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The motion of a falling particle in a Schwarzschild field

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In the wake of the problem of the motion of the perihelion of a planet [*1-3*] in the General Theory of Relativity it is interesting to consider the simpler but nevertheless instructive problem of a falling particle in a Schwarzschild field. Whittaker [*4*] discusses a similar rectilinear motion and gives expressions for the velocity and acceleration of a particle in terms of an arbitrary constant. We shall, however, obtain an expression for the velocity of such a falling particle which is projected with an initial velocity u and show that it reduces to the Newtonian formula in the limit $c \to \infty$ or what is the same thing, as $S \to 0$ where $S = \frac{2GM}{c^2}$ is the Schwarzschild radius. On the otherhand, in a differential region round a point of interest where the gravitational field could be treated as uniform, the second approximation wherein we neglect terms containing S^2 but retain those containing S, yields a motion which follows from the Special Theory of Relativity, thus providing an illustration of Einstein's Principle of Equivalence [*2*]. In particular, a further approximation leads to the well-known Hyperbolic motion [*5-6*].

Assuming that the particle is « falling » along the line $\theta = 0$ (or along φ = constant in the plane $\theta = \pi/2$ to keep correspondence with the problem of perihelion motion), the Schwarzschild metric becomes,

$$ds^2 = -\frac{dr^2}{1 - S/r} + (1 - S/r)c^2 d\tau^2 \qquad (1)$$

where, for convenience in notation we denote the coordinate time by τ rather than t which will denote physical time. Thus, if dl and dt are respectively the elements of length and time, we have [*4*],

$$dl^2 = \frac{dr^2}{1 - S/r}; \qquad dt^2 = (1 - S/r)d\tau^2 \tag{2}$$

It is clear that in the limit $S \to 0$, we can take $l = r$. With $l = l_0$ at $r = r_0$, we have

$$\begin{aligned} x = l_0 - l &= \int_r^{r_0} \frac{dr}{\sqrt{1 - S/r}}; \qquad r_0 > r > S \\ &= \frac{1}{\sqrt{1 - S/\xi}}(r_0 - r); \qquad r \leqq \xi \leqq r_0 \end{aligned} \tag{3}$$

from the mean value theorem of the integral calculus. If we regard S as so small that its square may be neglected, we can obviously write, approximately

$$r_0 - r = x\left(1 - \frac{S}{2\xi}\right) \tag{4}$$

The equations of the geodesic reduce, with $\theta = 0$, to

$$\frac{d^2r}{ds^2} + \frac{1}{2}\frac{d\lambda}{dr}\left(\frac{dr}{ds}\right)^2 + c^2\frac{e^{\nu-\lambda}}{2}\left(\frac{d\nu}{dr}\right)\left(\frac{d\tau}{ds}\right)^2 = 0 \tag{5}$$

$$\frac{d^2\tau}{ds^2} + \frac{d\nu}{ds}\frac{d\tau}{ds} = 0 \tag{6}$$

where

$$e^\nu = e^{-\lambda} = 1 - S/r \tag{7}$$

The integral of (6) is

$$\frac{d\tau}{ds} = \frac{k}{1 - S/r}; \qquad k \text{ a constant} \tag{8}$$

and one can check that (1) already provides an integral of (5). Thus from (1) and (8) we obtain

$$\left(\frac{dr}{d\tau}\right)^2 = c^2(1 - S/r)^2\left\{1 - \frac{1 - S/r}{c^2k^2}\right\} \tag{9}$$

If v is the velocity of the particle, we have, from (2)

$$\begin{aligned} v^2 = \left(\frac{dl}{dt}\right)^2 &= \frac{1}{(1 - S/r)^2}\left(\frac{dr}{d\tau}\right)^2 \\ &= c^2\left\{1 - \frac{1 - S/r}{c^2k^2}\right\}. \end{aligned} \tag{10}$$

Let the particle have the initial velocity u (i. e., at $t = 0$) at $r = r_0$. Evaluating the constant k and simplifying, we obtain

$$\frac{1 - u^2/c^2}{1 - v^2/c^2} = \frac{1 - S/r_0}{1 - S/r} \tag{11}$$

giving an exact expression for the velocity of the falling particle.

Bearing in mind that $S = \dfrac{2GM}{c^2}$, we can rewrite (11) as

$$v^2 = \frac{u^2(1 - S/r) + 2GM\left(\frac{1}{r} - \frac{1}{r_0}\right)}{1 - S/r_0} \tag{12}$$

which clearly goes over into the Newtonian formula (for $S \to 0$)

$$v^2 = u^2 + 2GM\left(\frac{1}{l} - \frac{1}{l_0}\right) \tag{13}$$

since $r \to l$ as $S \to 0$.

We now proceed to an approximation of (11) which yields the motion according to the Special Theory of Relativity. We recall [5] that the motion of a freely falling particle in a uniform field is the well known Hyperbolic motion given by

$$(x + c^2/g)^2 - c^2t^2 = c^4/g^2 \tag{14}$$

which is the solution of the differential equation

$$\frac{d}{dt}(mv) = \frac{d}{dt}\left(\frac{m_0 v}{\sqrt{1 - v^2/c^2}}\right) = m_0 g \tag{15}$$

If we drop the stipulation that the force on the particle is constant and equal to $m_0 g$, we should really have

$$\frac{d}{dt}\left(\frac{m_0 v}{\sqrt{1 - v^2/c^2}}\right) = mg = \frac{m_0 g}{\sqrt{1 - v^2/c^2}} \tag{16}$$

Simplifying, we get

$$\frac{1}{1 - v^2/c^2}\,\frac{dv}{dt} = g \tag{17}$$

Integration of (17) yields

$$\frac{1}{2}\log\frac{1 + v/c}{1 - v/c} = gt/c + \text{constant}.$$

If $v = u$ at $t = 0$, we obtain, on evaluating the constant

$$v/c = \frac{\sinh gt/c + u/c \cosh gt/c}{\cosh gt/c + u/c \sinh gt/c} \tag{18}$$

Writing $v = \frac{dx}{dt}$ and with $x = 0$ at $t = 0$, an integration of (18) immediately gives

$$x = c^2/g \log (\cosh gt/c + u/c \sinh gt/c). \tag{19}$$

We shall now arrive at (19) as an approximation from (11). On taking logarithms, we have

$$\log \frac{1 - u^2/c^2}{1 - v^2/c^2} = \log (1 - S/r_0) - \log (1 - S/r)$$

$$\approx S\left(\frac{1}{r} - \frac{1}{r_0}\right), \text{ neglecting } S^2 \tag{20}$$

Consider now a differential region at $r = r_0$ wherein second and higher powers of $\frac{r_0 - r}{r_0}$ may be regarded as negligible. We then have

$$\frac{1}{r} = \frac{1}{r_0}\left(1 - \frac{r_0 - r}{r_0}\right)^{-1} \approx \frac{1}{r_0} + \frac{r_0 - r}{r_0^2}$$

or

$$\frac{1}{r} - \frac{1}{r_0} = \frac{r_0 - r}{r_0^2} \approx \frac{x}{r_0^2}\left(1 - \frac{S}{2\xi}\right) \tag{21}$$

from (4)
Substituting into (20), we get

$$\log \frac{1 - u^2/c^2}{1 - v^2/c^2} \approx \frac{x}{r_0^2} S(1 - S/2\xi)$$

$$\approx \frac{xS}{r_0^2} \text{ neglecting } S^2 \text{ again}$$

$$= \frac{2GM}{c^2 r_0^2} x \tag{22}$$

With an obvious notation which has special reference to the earth's gravitational field, we define « g » by the relation $GM = g r_0^2$ and we get

$$\log \frac{1 - u^2/c^2}{1 - v^2/c^2} = \frac{2gx}{c^2}$$

or, equivalently

$$\frac{1 - u^2/c^2}{1 - v^2/c^2} = e^{2gx/c^2} \tag{23}$$

Since $x = l_0 - l$, $v^2 = \left(\frac{dl}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2$ and rewriting (23) we have

$$\frac{dx}{dt} = \frac{c\sqrt{e^{2gx/c^2} - 1 + (u^2/c^2)}}{e^{gx/c^2}}$$

i. e.,

$$ct = \int \frac{\gamma e^{gx/c^2} dx}{\sqrt{\gamma^2 e^{2gx/c^2} - 1}} + \text{constant}, \qquad \text{where} \qquad \gamma = \frac{1}{\sqrt{1 - u^2/c^2}}$$

or

$$ct = c^2/g \cosh^{-1} (\gamma e^{gx/c^2}) + \text{constant}$$

If $x = 0$ at $t = 0$, we get

$$gt/c = \cosh^{-1} (\gamma e^{gx/c^2}) - \cosh^{-1} (\gamma) \tag{24}$$

Since

$$(i) \sinh [\cosh^{-1} (\gamma e^{gx/c^2})] = \sqrt{\gamma^2 e^{2gx/c^2} - 1}$$

$$(ii) \sinh [\cosh^{-1} (\gamma)] = \sqrt{\gamma^2 - 1} = \frac{u\gamma}{c}$$

we have

$$\cosh gt/c = \gamma^2 e^{gx/c^2} - \frac{u\gamma}{c} \sqrt{\gamma^2 e^{2gx/c^2} - 1} \tag{25 a}$$

$$\sinh gt/c = \gamma\sqrt{\gamma^2 e^{2gx/c^2} - 1} - \frac{u\gamma^2}{c} e^{gx/c^2} \tag{25 b}$$

Multiplying (25 *b*) by u/c and adding to (25 *a*), we get

$$e^{gx/c^2} = \cosh gt/c + u/c \sinh gt/c \tag{26}$$

Taking logarithms, we obtain the formula (19) of the Special Theory of Relativity :

$$x = c^2/g \log (\cosh gt/c + u/c \sinh gt/c). \tag{19}$$

In order to arrive at the equation describing Hyperbolic motion, we start from (26) and retain terms, only upto the second power in x and t. Thus

$$1 + gx/c^2 + g^2x^2/2c^4 = \left(1 + \frac{g^2t^2}{2c^2}\right) + \frac{u}{c}(gt/c)$$

and on simplification, we have

$$(x + c^2/g)^2 - c^2(t + u/g)^2 = \frac{c^4}{\gamma^2 g^2} \tag{27}$$

which is clearly the equation of a hyperbola in $x - t$ space. With $u = 0$, equation (27) reduces to the familiar equation (14). We remark that (27) results from

$$\frac{d}{dt}\left(\frac{m_0 v}{\sqrt{1 - v^2/c^2}}\right) = m_u g \, ; \qquad m_v = \frac{m_0}{\sqrt{1 - v^2/c^2}} = m$$

showing that hyperbolic motion is a consequence, only of a *constant* [*6*] force acting on the particle.

Taking the square root, we can rewrite equation (23) as

$$\frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - u^2/c^2}} e^{gx/c^2} \tag{28}$$

Multiplying by m_0, we have

$$m_v = m_u e^{gx/c^2} \tag{29}$$

Thus $m_v = m_v(x)$, a function of position only and our reference system of interest is *conservative* [*7*]. Indeed, if we take

$$\varphi(x) = - m_u c^2 (e^{gx/c^2} - 1) + \varphi(0) \tag{30}$$

we have

$$- \frac{\partial \varphi}{\partial x} = m_v g$$

and

$$m_v c^2 + \varphi(x) = m_u c^2 + \varphi(0) = \text{constant} \tag{31}$$

which is the energy equation.

We finally observe that equations (18), (19) and (23) may be regarded as the relativistic analogues of the Galilean laws for a falling body. Equation (18), for example, would show that the « time of rise » of a particle thrown « vertically up » is

$$\mathrm{T} = \frac{c}{2g} \log \frac{1 + u/c}{1 - u/c} \tag{32}$$

while (23) would give, for the maximum height attained

$$\mathrm{H} = c^2/g \log \frac{1}{\sqrt{1 - u^2/c^2}} \tag{33}$$

One can similarly arrive at other results which are equivalent in content to those derivable from the Galilean laws.

SUMMARY

It is shown that the rectilinear motion of a particle in a Schwarzschild field reduces to the Newtonian law in the limit $S \to 0$ where $S = \frac{2GM}{c^2}$ is the Schwarzschild radius and that it reduces to a motion according to Special Relativity in a differential region if S^2 is neglected, thus illustrating Einstein's Principle of Equivalence. Relativistic analogues of the Galilean laws for a falling body are incidentally obtained.

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