

$$G_{a,d}(z_{0,r}) = \begin{pmatrix} 1 & & & & \\ & \alpha_0 & & & \\ & & \alpha_0\alpha_1 & & \\ & & & \ddots & \\ & & & & \alpha_0\alpha_1\ldots\alpha_{r-2} \end{pmatrix}^{-1}$$

we obtain the formula of Theorem 4.1. □

Thus, an element of the Galois group $G_{p,r}$ is a triple $\varphi = (t, \gamma, \lambda)$ where $t \in \mathbb{C}^*, \gamma \in E_q^\vee, \lambda \in \mathbb{C}$, λ acting on the unipotent part U_m.

The group law is given by $(t, \gamma, \lambda)(t', \gamma', \lambda') = (tt'\gamma(q_r)^{-k'}, \gamma\gamma', \lambda + \lambda')$ because the matrices T_r and $\mathrm{diag}(1, \gamma(q_r), \ldots)$ don't commute.

As $\gamma(q_r)$ is rth root of the unity, we have $\gamma(q_r)^{k'} = \gamma'(\gamma(q_r))$ (which is the composition $\gamma' \circ \gamma$ but not the multiplication in $E_q^\vee$). We set $\varepsilon_r(\gamma, \gamma') = \frac{1}{\gamma'(\gamma(q_r))} = \xi_r^{-lk'}$ if $\gamma(q_r) = \xi_r^l$ and $\gamma'(\xi_r) = \xi_r^{k'}$. So, the group law is:

$$(t, \gamma, \lambda)(t', \gamma', \lambda') = (tt'\varepsilon_r(\gamma, \gamma'), \gamma\gamma', \lambda + \lambda').$$

We easily verify that the group law is associative, as ε_r satisfies the following property (and the symmetric property):

$$\varepsilon_r(\gamma\gamma', \gamma'') = \varepsilon_r(\gamma, \gamma'')\varepsilon_r(\gamma', \gamma'').$$

In fact, $\varepsilon_r : E_q^\vee \times E_q^\vee \to \mu_r$ is bilinear, μ_r denoting the group of the rth roots of the unity.

Proposition 4.6. — *The group $\mathbb{C}^*$ is central in $G_{p,r}$ and we have: $G_{p,r} = H_r \times \mathbb{C}$ where*

$$1 \to \mathbb{C}^* \xrightarrow{u} H_r \xrightarrow{v} E_q^\vee \to 1, \quad u(t) = (t, 1),\ v((t, \gamma)) = \gamma$$

is a central extension.

Proof. — We easily verify that $(t, \gamma)(t', 1) = (t', 1)(t, \gamma)$. □

Remark 4.7. — The extension $1 \to \mathbb{C}^* \to H_r \to E_q^\vee \to 1$ is central but the exact sequence does not split. Indeed, to choose a section $s : E_q^\vee \to H_r$ such that $v \circ s = \mathrm{Id}_{E_q^\vee}$, the only possibility is to put $s(\gamma) = (1, \gamma)$, but it is not a morphism of groups because $(1, \gamma)(1, \gamma') = (\varepsilon_r(\gamma, \gamma'), \gamma\gamma')$.

Let $r, s \in \mathbb{N}^*$, if r divides s then $\mathcal{E}_{p,r}$ is a full subcategory of $\mathcal{E}_{p,s}$. Thanks to Proposition 2.21 of [4], we have the onto morphisms of groups:

$$\varphi_{r,s} : G_{p,s} \to G_{p,r},\ \varphi_{r,s}(t, \gamma, \lambda) = (t^{s/r}, \gamma, \lambda).$$

Proposition 4.8. — *The universal formal Galois group is then $G_p = H \times \mathbb{C}$ where $H = \varprojlim H_r$ and*

$$1 \to \mathbb{Q}^\vee \to H \to E_q^\vee \to 1$$

is a central extension. The morphisms $\varphi_r : G_p \to G_{p,r}$ are $\varphi_r(\alpha, \gamma, \lambda) = (\alpha(\frac{1}{r}), \gamma, \lambda)$.

An element of G_p is a triple $(\alpha, \gamma, \lambda)$, $\alpha \in \mathbb{Q}^\vee$, $\gamma \in E_q^\vee$ and $\lambda \in \mathbb{C}$. The group law is:

$$(\alpha, \gamma, \lambda)(\alpha', \gamma', \lambda') = (\alpha\alpha'\varepsilon(\gamma, \gamma'), \gamma\gamma', \lambda + \lambda')$$

where $\varepsilon : E_q^\vee \times E_q^\vee \to \mathbb{Q}^\vee$ is a morphism of groups such that $r \in \mathbb{N}^*$, $\varepsilon(\gamma, \gamma')(\frac{1}{r}) = \varepsilon_r(\gamma, \gamma')$.

Proof. — To prove Proposition 4.8, we need the two following lemmas.

Lemma 4.9. — $\varprojlim(H_r \times \mathbb{C}) = \varprojlim(H_r) \times \mathbb{C}$.

Lemma 4.10. — *If we consider $\mathbb{C}^*$ with the following projective system: let $r, s \in \mathbb{N}^*$, if r divides s, we put:*

$$\begin{aligned} \rho_{r,s} : \mathbb{C}^*_s &\to \mathbb{C}^*_r \\ t &\mapsto t^{s/r}. \end{aligned}$$

Then $\varprojlim(\mathbb{C}^*_r) = \mathbb{Q}^\vee$.

The proofs of the lemmas are left to the reader.

For all $r \in \mathbb{N}^*$, we have the following exact sequence:

$$1 \to \mathbb{C}^* \to H_r \to E_q^\vee \to 0\,.$$

Let us consider the following morphism of groups:

$$\begin{aligned} \varepsilon : E_q^\vee \times E_q^\vee &\to \mathbb{Q}^\vee \\ (\gamma, \gamma') &\mapsto \varepsilon(\gamma, \gamma')\,, \end{aligned}$$

where $\varepsilon(\gamma, \gamma') : \mathbb{Q} \to \mathbb{C}^*$ is a group morphism defined for all $r \in \mathbb{N}^*$ by $\varepsilon(\gamma, \gamma')(\frac{1}{r}) = \varepsilon_r(\gamma, \gamma')$.

So, we have $G_p = \varprojlim(H_r) \times \mathbb{C}$. One puts $H = \varprojlim H_r$. We have:

$$H = \left\{ (\alpha, \gamma) \in \mathbb{Q}^\vee \times E_q^\vee \;\middle|\; \begin{array}{l} \forall\, \alpha, \alpha' \in \mathbb{Q}^\vee, \forall\, \gamma, \gamma' \in E_q^\vee, \\ \qquad (\alpha, \gamma)(\alpha', \gamma') = (\alpha\alpha'\varepsilon(\gamma, \gamma'), \gamma\gamma') \end{array} \right\}$$

with the morphisms:

$$\begin{aligned} \psi_r : \qquad H &\to H_r \\ (\alpha, \gamma) &\mapsto \left(\alpha\left(\frac{1}{r}\right), \gamma\right), \end{aligned}$$

that satisfy $\varphi_{r,s|} \circ \psi_s = \psi_r$.

As a consequence, we have the following diagram of exact sequences:

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{Q}^\vee \arrow[r] \arrow[d, "\rho_r"] & \varprojlim H_r = H \arrow[r] \arrow[d, "\psi_r"] & E_q^\vee \arrow[r] \arrow[d, "id"] & 1 \\ 1 \arrow[r] & \mathbb{C}^* \arrow[r] & H_r \arrow[r] & E_q^\vee \arrow[r] & 1\,, \end{tikzcd}$$

$\rho_r : \mathbb{Q}^\vee \to \mathbb{C}^*$ is defined by $\alpha \mapsto \alpha\left(\frac{1}{r}\right)$. These extensions are central. □

We find the same result as van der Put and Reversat obtained by the Picard–Vessiot theory

4.2. Galois group

Let $(K',\sigma_{q'})$ be a q'-difference field, $(K',\sigma_{q'}) = (K,\sigma_q)$, (K_r,σ_{q_r}) or (K,σ_{q^r}), we denote by:

- $\mathcal{E}(K',q')$: the category of q'-difference modules over K'. If $(K',\sigma_{q'}) = (K,\sigma_q)$, we simply denote it by $\mathcal{E}$.
- $\mathcal{E}_r(K',q')$, $r\in\mathbb{N}^*$: the category of q'-difference modules over K' of slopes $\frac{k}{r}$, $k\in\mathbb{Z}$ and over (K,σ_q), we simply denote it by $\mathcal{E}_r$.

Every q-difference module admits a unique graded module. The functor gr from $\mathcal{E}$ to $\mathcal{E}_p$ (cf. [16]) associates with a q-difference module M its graded module and with a morphism of q-difference modules its graded morphism. It is faithful, exact and tensor compatible.

We can consider that an element of $\mathcal{E}$ has a matrix in standard form:

$$A_U = \begin{pmatrix} A_1 & & U_{i,j} \\ & \ddots & \\ 0 & & A_s \end{pmatrix},$$

where A_i is the matrix of a pure isoclinic module in normal form. According to the Section 2, $U_{i,j}$ are matrices with coefficients in $\mathbb{C}[z,z^{-1}]$.

We defined on $\mathcal{E}_p$, the functor ω_{z_0}. Now, we define the functor $\hat{\omega}_{z_0} := \omega_{z_0}\circ\mathrm{gr}$ on $\mathcal{E}$. It is a fiber functor on $\mathcal{E}$. The category $\mathcal{E}$ provided with this fiber functor is tannakian. Let $G := \mathrm{Aut}^{\otimes}(\hat{\omega}_{z_0})$ be its Galois group. On the other hand, $\mathcal{E}_p$ is a subcategory of $\mathcal{E}$, if we denote by i this inclusion, $\mathrm{gr}\circ i = \mathrm{id}$ and by tannakian duality, we have:

$$G \underset{\mathrm{gr}^*}{\overset{i^*}{\rightleftarrows}} G_p \qquad \text{and} \quad i^*\circ\mathrm{gr}^* = \mathrm{id}\,.$$

Let $S = \mathrm{Ker}\ i^*$ be the *Stokes group* (whose elements are trivial on pure modules), we obtain a split exact sequence:

$$1\to S\to G\to G_p\to 1.$$

Thus, the Galois group G is the semi direct product of S by G_p, $G = S\rtimes G_p$. The group G_p acts by conjugation on the elements of S: we will set for all $g_0\in G_p$ and $s\in S$, $s^{g_0} = g_0\,s\,g_0^{-1}\in S$.

For all $r\in\mathbb{N}^*$, the functor $\hat{\omega}_{z_0}$ restricts to the rigid abelian tensor subcategories $\mathcal{E}_r$ and the inclusion i restricts too: $\mathcal{E}_{p,r}\subset\mathcal{E}_r$, as a consequence, we can define: $G_r := \mathrm{Aut}^{\otimes}(\hat{\omega}_{z_0}|_{\mathcal{E}_r})$ the Galois group of the tannakian category $\mathcal{E}_r$ and $S_r = \mathrm{Ker}(i^*|_{G_r})$, the exact sequence of groups

$1 \to S_r \to G_r \to G_{p,r} \to 1$ is also split. We have the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & S & \longrightarrow & G & \longrightarrow & G_p & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & S_r & \longrightarrow & G_r & \longrightarrow & G_{p,r} & \longrightarrow & 1. \end{array}$$

The functor also restricts to the tannakian category which is generated by an object M of $\mathcal{E}$, in this case, we will denote the previous groups by S_M, G_M and $G_{p,M}$ and we have the split exact sequence of algebraic groups:

$$1 \to S_M \to G_M \to G_{p,M} \to 1.$$

By tannakian theory, every tannakian category is equivalent to the category of rational representations of its Galois group. With an object M of $\mathcal{E}$, we can associate the representation $\rho_M : G \to \mathrm{GL}(\hat{\omega}_{z_0}(M))$, $\varphi \mapsto \varphi(M)$. The group G_M is identified to the algebraic subgroup $\operatorname{Im} \rho_M = G(M) = \{\varphi(M) \mid \varphi \in G_M\}$ of $\mathrm{GL}(\hat{\omega}_{z_0}(M))$.

4.3. Extensions of representations

We consider $\mathcal{C}$ the tannakian category generated by q-difference modules with two slopes whose graded module is fixed. Precisely, we fix $M_0 = (K^{n_1+n_2}, \Phi_{A_0})$ which is a pure module with two slopes such that:

$$A_0 = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}.$$

The modules (K^{n_1}, Φ_{A_1}) and (K^{n_2}, Φ_{A_2}) are pure isoclinic of slopes $\mu_1 < \mu_2$. The category $\mathcal{C}$ is generated by the q-difference modules $M = (K^{n_1+n_2}, \Phi_{A_U})$ where:

$$A_U = \begin{pmatrix} A_1 & U \\ 0 & A_2 \end{pmatrix} \quad U \in M_{n_1,n_2}(K).$$

Abusing the notation, a matrix denotes its associated module. Thus, for all $U \in M_{n_1,n_2}(K)$, we have an exact sequence of q-difference modules:

$$0 \to A_1 \to A_U \to A_2 \to 0.$$

To simplify, we set $\omega := \omega_{z_0}$ and $\hat{\omega} = \omega \circ \mathrm{gr}$, and here G is the Galois group of the category $\mathcal{C}$ obtained by restriction of the functor $\hat{\omega}$, S is the Stokes group and G_0 the Galois group of $\langle M_0 \rangle$, which is also the formal Galois group associated to the category $\mathcal{C}$. The following split exact sequence of groups is still true:

$$1 \to S \to G \to G_0 \to 1 .$$

For the properties of vector groups used in the following, see [2, §1.2] or [6].

Proposition 4.11. — *The Stokes group of the category $\mathcal{C}$ is a vector group, that is to say, pro-unipotent and commutative.*

Proof. — Let N be an object of $\mathcal{C}$, $\langle N \rangle$ is a full subcategory of $\mathcal{C}$, then there is an onto morphism of groups $j_N^* : G \to G_N$ and $N' > N$ if there is an onto morphism of algebraic groups $j_{N,N'}^* : G_{N'} \to G_N$. The following diagram is commutative:

$$\begin{tikzcd}
1 \arrow[r] & S \arrow[r] \arrow[d, "j_N^*"] & G \arrow[r] \arrow[d, "j_N^*"] & G_0 \arrow[r] \arrow[d, "j_N^*"] & 1 \\
1 \arrow[r] & S_N \arrow[r] & G_N \arrow[r] & G_{0,N} \arrow[r] & 1 .
\end{tikzcd}$$

In the basic case $M = (K^{n_1+n_2}, \Phi_{A_U})$, the Stokes group S_M is isomorphic to the algebraic subgroup $\mathfrak{S}_{n_1,n_2}(\mathbb{C})$ of $\mathrm{GL}_{n_1+n_2}(\mathbb{C})$ (defined in Section 2.1), so S_M is a vector space. Generally,

$$S = \varprojlim S_N$$

for the projective system: $N' > N$, if there exists X such that $N' = N \oplus X$, and by restriction, there is a linear morphism $j_{N,N'}^* : S_{N'} \to S_N$.

According to [4], $G = \varprojlim G_N$, for all N, S_N is a subgroup of G_N and the morphisms $j_{N,N'}^*$ restrict to S. Therefore, $S = \varprojlim S_N$ is a commutative proalgebraic subgroup and it is pro-unipotent. □

For $i = 1, 2$, let V_i denote the $\mathbb{C}$-vector space $\hat{\omega}(A_i)$. Let $\rho_1 : G \to \mathrm{GL}(V_1)$ be the representation of the group G associated with A_1 and $\rho_2 : G \to \mathrm{GL}(V_2)$ the representation associated with A_2. If we denote by ρ the representation of G associated with A_U, the following sequence of representations is exact:

$$0 \to \rho_1 \to \rho \to \rho_2 \to 0 .$$

We have:

$$\rho(g) = \begin{pmatrix} \rho_1(g) & \alpha(g) \\ 0 & \rho_2(g) \end{pmatrix}$$

where α is a map $G \to \mathcal{L}(V_2, V_1)$ such that ρ is a representation of group, which means that α verifies:

$$\forall\, g, g' \in G,\ \alpha(gg') = \rho_1(g)\alpha(g') + \alpha(g)\rho_2(g') . \tag{4.1}$$

Theorem 4.12. — *There is an isomorphism of $\mathbb{C}$-vector spaces:*

$$\mathit{Ext}(\rho_2,\rho_1) \xrightarrow{\sim} \mathcal{L}_{G_0}(S,\mathcal{L}(V_2,V_1))\,.$$

Here, the group G_0 acts on S and on $\mathcal{L}(V_2,V_1))$ which are G_0-modules.

We define the $\mathbb{C}$-vector space $\mathcal{L}_{G_0}(S,\mathcal{L}(V_2,V_1))$ to be the set of linear maps α from S to $\mathcal{L}(V_2,V_1)$ such that for all $g_0 \in G_0$, $\alpha(s^{g_0}) = \rho_1(g_0)\alpha(s)\rho_2(g_0)^{-1}$ where $s^{g_0} := g_0 s g_0^{-1}$.

Remark 4.13. — Let us notice the analogy with the Corollary 5.11 of [7], the authors obtain a similar result with a Picard–Vessiot theory.

Proof. — The group G acts on $\mathcal{L}(V_2,V_1)$ in the following way:

$$\forall\, g\in G, \forall\, A\in\mathcal{L}(V_2,V_1)\,,\quad g.A := \rho_1(g)A\rho_2(g)^{-1}.$$

We define $H^1(G,\mathcal{L}(V_2,V_1))$ as the quotient

$$Z^1(G,\mathcal{L}(V_2,V_1))/B^1(G,\mathcal{L}(V_2,V_1))$$

where Z^1 is the set of cocycles and B^1 the set of coboundaries, they are defined by:

$$Z^1(G,\mathcal{L}(V_2,V_1)) = \left\{ f: G\to\mathcal{L}(V_2,V_1) \;\middle|\; \begin{array}{l} \forall\, g,g'\in G,\\ \quad f(gg') = g.f(g') + f(g)\end{array}\right\}$$

$$B^1(G,\mathcal{L}(V_2,V_1)) = \left\{ f\in Z^1(G,\mathcal{L}(V_2,V_1)) \;\middle|\; \begin{array}{r} \exists\, A\in\mathcal{L}(V_2,V_1), \forall\, g\in G,\\ f(g) = g.A - A\end{array}\right\}$$

and all f in $Z^1(G,\mathcal{L}(V_2,V_1))$ and $B^1(G,\mathcal{L}(V_2,V_1))$ are rational.

Every matrix A_U gives a representation of the group G, $\rho : G \to \mathrm{GL}(V_1\times V_2)$:

$$\rho(g) = \begin{pmatrix} \rho_1(g) & \alpha(g) \\ 0 & \rho_2(g)\end{pmatrix}.$$

We put for all $g\in G$, $Z_\rho(g) = \alpha(g)\rho_2(g)^{-1}$. Then, Z_ρ is a cocycle of dimension 1 for the cohomology of the group G with values in $\mathcal{L}(V_2,V_1)$. Indeed,

$$\begin{aligned}
&dZ_\rho(gg')\\
&= g.Z_\rho(g') - Z_\rho(gg') + Z_\rho(g)\\
&= \rho_1(g)\alpha(g')\rho_2(gg')^{-1} - \alpha(gg')\rho_2(gg')^{-1} + \alpha(g)\rho_2(g)^{-1}\\
&= \rho_1(g)\alpha(g')\rho_2(gg')^{-1} - \rho_1(g)\alpha(g')\rho_2(gg')^{-1} - \alpha(g)\rho_2(g)^{-1} + \alpha(g)\rho_2(g)^{-1}\\
&= 0\,.
\end{aligned}$$

As a consequence, $Z_\rho \in Z^1(G, \mathcal{L}(V_2, V_1))$. Let B be a coboundary, there exists $A \in \mathcal{L}(V_2, V_1)$ such that for all $g \in G$, $B(g) = g.A - A$. So, Z_ρ and $Z_{\rho'}$ are in the same cohomology class if, and only if,

$$\begin{aligned} \forall\, g \in G,\ \ Z_\rho(g) = Z_{\rho'}(g) + g.A - A& \\ &\Leftrightarrow \alpha(g) = \alpha'(g) + \rho_1(g)A - A\rho_2(g) \\ &\Leftrightarrow \rho(g) = \begin{pmatrix} \mathrm{Id} & -A \\ 0 & \mathrm{Id} \end{pmatrix} \begin{pmatrix} \rho_1(g) & \alpha'(g) \\ 0 & \rho_2(g) \end{pmatrix} \begin{pmatrix} \mathrm{Id} & A \\ 0 & \mathrm{Id} \end{pmatrix}, \end{aligned}$$

that is to say, the representation ρ is isomorphic to the representation ρ', they have the same class in $\mathrm{Ext}(\rho_2, \rho_1)$. We prove in the same way that $[Z_\rho] = 0$ if, and only if, ρ is in the class of $\rho_1 \oplus \rho_2$. Thus, we obtain an well defined injective map:

$$\mathrm{Ext}(\rho_2, \rho_1) \to H^1(G, \mathcal{L}(V_2, V_1)),\ \rho \mapsto [Z_\rho].$$

This is a bijection because if Z is a cocycle, we set for all $g \in G$, $\alpha(g) := Z(g)\rho_2(g)$ and α is in $\mathcal{L}(V_2, V_1)$. As Z is a cocycle α satisfies the property (4.1), so ρ defined by

$$\rho(g) = \begin{pmatrix} \rho_1(g) & \alpha(g) \\ 0 & \rho_2(g) \end{pmatrix}$$

is a representation of the group G in $\mathrm{Ext}(\rho_2, \rho_1)$. We have constructed a bijection of sets:

$$\mathrm{Ext}(\rho_2, \rho_1) \xrightarrow{\sim} H^1(G, \mathcal{L}(V_2, V_1)). \tag{4.2}$$

There is left to prove:

(1) $H^1(G, \mathcal{L}(V_2, V_1)) \cong \mathcal{L}_{G_0}(S, \mathcal{L}(V_2, V_1))$,
(2) the previous bijection (4.2) is a linear isomorphism.

Lemma 4.14. — *$H^1(G, \mathcal{L}(V_2, V_1)) \cong \mathcal{L}_{G_0}(S, \mathcal{L}(V_2, V_1))$ by $[Z_\rho] \mapsto Z_{\rho_{|S}}$.*

Proof of Lemma 4.14. — Let $g \in G$, we can write $g = sg_0$ where $s \in S$ and $g_0 \in G_0$ because $G = S \rtimes G_0$. Let Z be a cocycle of $Z^1(G, \mathcal{L}(V_2, V_1))$, we have:

$$Z(g) = Z(sg_0) = s.Z(g_0) + Z(s) = \rho_1(s)Z(g_0)\rho_2(s)^{-1} + Z(s).$$

But $\rho_i(s) = s(A_i) = \mathrm{Id}$ for $i = 1, 2$ because A_i is a matrix of a pure isoclinic module. Thus, $Z(g) = Z(g_0) + Z(s)$ and a coboundary for S is necessarily zero because $B(s) = s.A - A = A - A = 0$.

We have $H^1(G, \mathcal{L}(V_2, V_1)) \subset H^1(G_0, \mathcal{L}(V_2, V_1)) \times \mathcal{L}_{G_0}(S, \mathcal{L}(V_2, V_1))$ defined by $[Z] \mapsto ([Z_{|G_0}], Z_{|S})$. Indeed for all $s \in S$, we have $Z(s^{g_0}) =$

$\rho_1(g_0)Z(s)\rho_2(g_0)^{-1}$ so $Z_{|S}$ is in $\mathcal{L}_{G_0}(S,\mathcal{L}(V_2,V_1))$. We have a well defined and injective map.

Let us prove that $H^1(G_0,\mathcal{L}(V_2,V_1)) = 0$. Let Z' be a cocycle in $Z^1(G_0,\mathcal{L}(V_2,V_1)) = 0$, Z' gives a rational representation of G_0:

$$\rho : G_0 \to \mathrm{GL}(V_1\times V_2),\ g_0 \mapsto \begin{pmatrix}1 & Z'(g_0)\\ 0 & 1\end{pmatrix}\begin{pmatrix}\rho_1(g) & 0\\ 0 & \rho_2(g)\end{pmatrix},$$

such that the sequence of representations of G_0, $1\to\rho_1\to\rho\to\rho_2\to 1$ is exact. By tannakian duality, the category of rational representations of G_0 is equivalent to the category $\langle M_0\rangle$. This exact sequence corresponds to the exact sequence in $\langle M_0\rangle$: $1\to M_1\to M\to M_2\to 1$. Necessarily, $M = M_1\oplus M_2$ since $\langle M_0\rangle$ has only pure modules and M_1 and M_2 have not the same slope. Thus, we have $\rho=\rho_1\oplus\rho_2$ and $Z_\rho(g_0) = Z'(g_0) = 0$.

If $g = sg_0$, and $Z\in H^1(G,\mathcal{L}(V_2,V_1))$, then $Z(g)=Z(s)$. The map described by the lemma is injective. It is onto: we put $\alpha\in\mathcal{L}_{G_0}(S,\mathcal{L}(V_2,V_1))$, for all $g\in G$, $g=sg_0$, $Z(g)=Z(sg_0):=\alpha(s)$, Z is a cocycle, indeed:

$$\begin{aligned}Z(gg') = Z(sg_0s'g_0') = Z(ss'^{g_0}g_0g_0') = \alpha(ss'^{g_0}) &= \alpha(s)+g_0.\alpha(s')\\ &= Z(g)+g.Z(g').\qquad\square\end{aligned}$$

To finish the proof of the theorem, we have to prove that the bijection between $\mathrm{Ext}(\rho_2,\rho_1)$ and $\mathcal{L}_{G_0}(S,\mathcal{L}(V_2,V_1))$ is a linear isomorphism.

The structure of $\mathbb{C}$-vector space of $\mathrm{Ext}(\rho_2,\rho_1)$ is described in [12] at Section A.4 (for the extensions of modules). An element of $\mathrm{Ext}(\rho_2,\rho_1)$ is a representation of G of the form:

$$\rho : G\to\mathrm{GL}(V),\quad \forall\, g\in G,\ \rho(g) = \begin{pmatrix}\rho_1(g) & \alpha(g)\\ 0 & \rho_2(g)\end{pmatrix},$$

addition is given by:

$$\rho+\rho' : g\mapsto \begin{pmatrix}\rho_1(g) & \alpha(g)+\alpha'(g)\\ 0 & \rho_2(g)\end{pmatrix},$$

the identity element is the representation $\rho_1\oplus\rho_2$ where for all $g\in G$, $\alpha(g)=0$. And scalar multiplication by $\lambda\in\mathbb{C}$ is defined by:

$$\lambda\rho : g\mapsto\begin{pmatrix}\rho_1(g) & \lambda\alpha(g)\\ 0 & \rho_2(g)\end{pmatrix}.$$

Our map:

$$\kappa : \mathrm{Ext}(\rho_2,\rho_1)\to\mathcal{L}_{G_0}(S,\mathcal{L}(V_2,V_1)),\ \rho\mapsto Z_\rho|_S = \alpha|_S$$

is $\mathbb{C}$-linear and bijective. $\square$

Remark 4.15. — The theory of [18, Chapter VII] can be extended to the context of algebraic groups and rational maps. The Theorem 4.12 is then a particular case of the sequence of inflation restriction of [18, Chapter VII p. 126]. Indeed, by setting $A = \mathcal{L}(V_2, V_1)$, we have the following exact sequence:

$$0 \to H^1(G/S, A^S) \to H^1(G, A) \to H^1(S, A)^{G_0} \to H^2(G/S, A),$$

induced by $S \subset G$ and $i^* : G \to G_0$.

But the action of S on A is trivial, as a consequence $A^S = A$ and $H^1(S, A)$ is $\mathrm{Hom}(S, A)$, thus $H^1(S, A)^{G_0} = \mathcal{L}_{G_0}(S, A)$. In fact, we have:

$$0 \to H^1(G_0, A) \to H^1(G, A) \to \mathcal{L}_{G_0}(S, A) \to 0.$$

The map $H^1(G, A) \to \mathcal{L}_{G_0}(S, A)$ is onto by the Lemma 4.14, we proved by hand but not using the fact $H^2 = 0$, the principal argument is $G = S \rtimes G_0$. By tannakian theory, we prove that $H^1(G_0, A)$ is zero.

4.3.1. Link between the Galois group of an element of $\mathcal{F}(P_1, P_2)$ and its corresponding element in $\mathcal{F}(P_1 \otimes P_2^\vee, 1)$

If $P_1 = (K^{n_1}, \Phi_{A_1})$ and $P_2 = (K^{n_2}, \Phi_{A_2})$ are two pure isoclinic modules of slopes $\mu_1 < \mu_2$, we saw that $\mathcal{F}(P_1, P_2)$ and $\mathcal{F}(P_1 \otimes P_2^\vee, \underline{1})$ are isomorphic (Theorem 2.4). If we take a representative of a class of $\mathcal{F}(P_1, P_2)$, $M = (K^{n_1+n_2}, \Phi_{A_U})$ with:

$$A_U = \begin{pmatrix} A_1 & U \\ 0 & A_2 \end{pmatrix}$$

and the corresponding element in $\mathcal{F}(P_1 \otimes P_2^\vee, 1)$ is $M' = (K^{r_1 r_2 + 1}, \Phi_{A'_{U'}})$ where:

$$A'_{U'} = \begin{pmatrix} B_1 \widehat{\otimes} B_2^\vee & U' \\ 0 & 1 \end{pmatrix} \quad U' = \widehat{U B_2^{-1}}.$$

Now we shall describe the relationship between their Galois groups.

According to Proposition 2.2 established during the study of $\mathcal{F}(P_1, P_2)$, the module M' is a pullback, we have:

$$M' = (M \otimes P_2^\vee) \times_{P_2 \otimes P_2^\vee} \underline{1}.$$

There is an equivalence of categories between the category of q-difference modules and the category of the rational representations of the Galois group, so we study the pullback of the associated representations.

We follow the scheme of Proposition 2.2. We denote by $W_1 = \hat{\omega}_{z_0}(P_1)$ and $W_2 = \hat{\omega}_{z_0}(P_2)$ the vector spaces associated with P_1 and P_2. The representations of the Galois group of P_1 and P_2 are $\rho_1 : G \to \mathrm{GL}(W_1)$ and $\rho_2 : G \to \mathrm{GL}(W_2)$.

The representation which is associated with M is an extension of ρ_2 by ρ_1, we denote it by: $\rho_v : G \to \mathrm{GL}(W_1 \times W_2)$ such that for all $(x_1, x_2) \in W_1 \times W_2$, $\rho_v(g)(x_1, x_2) = (\rho_1(g)(x_1) + v(g)(x_2), \rho_2(g)(x_2))$, where $v : G \to \mathcal{L}(W_2, W_1)$ satisfies:

$$\forall\, g, g' \in G,\ v(gg') = \rho_1(g)v(g') + v(g)\rho_2(g')\,.$$

We have an exact sequence of representations:

$$0 \to \rho_1 \to \rho_v \to \rho_2 \to 0$$

by identifying $\rho \otimes \rho_2^\vee$ with $\mathrm{Hom}(\rho_2, \rho)$, we have the exact sequence

$$0 \to \underline{\mathrm{Hom}}(\rho_2, \rho_1) \to \underline{\mathrm{Hom}}(\rho_2, \rho_v) \to \underline{\mathrm{Hom}}(\rho_2, \rho_2) \to 0$$

where $\underline{\mathrm{Hom}}(\rho_2, \rho_1)$ corresponds to the representation $G \to \mathcal{L}(W_2, W_1)$ such that $g \mapsto (p \in \mathcal{L}(W_2, W_1) \mapsto \rho_1(g) \circ p \circ \rho_2(g)^{-1})$; $\underline{\mathrm{Hom}}(\rho_2, \rho_2)$ corresponds to the representation $G \to \mathcal{L}(W_2, W_2)$ such that $g \mapsto (q \in \mathcal{L}(W_2, W_2) \mapsto \rho_2(g) \circ q \circ \rho_2(g)^{-1})$ and $\underline{\mathrm{Hom}}(\rho_2, \rho_v)$ corresponds to the representation $G \to \mathcal{L}(W_2, W_1) \times \mathcal{L}(W_2, W_2)$ such that

$$g \mapsto \big((p, q) \mapsto (\rho_1(g) \circ p \circ \rho_2(g)^{-1} + v(g) \circ q \circ \rho_2(g)^{-1}, \rho_2(g) \circ q \circ \rho_2(g)^{-1})\big)\,.$$

We make the pullback by $\underline{1} \to \underline{\mathrm{Hom}}(\rho_2, \rho_2)$ where $\underline{1}$ is the trivial representation and this morphism is described by $\mathbb{C} \to \mathcal{L}(W_2, W_2)$, $\lambda \mapsto \lambda\,\mathrm{Id}$. Then, we obtain a representation

$$G \to (\mathcal{L}(W_2, W_1) \times \mathcal{L}(W_2, W_2)) \times_{\mathcal{L}(W_2, W_2)} \mathbb{C}$$

defined by:

$$g \mapsto \ \big((p, \lambda) \mapsto (\rho_1(g) \circ p \circ \rho_2(g)^{-1} + \lambda v(g) \circ \rho_2(g)^{-1}, \lambda)\big) \qquad \text{by identifying } q = \lambda\,\mathrm{Id}\,.$$

So, we have:

$$G(M) = \left\{ \rho_v(g) = \begin{pmatrix} \rho_1(g) & v(g) \\ 0 & \rho_2(g) \end{pmatrix} \right\}$$

and

$$G(M') = \left\{ \begin{pmatrix} (\rho_1 \otimes \rho_2^\vee)(g) & V(g) \\ 0 & 1 \end{pmatrix} \right\}$$

where

$$(\rho_1 \otimes \rho_2^\vee)(g))(p) = \rho_1(g) \circ p \circ \rho_2(g)^{-1}$$

where $V(g) \in \mathcal{L}(W_2, W_1)$ and $V(g)(p) = v(g) \circ p \circ \rho_2(g)^{-1}$.

As far as the Stokes groups are concerned, we have:

$$S(M)=\left\{\begin{pmatrix}\mathrm{Id}_{W_1} & v(s)\\ 0 & \mathrm{Id}_{W_2}\end{pmatrix},\ s\in S\right\},$$
$$S(M')=\left\{\begin{pmatrix}\mathrm{Id}_{\mathcal{L}(W_2,W_1)} & V(s)\\ 0 & 1\end{pmatrix},\ s\in S\right\}$$

and $V(s)$ is identified with $v(s)$ through the identification of $\mathcal{L}(W_2,W_1)$ with $W_1\otimes W_2^{\vee}$. We obtain the following proposition:

Proposition 4.16. — *The isomorphism of* $\mathbb{C}$*-vector spaces which identifies* $M_{r_1,r_2}(\mathbb{C})$ *and* $M_{r_1r_2,1}(\mathbb{C})$ *induces the isomorphism:*

$$S(M)\cong S(M').$$

4.3.2. The functor it_r

There are many processes to make the slopes of a q-difference module integral, we could use ramification as in [7]. To compute the Stokes operators in Section 3, we did not use ramification. An iteration process appears in the calculations and we will see that is a way to have integral slopes.

Let $r\in\mathbb{N}^*$, we work on the category $\mathcal{E}_r(K,q)$. We define the functor iteration of order r, denoted by it_r. It is a functor from the category $\mathcal{E}_r(K,q)$ to the category $\mathcal{E}_1(K,q^r)$:

$$\begin{array}{rrcl}\mathrm{it}_r: & \mathcal{E}_r(K,q) & \to & \mathcal{E}_1(K,q^r)\\ & M=(K^n,\Phi_A) & \rightsquigarrow & \mathrm{it}_r(M)=(K^n,\Phi_A^r)\\ & F:M\to N & \rightsquigarrow & F:\mathrm{it}_r(M)\to\mathrm{it}_r(N).\end{array}$$

Remark 4.17. — In fact, $\Phi_A^r=\Phi_{\sigma_q^{r-1}(A)\ldots\sigma_q(A)A}$ and the morphisms are invariant by it_r because:

$$\Phi_B\circ F=F\circ\Phi_A\Rightarrow\Phi_B^r\circ F=F\circ\Phi_A^r.$$

Lemma 4.18. — *The functor* it_r *makes the slopes of modules in* $\mathcal{E}_r(K,q)$ *integral.*

Proof. — The functor it_r multiplies the slopes by r. □

Proposition 4.19. — *The functor* it_r *is exact, faithful and tensor compatible.*

We notice that $\hat{\omega}_{z_0}|_{\mathcal{E}_r(K,q)} = \hat{\omega}_{z_0}|_{\mathcal{E}_1(K,q^r)} \circ \mathrm{it}_r$, by tannakian duality, we obtain the following commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & S_r(K,q) \arrow[r] & G_r(K,q) \arrow[r] & G_{p,r}(K,q) \arrow[r] & 1 \\
1 \arrow[r] & S_1(K,q^r) \arrow[r] \arrow[u, "\mathrm{it}_r^*"] & G_r(K,q^r) \arrow[r] \arrow[u, "\mathrm{it}_r^*"] & G_{p,1}(K,q^r) \arrow[r] \arrow[u, "\mathrm{it}_r^*"] & 1.
\end{tikzcd}$$

Remark 4.20. — The morphism it_r^* is not a closed immersion, for example, the module (K, Φ_z) of $\mathcal{E}_r(K,q)$ cannot be a $\mathrm{it}_r(K, \Phi_a)$: otherwise $z = \sigma_q^{r-1}(a) \dots \sigma_q(a)a$ but it is not possible in K.

Nevertheless, $\mathrm{it}_r^*(S_1(K,q^r))$ is a subgroup of $S_r(K,q)$. As a consequence, if we find a morphism $F : \mathrm{gr}(A_U) \to \mathrm{gr}(A_U)$ such that $F(z_0) \in S_1(K,q^r)(\mathrm{it}_r(A_U))$ then $F(z_0) \in S_r(K,q)(A_U)$. We will see in the next paragraph, that this argument enables us to show that our Stokes operators are galoisian.

4.3.3. Stokes operators for two slopes

According to [17],the Stokes operators for a module $M = (K^n, \Phi_{A_U})$ with integral slopes are for all $\bar{c}, \bar{d} \notin \Sigma(A_0)$ the meromorphic morphisms over $\mathbb{C}^*$, $S_{\bar{c},\bar{d}}\hat{F}(A_U)$ and their poles are the q-spirals $[-c;q]$ and $[-d;q]$. We fix $c_0 \notin \Sigma(A_0) \cup \{\bar{\bar{z}}_0\}$ and we put as σ_{q^r}-modules:

$$\forall\, \bar{c} \in E_q\,, \quad \dot{\Delta}_{\bar{c}}(A_U) := Res_{\bar{d}=\bar{c}} \log S_{\bar{c}_0,\bar{d}}\hat{F}(A_U)(z_0)\,.$$

The $\dot{\Delta}_{\bar{c}}$ are called the q-alien derivations (see the introduction of [9] for the explanation of the analogy). According to [9], the q-alien derivation are "Lie-like" morphisms of the Lie algebra $\mathfrak{s}_1$ of S_1.

When the slopes are not integral, we introduced similar notations for Stokes operators in the case of two slopes: $\tilde{S}_{\bar{\bar{c}},\bar{\bar{d}}}F(A_U)$ is the Stokes operator, it is a meromorphic automorphism over $\mathbb{C}^*$ of A_0, its poles are, for $i = 0, \dots, r$, the q^r-spirals $[-cq^{-i+1}; q^r]$ and $[-dq^{-i+1}; q^r]$. And $\tilde{\Delta}_{\bar{\bar{c}}}(A_U) := Res_{\bar{\bar{d}}=\bar{\bar{c}}} \log \tilde{S}_{\bar{c}_0,\bar{d}}\hat{F}(A_U)(z_0)$ is the residue in $\bar{\bar{c}} \in \Sigma(A_0)$.

One non integral slope and one null slope. — Let $M = (K^{r+1}, \Phi_{A_U})$ be a q-difference module whose graded module is $E(r, -d, b^r) \oplus \underline{1}$, the matrix A_U has the following form:

$$A_U = \begin{pmatrix} B & U \\ 0 & 1 \end{pmatrix}, \quad B := \begin{pmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & & & 1 \\ b' z^{-d} & 0 & \dots & 0 \end{pmatrix}$$

where $b' = q^{\frac{-d(r-1)}{2}} b^r \in \mathbb{C}^*$.

Let $h_l = (1, \gamma_l, 0)$, $l \in \mathbb{Z}$, be elements of the formal Galois group $G_{p,r}$, $\gamma_l \in E_q^\vee$. We suppose $\gamma_0 = 1$, the trivial morphism. Moreover, we have $\mathbb{C}^* = U \times q^{\mathbb{R}}$ (U is the set of complex numbers of module 1), we define $\gamma_1 \in E_q^\vee$ such that γ_1 is trivial over U and for all $x \in \mathbb{R}$, $\gamma_1(q^x) = e^{2i\pi x}$. In particular, $\gamma_1(\xi_r) = 1$ and $\gamma_1(q_r) = \xi_r$.

Finally, we define for $l \geqslant 2$, $\gamma_l = (\gamma_1)^l$, that is for all $c \in E_q$, $\gamma_l(c) := (\gamma_1(c))^l$.We notice that for all $l \in \mathbb{Z}$, $\gamma_l(q_r) = \xi_r^l$ and for all $l, l' \in \mathbb{Z}$, $\gamma_l \gamma_{l'} = \gamma_{l+l'}$.

For all $\bar{\bar{c}} \in E_{q^r} \setminus \Sigma(A_0)$, we find a meromorphic isomorphism on $\mathbb{C}^*$ from M_0 to M denoted by:

$$\tilde{S}_{\bar{\bar{c}}} \hat{F}(A_U) = \begin{pmatrix} I_r & \tilde{F}_{\bar{\bar{c}}}(A_U) \\ 0 & 1 \end{pmatrix}.$$

The map $c \in \mathbb{C}^* \mapsto \tilde{F}_{\bar{\bar{c}}_0, \bar{\bar{c}}}(A_U)(z_0)$ is meromorphic on $\mathbb{C}^*$ and the poles are $\Sigma(A_0)$. Thus the residues $\tilde{\Delta}_{\bar{\bar{c}}}(A_U) = Res_{\bar{\bar{d}} = \bar{\bar{c}}} \log \tilde{S}_{\bar{\bar{d}}} \hat{F}(A_U)$ are such that:

$$\tilde{\Delta}_{\bar{\bar{c}}}(A_U) = \begin{pmatrix} 0 & Res_{\bar{\bar{d}} = \bar{\bar{c}}} \tilde{F}_{\bar{\bar{d}}}(A_U) \\ 0 & 0 \end{pmatrix}.$$

And for the q^r-difference module with integral slopes and matrix $\mathrm{it}_r(A_U)$, we have, for all $\bar{\bar{c}} \in E_{q^r} \setminus \Sigma(\mathrm{it}_r(A_0))$,

$$S_{\bar{\bar{c}}} \hat{F}(\mathrm{it}_r(A_U)) = \begin{pmatrix} I_r & F_{\bar{\bar{c}}}(\mathrm{it}_r(A_U)) \\ 0 & 1 \end{pmatrix}$$

and for all $\bar{\bar{c}} \in \Sigma(\mathrm{it}_r(A_0))$,

$$\dot{\Delta}_{\bar{\bar{c}}}(\mathrm{it}_r(A_U)) = \begin{pmatrix} 0 & Res_{\bar{\bar{d}} = \bar{\bar{c}}} F_{\bar{\bar{d}}}(\mathrm{it}_r(A_U)) \\ 0 & 0 \end{pmatrix}.$$

Proposition 4.21. — *For all $\bar{\bar{c}} \in E_{q^r} \setminus \Sigma(\mathrm{it}_r(A_0))$,*

$$\tilde{F}_{\bar{\bar{c}}}(A_U) = \sum_{j=0}^{r-1} \sum_{l=0}^{r-1} \frac{1}{r} \frac{\gamma_l(q_r^{dj})}{\gamma_l(b)} h_l(B) F_{\overline{\overline{c_j}}}(\mathrm{it}_r(A_U)),$$

where $c_j := cq^{-j}$.

Proof. — We remember that $\tilde{F}_{\bar{\bar{c}}}(A_U) = {}^t(f_1, \dots, f_r)$ satisfies the following system (cf. system (2.8)):

$$\begin{cases} \quad f_2 = \sigma_q(f_1) - u_1 \\ \quad f_3 = \sigma_q^2(f_1) - (\sigma_q(u_1) + u_2) \\ \qquad \vdots \\ \quad f_r = \sigma_q^{r-1}(f_1) - (\sigma_q^{r-2}(u_1) + \dots + \sigma_q(u_{r-2}) + u_{r-1}) \\ \sigma_q^r(f_1) = b' z^{-d} f_1 + \sigma_q^{r-1}(u_1) + \dots + \sigma_q(u_{r-1}) + u_r \end{cases}$$

and $f_1 = \frac{g}{\theta^d_{q^r,c}}$ with g holomorphic on $\mathbb{C}^*$.

We have

$$\mathrm{it}_r(B) = z^{-d} \begin{pmatrix} b' & 0 & \dots & 0 \\ 0 & b'q^{-d} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & b'q^{-(r-1)d} \end{pmatrix}$$

and

$$\mathrm{it}_r(A_U) = \begin{pmatrix} \mathrm{it}_r(B) & V \\ 0 & 1 \end{pmatrix}, \quad V = \sum_{k=0}^{r-2} \sigma_q^{r-1}(B) \dots \sigma_q^{k+1}(B) \sigma_q^k(U) + \sigma_q^{r-1}(U).$$

We have $\Sigma(\mathrm{it}_r(A_U)) = \{c \in \mathbb{C}^* \mid \exists\, i = 0, \dots, r-1,\ c^d = b' q^{-\,\mathrm{id}}\} \mod q^r$. Moreover, we easily verify that

$$\sigma_q(F) - BF = U \Rightarrow \sigma_q^r(F) - \mathrm{it}_r(B) F = V.$$

Let us write, $F_{\bar{\bar{c}}}(\mathrm{it}_r(A_U)) = {}^t(f_{1,\bar{\bar{c}}}, f_{2,\bar{\bar{c}}}, \dots, f_{r,\bar{\bar{c}}})$, by the uniqueness of the $f_{i,\bar{\bar{c}}}$, we notice that:

$$f_1 = f_{1,\bar{\bar{c}}},\ f_2 = f_{2,\overline{\overline{c_1}}},\ \dots,\ f_r = f_{r,\overline{\overline{c_{r-1}}}}.$$

Thus,

$$\tilde{F}_{\bar{\bar{c}}}(A_U) = E_{1,1} F_{\bar{\bar{c}}}(\mathrm{it}_r(A_U)) + E_{2,2} F_{\overline{\overline{c_1}}}(\mathrm{it}_r(A_U)) + \dots + E_{r,r} F_{\overline{\overline{c_{r-1}}}}(\mathrm{it}_r(A_U)),$$

where $E_{i,j}$ is the elementary matrix with zeroes everywhere except at the place (i,j).

According to Lemma 4.5:

$$\sum_{l=0}^{r-1} \frac{1}{r} \gamma_l(q_r^{dj}) \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \gamma_l(q_r)^{-d} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \gamma_l(q_r)^{-(r-1)d} \end{pmatrix} = E_{j+1,j+1}.$$

As

$$h_l(B) = \gamma_l(b) \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \gamma_l(q_r)^{-d} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \gamma_l(q_r)^{-(r-1)d} \end{pmatrix},$$

we obtain the formula. □

Lemma 4.22. — *The only vector spaces which are stable under $G_p(E(r,-d,b^r))$ are $\{0\}$ and $\mathbb{C}^r$.*

Proof. — Let us suppose that $X = {}^t(x_1, x_2, \dots, x_r) \in \mathfrak{s}(M) \subset \mathbb{C}^r$ then for all $l \in \{0, \dots, r-1\}$,

$$\begin{pmatrix} 1 & & & \\ & \xi_r^l & & \\ & & \ddots & \\ & & & \xi_r^{l(r-1)} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \end{pmatrix} = \begin{pmatrix} x_1 \\ \xi_r^l x_2 \\ \vdots \\ \xi_r^{(r-1)l} x_r \end{pmatrix} \in \mathfrak{s}(M),$$

because r and d are coprime (ξ_r^d is a primitive rth root of the unity).

According to Lemma 4.5, for all $j \in \{0, \dots, r-1\}$ we have:

$$\sum_{j=0}^{r-1} \xi_r^{r-jl} \begin{pmatrix} x_1 \\ \xi_r^l x_2 \\ \vdots \\ \xi_r^{(r-1)l} x_r \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ r x_{j+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathfrak{s}(M).$$

And thanks to the permutation matrices

$$\begin{pmatrix} 0 & \alpha_0^{-1} & & \\ \vdots & & \ddots & \\ 0 & & & \alpha_{r-2}^{-1} \\ \alpha_0 \dots \alpha_{r-2} & & & 0 \end{pmatrix}^k$$

for all $k \in \{1, \dots, r\}$, we have $x_j E_k \in \mathfrak{s}(M)$ where E_k is the elementary vector whose coordinates are equal to zero except the kth which is 1.

As a conclusion, if there exists $X \neq 0$ in $\mathfrak{s}(M)$ then $\mathfrak{s}(M) = \mathbb{C}^r$. □

We can extend the previous lemma to the case of an arbitrary indecomposable module.

Lemma 4.23. — *The vector spaces which are stable by $G_p(E(r,-d,b^r)\otimes U_m)$ are $\{0\}$ and $\mathbb{C}^{rm}$.*

Proof. — An element of $G_p(E(r,-d,b^r)\otimes U_m)$ is a matrix $C\otimes W_m^\lambda$ where C is in $G_p(E(r,-d,b^r))$ and $\lambda\in\mathbb{C}$. Let V be a vector space stable by $G_p(E(r,-d,b^r)\otimes U_m)$ and $X\in V$ such that

$$X=\begin{pmatrix}X_1\\ \vdots\\ X_m\end{pmatrix}\in\mathbb{C}^{rm},\ X_i\in\mathbb{C}^r.$$

Let us take $\lambda=0$. Then,

$$C\otimes I_m X=\begin{pmatrix}CX_1\\ \vdots\\ CX_m\end{pmatrix}\in V\,.$$

As in the previous proof, we show that for $j=1,\dots,r$,

$$\begin{pmatrix}E_{j,j}X_1\\ \vdots\\ E_{j,j}X_m\end{pmatrix}\in V\,.$$

Let us take $\lambda=1$, then,

$$\begin{pmatrix}E_{j,j}X_1+E_{j,j}X_2\\ \vdots\\ E_{j,j}X_m\end{pmatrix}\in V\,.$$

We prove the following by induction: for all $k=1,\dots,m$,

$$\begin{pmatrix}E_{j,j}X_k\\ \vdots\\ E_{j,j}X_m\\ 0\\ \vdots\\ 0\end{pmatrix}\in V\,.$$

Consequently, for all j, for all $x \in \mathbb{C}$

$$\begin{pmatrix} E_j x \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} \in V \quad \text{and} \quad \begin{pmatrix} \vdots \\ 0 \\ E_j x \\ 0 \\ \vdots \end{pmatrix} \in V\,.$$

By the permutation matrices, $X = 0$ is the unique vector of V or $V = \mathbb{C}^{rm}$. □

Proposition 4.24. — *For all $\bar{\bar{c}} \in E_{q^r} \setminus (\Sigma(\mathrm{it}_r(A_0)) \cup \{\bar{\bar{z}}_0\})$,*

$$\tilde{\Delta}_{\bar{\bar{c}}}(A_U) = \sum_{j=0}^{r-1}\sum_{l=0}^{r-1} \frac{1}{r}\frac{\gamma_l(q_r^{dj})}{\gamma_l(b)}\; h_l(A_0).\dot{\Delta}_{\overline{\overline{cq^{-j}}}}(\mathrm{it}_r(A_U)) \in \mathfrak{s}(M)\,.$$

Proof. — It is a consequence of the Proposition 4.21 with $z = z_0$ and taking the residue in $\bar{\bar{c}}$, and because $S_{\bar{\bar{c}},\bar{\bar{d}}}\hat{F}(\mathrm{it}_r(A_U)) \in S(M)$. □

Remark 4.25. — This formula remains the same if we replace A_U by the matrix in standard form $\begin{pmatrix} 1 & V \\ 0 & B^\vee \end{pmatrix}$, $V = -UB^\vee$, of the dual of M.

Any object of the tannakian category generated by the q-difference module M, denoted by $\langle M\rangle$, belongs in particular to the category $\mathcal{E}_r(K)$, so the functor it_r makes the slopes integral. Thus, we can define for any object $N = (K^n, \Phi_A)$ of $\langle M\rangle$, an element of the Lie algebra $\mathfrak{s}(N)$:

$$\forall\, \bar{\bar{c}} \in E_{q^r} \setminus (\Sigma(\mathrm{it}_r(\mathrm{gr}\,A)) \cup \{\bar{\bar{z}}_0\})\,,$$
$$\tilde{\Delta}_{\bar{\bar{c}}}(A) := \sum_{j=0}^{r-1}\sum_{l=0}^{r-1} \frac{1}{r}\frac{\gamma_l(q_r^{dj})}{\gamma_l(b)}\; h_l(\mathrm{gr}\,A).\dot{\Delta}_{\overline{\overline{cq^{-j}}}}(\mathrm{it}_r(A)) \in \mathfrak{s}(N).$$

Proposition 4.26. — *The operators $\tilde{\Delta}_{\bar{\bar{c}}}$ are "Lie-like" morphisms, in the Lie algebra $\mathfrak{s}_M$. Moreover,*

$$\tilde{\Delta}_{\bar{\bar{c}}}(\,\cdot\,) := \sum_{j=0}^{r-1}\sum_{l=0}^{r-1} \frac{1}{r}\frac{\gamma_l(q_r^{dj})}{\gamma_l(b)}\; h_l.\dot{\Delta}_{\overline{\overline{cq^{-j}}}}(\mathrm{it}_r(\,\cdot\,))$$

and we have the inversion formula:

$$\dot{\Delta}_{\bar{\bar{c}}}(\mathrm{it}_r(\,\cdot\,)) = \sum_{j=0}^{r-1}\sum_{l=0}^{r-1} \frac{1}{r}\frac{\gamma_l(q_r^{dj})}{\gamma_l(b)}\; h_l.\tilde{\Delta}_{\overline{\overline{cq^{j}}}}(\,\cdot\,)$$

Remark 4.27. — I have no conceptual explanation for the complicated calculations that follow, nor for the apparent "self-duality" in the above formulas.

Proof. — The operators $\dot{\Delta}_{\overline{\overline{cq^{-j}}}}(\mathrm{it}_r(\cdot))$ are "Lie-like" morphisms, in the Lie algebra $\mathfrak{s}_M$, that is to say

$$\dot{\Delta}_{\overline{\overline{cq^{-j}}}}(\mathrm{it}_r(A\otimes B)) = \dot{\Delta}_{\overline{\overline{cq^{-j}}}}(\mathrm{it}_r(A))\otimes 1 + 1\otimes \dot{\Delta}_{\overline{\overline{cq^{-j}}}}(\mathrm{it}_r(B)).$$

The formula linking $\tilde{\Delta}_{\bar{\bar{c}}}(\cdot)$ and $\dot{\Delta}_{\overline{\overline{cq^{-j}}}}(\mathrm{it}_r(\cdot))$ is linear so the morphisms $\tilde{\Delta}_{\bar{\bar{c}}}(\cdot)$ are "Lie-like" morphisms in the Lie algebra $\mathfrak{s}_M$.

Let us prove the inversion formula:

$$\begin{aligned}
&\sum_{j'=0}^{r-1}\sum_{l'=0}^{r-1}\frac{1}{r}\frac{\gamma_{l'}(q_r^{dj'})}{\gamma_{l'}(b)}\,h_{l'}.\tilde{\Delta}_{\overline{\overline{cq^{j'}}}}(\cdot)\\
&=\sum_{j'0}^{r-1}\sum_{l'=0}^{r-1}\frac{1}{r}\frac{\gamma_{l'}(q_r^{dj'})}{\gamma_{l'}(b)}\,h_{l'}.\sum_{j=0}^{r-1}\sum_{l=0}^{r-1}\frac{1}{r}\frac{\gamma_{l}(q_r^{dj})}{\gamma_{l}(b)}\,h_{l}.\dot{\Delta}_{\overline{\overline{cq^{j'-j}}}}(\mathrm{it}_r(\cdot))\\
&=\sum_{j=0}^{r-1}\sum_{j'=0}^{r-1}\frac{1}{r^2}\sum_{l=0}^{r-1}\sum_{l'=0}^{r-1}\frac{\gamma_l(q_r^{dj})\gamma_{l'}(q_r^{dj'})}{\gamma_{l+l'}(b)}h_lh_{l'}.\dot{\Delta}_{\overline{\overline{cq^{j'-j}}}}(\mathrm{it}_r(\cdot))\\
&=\sum_{j=0}^{r-1}\sum_{j'=0}^{r-1}\frac{1}{r^2}\left(\sum_{k=0}^{r-1}\sum_{l=0}^{k}\frac{\gamma_l(q_r^{dj})\gamma_{k-l}(q_r^{dj'})}{\gamma_k(b)}h_k\right.\\
&\qquad\qquad\left.+\sum_{k=r}^{2r-2}\sum_{l=k-r+1}^{r-1}\frac{\gamma_l(q_r^{dj})\gamma_{k-l}(q_r^{dj'})}{\gamma_k(b)}h_k\right).\dot{\Delta}_{\overline{\overline{cq^{j'-j}}}}(\mathrm{it}_r(\cdot))\\
&=\sum_{j=0}^{r-1}\sum_{j'=0}^{r-1}\frac{1}{r^2}\left(\sum_{k=0}^{r-1}\sum_{l=0}^{k}\frac{\gamma_l(q_r^{dj})\gamma_{k-l}(q_r^{dj'})}{\gamma_k(b)}h_k\right.\\
&\qquad\qquad\left.+\sum_{k=0}^{r-1}\sum_{l=k+1}^{r-1}\frac{\gamma_l(q_r^{dj})\gamma_{k-l}(q_r^{dj'})}{\gamma_k(b)}h_k\right).\dot{\Delta}_{\overline{\overline{cq^{j'-j}}}}(\mathrm{it}_r(\cdot))\\
&\qquad\qquad\text{because } h_k(A_0)=h_{k-r}(A_0)\\
&=\sum_{j=0}^{r-1}\sum_{j'=0}^{r-1}\frac{1}{r^2}\left(\sum_{k=0}^{r-1}\sum_{l=0}^{r-1}\frac{\gamma_l(q_r^{dj})\gamma_{k-l}(q_r^{dj'})}{\gamma_k(b)}h_k\right).\dot{\Delta}_{\overline{\overline{cq^{j'-j}}}}(\mathrm{it}_r(\cdot))\\
&=\sum_{j=0}^{r-1}\sum_{j'=0}^{r-1}\frac{1}{r^2}\left(\sum_{k=0}^{r-1}\frac{\gamma_k(q_r^{dj'})}{\gamma_k(b)}\sum_{l=0}^{r-1}\gamma_l(q_r^{d(j-j')})\,h_k\right).\dot{\Delta}_{\overline{\overline{cq^{j'-j}}}}(\mathrm{it}_r(\cdot))
\end{aligned}$$

$$= \sum_{j=0}^{r-1} \frac{1}{r} \left(\sum_{k=0}^{r-1} \frac{\gamma_k(q_r^{dj})}{\gamma_k(b)} h_k \right) \dot{\Delta}_{\overline{\overline{c}}}(\mathrm{it}_r(\,\cdot\,))$$

$$\text{because } \textstyle\sum_{l=0}^{r-1} \gamma_l(q_r^{d(j-j')}) = 0$$

$$\text{even if } j = j' \text{ according to Lemma 4.5}$$

$$= \sum_{k=0}^{r-1} \frac{1}{r} \frac{1}{\gamma_k(b)} \left(\sum_{j=0}^{r-1} \gamma_k(q_r^{dj}) \right) h_k.\dot{\Delta}_{\overline{\overline{c}}}(\mathrm{it}_r(\,\cdot\,))$$

$$= \dot{\Delta}_{\overline{\overline{c}}}(\mathrm{it}_r(\,\cdot\,)) \qquad \text{because } \sum_{j=0}^{r-1} \gamma_k(q_r^{dj}) = 0 \text{ except for } k = 0 \qquad \square$$

In the previous section, we computed the Stokes operators for a module with two non integral slopes, of the form $M = (K^{r_1+r_2}, \Phi_{A_U})$, where:

$$A_U = \begin{pmatrix} B_1 & U \\ 0 & B_2 \end{pmatrix}$$

and for $i = 1, 2$, B_i is the matrix associated with the irreducible module $E(r_i, d_i, b_i^r)$ of rank r_i and slope $\frac{d_i}{r_i}$ such that $\frac{d_1}{r_1} < \frac{d_2}{r_2}$. We notice that it was equivalent to compute those of the module $M' = (K^{r_1 r_2+1}, \Phi_{A'_{U'}})$ where:

$$A'_{U'} = \begin{pmatrix} B_1 \widehat{\otimes} B_2^{\vee} & U' \\ 0 & 1 \end{pmatrix} \quad U' = \widehat{UB_2^{-1}}.$$

We saw in Proposition 4.16 that $S(M)$ and $S(M')$ can be identified, consequently, we just have to prove the Stokes operators associated with M' are galoisian.

We can generalize the previous results to the module M' whose graded module is $E(r_1, d_1, b_1^{r_1}) \otimes E(r_2, -d_2, b_2^{-r_2}) \oplus \underline{1}$. According to Proposition 1.14, formula are nearly the same. We obtain the following corollary:

Corollary 4.28. — *The Stokes operators of the Galois group of a module with two non integral slopes are galoisian.*

The residues $\tilde{\Delta}_{\overline{\overline{c}}}(A_U)$ are in the Lie algebra $\mathfrak{s}(M)$. It will give us generators of the Stokes group associated with the module M.

4.4. Density theorem

In the case of integral slopes, Ramis and Sauloy proved the following theorem:

Theorem 4.29 ([10, Theorem 3.5]). — *The formal Galois group $G_{p,1}$ and the Stokes subgroup associated with q-alien derivations $\dot{\Delta}$ generate a Zariski dense subgroup of the Galois group of q-difference modules with integral slopes G_1.*

Here, we generalize this theorem to the Galois group G_p taking inspiration from the results of the previous paragraph, where the analogue of the q-alien derivations seems to be the $\dot{\Delta}_{\bar{\bar{c}}} \circ \mathrm{it}_r$.

Theorem 4.30. — *Let $r \in \mathbb{N}^*$. The subgroup of $S_r(K,q)$ associated with the residues $\dot{\Delta}_{\bar{\bar{c}}} \circ \mathrm{it}_r$ and the formal Galois group $G_{p,r}(K,q)$ generate a Zariski-dense subgroup of $G_r(K,q)$.*

Remark 4.31. — If $r = 1$, we recover Theorem 4.29.

Proof. — We use a density theorem of Chevalley, the same as the proof of Theorem 3.5 of [10], it may be formulated by:

Theorem. — *Let H be a subgroup of $G_r(K,q)$. For H to generate Zariski-dense subgroup of $G_r(K,q)$, it is sufficient that for each object M of $\mathcal{E}_r(K,q)$ and each line D of $\hat{\omega}_{z_0}(M)$ invariant under the action of each element of H, then D is invariant under the action of $G_r(K,q)$.*

We take $H = G_{p,r}(K,q) \times \exp(\{\dot{\Delta}_{\bar{\bar{c}}} \circ \mathrm{it}_r \mid c \in \mathbb{C}^*\})$. Let (K^n, Φ_{A_U}) be an object of $\mathcal{E}_r(K,q)$. Thanks to the canonical filtration by the slopes, we may suppose that A_U has the following form:

$$A_U = \begin{pmatrix} A_1 & & & \\ & A_2 & & U_{i,j} \\ 0 & & \ddots & \\ & & & A_s \end{pmatrix}$$

the matrices A_i correspond to pure isoclinic modules of slopes k_i/r, $k_i \in \mathbb{Z}$, such that $k_1 < k_2 < \cdots < k_s$. The matrix of the associated graded module is A_0.

Let D be a line of $\hat{\omega}_{z_0}(M) = \mathbb{C}^n$ and X a generator of this line, we write it by blocks:

$$X = \begin{pmatrix} X_1 \\ \vdots \\ X_s \end{pmatrix}.$$

The line D is supposed to be invariant under the action of $G_{p,r}(K,q)$, which means that for each matrix $B \in G_{p,r}(K,q)$, BX and X are colinear.

To be invariant by $\dot{\Delta}_{\bar{\bar{c}}} \circ \mathrm{it}_r$ means that $\dot{\Delta}_{\bar{\bar{c}}} \circ \mathrm{it}_r(A_U)\, X = 0$ because $\dot{\Delta}_{\bar{\bar{c}}} \circ \mathrm{it}_r$ is in the Lie algebra of $S(M)$. We have to prove that $D \in \mathfrak{s}_r(K,q)$, $D(M)X = 0$.

The slopes are distinct, the action of $\mathbb{C}^*$ of the formal Galois group implies that only one X_i corresponding to a unique slope μ_i is non zero. Indeed,for all $t \in \mathbb{C}^*$,

$$\begin{pmatrix} t^{k_1} X_1 \\ \vdots \\ t^{k_s} X_s \end{pmatrix}$$

must be colinear to X.

We proved in Lemma 4.23 that the vector spaces stabilized by the formal Galois group of an indecomposable module of rank n and of non integral slope are $\{0\}$ and $\mathbb{C}^n$. A module with non inegral slopes is a direct sum of indecomposable modules, the same proof shows that if μ_i is non integral then $X_i = 0$. In this case, we are done.

On the other hand, if μ_i is integral, then $A_i = z^{\mu_i} A'_i$, Ramis and Sauloy proved in [10, Lemma 2.2] that X_i is an eigenvector of A'_i. Then, there exists λ an eigenvalue of A'_i such that $A_0 X = \lambda z^{\mu_i} X$.

We may suppose that $i = s$ because if we denote by M' the submodule of M whose slopes are smaller than k_i/r and of rank $n' = n_1 + \cdots + n_i$, the inclusion given by the matrix $\mathrm{Inc} = \left(\begin{smallmatrix} I_{n'} \\ 0 \end{smallmatrix}\right)$ is a morphism of q-difference module. By functoriality, the vector $X' = \mathrm{Inc}\, X$ and the matrix A' corresponding to M' verify the same hypothesis as X and A_U.

Then, we have an analytic morphism:

$$\lambda z^{\mu_s} \xrightarrow{X} A_0 \,.$$

Moreover, there exists a unique formal morphism tangent to identity $\hat{F} : A_0 \to A_U$ (that is to say in $\mathfrak{G}_{n_1,\ldots,n_s}(\hat{K})$), hence a formal morphism $G = \hat{F}X : \lambda z^{\mu_s} \to A_U$. If we prove that this morphism is analytic then we will have the following commutative diagram for all $D \in S$:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{G_0(z_0)=X} & \mathbb{C}^n \\ {\scriptstyle D(\lambda z^{\mu_s})}\downarrow & & \downarrow{\scriptstyle D(A_U)} \\ \mathbb{C} & \xrightarrow{G_0(z_0)} & \mathbb{C}^n \end{array}$$

where G_0 is the graded morphism associated with G. But $D(\lambda z^{\mu_s}) = 0$ because the module is pure, as a consequence $D(A_U)X = 0$.

Now, we just have to prove that $G = \hat{F}X$ is analytic. By hypothesis $\dot{\Delta}_{\bar{\bar{c}}} \circ \mathrm{it}_r(A_U)\, X = 0$. The functor it_r does not change morphisms, by applying it on G we have:

$$\mathrm{it}_r(\lambda z^{\mu_s}) \xrightarrow{X} \mathrm{it}_r(A_0) \xrightarrow{\hat{F}} \mathrm{it}_r(A_U).$$

Now, the slopes are integral and we have the same hypothesis as Lemma 3.6 of [10]. This lemma shows that G is an analytic morphism. □

Corollary 4.32. — *Let $M = (K^n, \Phi_{A_U})$, with $A_U = \left(\begin{smallmatrix} B_1 & U \\ 0 & B_2 \end{smallmatrix}\right)$ where for $i = 1, 2$, B_i is the matrix associated with the irreducible module $E(r_i, d_i, b_i^r)$. The $\tilde{\Delta}_{\bar{\bar{c}}}$ and their conjugates by the action of the formal Galois group $G_{p,M}$ generate a Zariski-dense Lie subalgebra of $\mathfrak{s}_M$.*

Proof. — It is a consequence of the previous theorem and Proposition 4.26. □

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