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## Group of invariance of a relativistic supermultiplet theory (\*)

par

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Recently Sakita [1], Gürsey and Radicati [2] [4] and Pais [3] [4] have proposed a generalization of Wigner supermultiplet theory [5] for the nucleus to baryons and mesons [6]. This raises the question: what is a relativistic supermultiplet theory ? In this paper we shall consider only the problem of defining the invariance group  $G$  for such a theory [7].

We denote by  $P$  the connected Poincaré group. It is the semi-direct product  $P = T \times L$  where  $T$  is the translation group and  $L$  is the homogeneous Lorentz group.

CONDITION 1. — The invariance group  $G$  of a relativistic theory contains  $P$ . We shall not discuss here the discrete invariance  $P, C, T$ , so we shall add.

CONDITION 2. —  $G$  is a connected topological group (with  $P$  as topological subgroup) [8].

Invariance under  $G$  is considered as the largest symmetry for strong coupling physics [1] [2] [3] [4]. The particles of a supermultiplet have

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the same mass and for a given energy momentum  $p$ , all possible states (spin, charges, etc.) of these particles form a finite dimensional Hilbert space which is the space of an irreducible unitary representation of a compact group  $S_p$ , the « little group » of  $p$ . From the classical Wigner analysis it is easy to translate this as conditions on  $G$ .

CONDITION 3. — The translation group  $T$  is invariant subgroup of  $G$ . The action of  $G$  on  $T$  (by its inner automorphisms) preserves the Minkowski metric and the little group (mathematicians say stabiliser or isotopy group) in  $G/T$  of a time-like translation  $a \in T$  is a compact group  $S$ .

THEOREM. — If a group  $G$  satisfies condition 1, 2 and 3, then  $G/T$  is a direct product of  $H \times L$ .

*Proof :*

The condition 3 implies that for every  $g \in G$ ,  $a \in T$  and its transformed  $g(a) = gag^{-1}$  have same Minkowski length :  $a \cdot a = g(a) \cdot g(a)$ . In the dual of  $T$  [i. e., the four dimensional vector space of energy momentum] the orbits of  $G$  are the connected sheets of mass hyperboloid. Denote  $f: G \rightarrow \text{Aut } T$ , the homomorphism of  $G$  which describes its action, by inner automorphisms, on its invariant subgroup  $T$ . It is easy to prove [9] that the connected group of continuous automorphisms of  $G$  which preserves the Minkowski metric is  $L$ . So the image of  $f$  is  $L$ :  $\text{Im } f = L$ . Since  $T$  is abelian  $T \subset K = \text{Ker } f$ , the kernel of  $f$ , and  $f$  is factorized into  $g \circ p$ , where  $p: G \rightarrow G/T$  and  $g: G/T \xrightarrow{g} L$ . The restriction of  $g$  to the subgroup  $L = P/T \subset G/T$ , is an identity transformation. By definition of the semi-direct product, therefore,  $G/T$  is the semi-direct product  $H \times L$  where  $H = \text{Ker } g = K/T$ . Furthermore, by definition of  $H = \text{Ker } g$ ,  $H$  is an invariant subgroup of every stabilizer (little group)  $S$  for any  $a$ . For  $a$  time-like  $a$ ,  $S_a$  is compact, this implies that its invariant subgroup  $H$  is compact, and from a theorem of Iwasawa [10]  $H$  compact and  $G/T$  connected implies that it is a central extension of kernel  $H$ . As we have seen, it is also the semi-direct product  $H \times L$ . Hence it is a direct product:

$$G/T = H \otimes L. \quad (1)$$

The proof of the theorem also gives conditions on the little group  $S$ , for a time-like translation. Indeed it must be isomorphic to the direct product  $H \otimes R$  where  $R$  is the three dimensional rotation group. Of course, this excludes  $SU(6)$  or any simple Lie group for  $S$ .

A possible way to have a relativistic theory with supermultiplets of particles classified by irreducible unitary representation of  $SU(6)$ , is to find a (connected Lie) group  $\bar{G}$  with irreducible unitary representations characterized by  $m > 0$  and those of  $SU(6)$ , and such that  $\bar{G} \supset \bar{P}$ , the covering of the Poincaré group. We proceed now to build such a group  $\bar{G}$ .

Among all subgroups of the linear group with enumerable dimension let us look for the smallest group  $H$  such that:

$$SU(6) \subset H, SL(2, C) \subset H, SU(6) \cap SL(2, C) = SU(2) \quad (2)$$

where  $SU(2)$  is the covering of  $R$  and  $SL(2, C)$  the covering of  $L$ . The smallest group  $H$  is the intersection of all groups which satisfy (2).  $SL(6, C)$  is one of them, so  $SU(6) \subset H \subset SL(6, C)$ . But  $SU(6)$  is maximal subgroup of  $SL(6, C)$ ; this implies:  $H = SL(6, C)$ . The elements  $x \in H$  are  $6 \times 6$  matrices with determinant 1. They can be decomposed in a unique way into the product  $x = hu$  where  $h$  is a  $6 \times 6$  hermitian positive matrix of determinant 1 and  $u$  is a  $6 \times 6$  unitary matrix with determinant 1. The matrices  $u$  generates  $SU(6)$  and the set  $\{h\}$  of matrices  $h$  is the homogenous space  $SL(6, C)/SU(6)$ . The smallest Lie group generated by  $\{h\}$  is the additive group of the  $6 \times 6$  hermitian matrices; it is the 36 real parameter simply connected abelian Lie group. We shall denote it  $T_{36}$ .

The group  $\bar{G}$  is the semi-direct product of  $SL(6, C)$  by  $T_{36}$  with the action:  $h \rightarrow xhx^*$  (indeed this contains the action of  $SL(2, C)$  on  $T_4$ , hence  $\bar{G} \supset \bar{P}$ ). The orbit of  $\bar{G}$  on  $T_{36}$  are characterized by  $\det h$  and the sign of the eigenvalues. If  $h > 0$ , one can take as representative  $h = m1$ . Its little group is that of the matrices with determinant 1 such that  $xhx^* = mxm^* = m1$ ; it is  $SU(6)$ .

Hence the smallest connected Lie group which contains  $P$  and has unitary irreducible linear representations characterized by:  $m > 0$  and the unitary representations of  $SU(6)$ , is the group  $G$  we just defined. It is a 106 parameter Lie group [11]. As we shall explain elsewhere the use of such  $G$  as invariance group for a relativistic supermultiplet theory of elementary particle is possible, but we do not like it.

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- [4] F. GÜRSEY, A. PAIS and L. RADICATI, *Phys. Rev. Letters*, t. 13, 1964, p. 299.
- [5] E. P. WIGNER, *Phys. Rev.*, t. 51, 1937, p. 106.
- [6] More than twenty papers on this subject have been published or mimeographed.
- [7] For physicists who may find our simple rigorous proof too abstract we are writing a more detailed paper on the subject in terms of Lie algebra. We will also show in this paper that the invariance properties of such a theory cannot be reduced to the study of a group.
- [8] This is a purely technical condition; with the work of E. C. ZEEMAN, *J. Math. Phys.*, t. 5, 1964, p. 491, we can obtain that  $G/T$  is the semi-direct product  $H \times L$  without condition 2.
- [9] Indeed ZEEMAN [8], has proven it without the assumptions of continuity and automorphisms.
- [10] K. IWASAWA, *Ann. Math.*, t. 50, 1949, p. 507.
- [11] This group has been mentioned at the end of reference [1].

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