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Gravitational motions of collapse or of expansion in general relativity

by

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ABSTRACT. — Einstein's equations of general relativity are solved exactly for a class of spherically symmetric distributions of matter in motion. The solutions for the interior of the distribution are obtained by using the notions of a co-moving coordinate system and the isotropy of internal stress, which is reduced to a pressure. This is zero at the outer boundary of the configuration and density and pressure gradients exist within the distribution. The speed of the motion is calculated. The theory has analogies with that of spherical blast waves in classical gas-dynamics; it is also applicable to gravitational collapse.

The object of this paper is to describe a class of solutions of Einstein's equations of general relativity. They refer to a spherically symmetric mass of material in radial motion which may either be collapsing onto the central point (gravitational collapse) or be exploding from that point (spherical blast). There is no limitation on the speed of the motion, apart, of course, from the restriction that the local speed of the material must be less than that of light. The results have analogies with the spherical blast waves of classical gas-dynamics (Courant and Friedrichs, 1948) but with the additional feature that the gravitational self-attraction of the material is a controlling element of the motion. The other controlling force is the pressure gradient. A by-product of the investigation is the discovery of the class of solutions of Einstein's equations to which belongs a particular case obtained in 1933 by a different method (McVittie, 1933, 1966).

The solutions are obtained from three general conditions: (*i*) from a preassigned form for the coefficients of the metric; (*ii*) from the existence of co-moving coordinates; and (*iii*) from the assumption that the stress in the material is isotropic. There is no *a priori* assumption that the energy-tensor is of the form appropriate to a fluid though presumably most of the applications of the theory will refer to this case.

Condition (*i*) consists of the assertion that an orthogonal coordinate system (t, r, θ, φ) exists in terms of which the metric inside the material has the form

$$\left.\begin{aligned} ds^2 &= e^{\lambda}dt^2 - R_0^2 S^2 \left\{ e^{\zeta}dr^2 + e^{\eta}f^2 d\omega^2 \right\} / c^2, \\ d\omega^2 &= d\theta^2 + \sin^2\theta \, d\varphi^2. \end{aligned}\right\} \tag{1}$$

Here R_0, c are constants, the function f is a dimensionless function of r and S is a function of t. The functions λ, ζ, η are also dimensionless, functions of a variable z which is given by

$$e^z = Q(r)/S(t),$$

Q being still another function of r. The six functions are unknown *a priori* and are to be found. The indices (4, 1, 2, 3) are associated with (t, r, θ, φ), respectively.

The Einstein equations with zero cosmical constant are

$$-(8\pi G/c^2)T^{ij} = R^{ij} - \frac{1}{2}Rg^{ij}, \tag{2}$$

where G is the constant of gravitation. It is well-known that, for all spherically symmetric metrics such as (1), the right-hand sides of (2) vanish except for

$$ij = 14, 41, 11, 22, 33, 44,$$

and this means that the only non-zero components of T^{ij} are $T^{14} (= T^{41})$, T^{11}, T^{22}, T^{33} and T^{44}. Of these T^{14} represents the momentum of the material relative to the coordinate system in the radial direction. If co-moving coordinates exist, as is demanded by condition (*ii*), then $T^{14} = 0$. Incidentally, this expresses more than a « coordinate condition » as may be seen from the following example. Suppose that the matter is collapsing onto the central point while, at the same time, radiation is moving radially outwards through the matter. Then T^{14} would consist of the sum of two parts, namely $(T^{14})_m$ due to the matter, and $(T^{14})_r$ due to the flux of radiation. Clearly it would be impossible to find a coordinate system which followed the matter inwards and simultaneously followed the radiation outwards. Therefore the condition $T^{14} = 0$ not only defines a type of coordinate system but also implies that *all* the material has a uni-directional motion at each point of space and at each instant of time.

The equation (2) for T^{14} may be computed by the usual methods (McVittie, 1964 *a*) and leads to the equation (A.1) of the Appendix. Inspection of this equation shows that $T^{14} = 0$ can be secured for arbitrary S, Q and f, if

$$\eta = \zeta, \qquad e^{\lambda/2} = 1 - \eta_z/2. \tag{3}$$

Thus the metric (1) now takes the form

$$\left.\begin{aligned} ds^2 &= y^2 dt^2 - R_0^2 S^2 e^{\eta}(dr^2 + f^2 d\omega^2)/c^2, \\ y &= e^{\lambda/2} = 1 - \eta_z/2. \end{aligned}\right\} \tag{4}$$

The equations (2) for the stress components of the energy-tensor can now be calculated and are shown in their mixed forms in equations (A.3) and (A.4). Condition (*iii*) is expressed by

$$T_1^1 = T_2^2 = T_3^3. \tag{5}$$

This is a consistency relation (McVittie, 1964 *b*) which, though it is not a tensor equation, is a relation that must exist when isotropy of stress is combined with the form of the metric shown in (4). The equation (5) yields (A.5) which is of the form

$$F_1(r)Y_1(y) + F_2(r)Y_2(y) + F_3(r)Y_3(y) = 0. \tag{6}$$

Since r and $y(z)$ are independant variables—z involves t as well as r—it must be the case that F_1, F_2, F_3 differ from one another by constant multiples only. Thus are obtained the two ordinary second-order differential equations (A.6) and (A.7), that involve the two fundamental constants a and b, and give f and Q as functions of r. There also arises equation (A.8), another ordinary second-order differential equation. It gives y as a function of z, which also involves the constants a and b. Thus the problem of determining y, η, f and Q is reduced to the solution of three second-order ordinary differential equations. Moreover the condition for the isotropy of stress does not involve S which, at this stage, is still arbitrary.

The determinations of f and Q are discussed in Sec. II of the Appendix. It turns out that they are expressible in terms of elementary functions and that there are many different cases. These arise, not only because different numerical values may be assigned to the constants a and b, but also because different combinations of the constants of integration of the equations (A.6) and (A.7) are possible. An exhaustive analysis of all possible functions f and Q has not been attempted; instead the methods of integration have been indicated and specimen solutions have been worked out. A main concern is the nature of the 3-space whose metric is

$$d\sigma^2 = dr^2 + f^2 d\omega^2, \tag{7}$$

to which the 3-space in (4) is conformal. The radial coordinate q defined by (A.10) is sometimes advantageously employed in place of r and, indeed, f itself may on occasion serve as the radial coordinate as is indicated in equations (A.15) and (A.16).

A particularly interesting sub-class of functions f is obtained by setting $b = 0$. Then f and Q in (A.6) and (A.7) cease to be interlocked and f is found to have one or other of the expressions

$$f = \mathrm{C}^{-1} \sin \mathrm{C}r,\ r,\ \mathrm{C}^{-1} \sinh \mathrm{C}r, \tag{8}$$

where C is the constant in (A.20). Therefore $b = 0$ is the condition that the 3-space (7) should be of constant curvature. Conversely, if f is given by (8) then $b = 0$ from (A.7) provided, of course, that the case Q = *constant* is excluded. It is easily seen that Q = *constant* and $b = 0$ together lead, by a transformation of the time-coordinate from t to $\bar{\tau}$, to

$$ds^2 = d\bar{\tau}^2 - \mathrm{R}^2(\bar{\tau})\,(dr^2 + f^2 d\omega^2)/c^2, \tag{9}$$

where f is given by (8). Thus the metrics used in the cosmological models and in uniform gravitational collapse (McVittie, 1964 *c*) constitute special cases of (4) which could also be derived from $b = 0$, $\eta = 0$, $y = 1$ in terms of the original coordinate t.

The integration of equation (A.8) for y as a function of z is not, in general, possible in terms of elementary functions (see Sec. III of the Appendix). To obtain its first integral it would be necessary to integrate (A.25) which is Abel's equation. Thus in general numerical integration would be needed to find y as a function of z. Nevertheless there exist a number of elementary particular integrals of (A.8), in which one of the two constants of integration is zero. Their defining first integrals are listed in equations (A.26) to (A.29). Two of these cases will be considered in more detail presently.

The use of co-moving coordinates might lead to the (incorrect) conclusion that the speed of the moving material could not be stated. A more serious source of ambiguity lies in the fact that speed is the rate of change of distance with time. Since there is no absolute definition of distance in general relativity, and more than one time is definable, it is clear that speed must be to some extent a matter of definition also. With the metric in the form (4), consider all the material which lies on the pseudo-spherical surface Σ whose radial coordinate is r. As the motion proceeds, the material will continue to lie on Σ because r is co-moving. However the « radius » of Σ at time t may be defined to be

$$u = R_0 S e^{\eta/2} f, \tag{10}$$

and this could also be regarded as the « distance » of the matter on Σ from the origin $r = 0$. At time $t + dt$, this distance is $u + du$ where

$$du = R_0 f S_t y e^{\eta/2} dt$$

by the differentiation of (10) treating r as a constant. One definition of the speed of the material on Σ is obtained by asserting that it is the rate of change of u with respect to t. The speed is thus $v = du/dt$ where

$$v = R_0 f S_t y e^{\eta/2}. \tag{11}$$

Alternatively, however, it could be said that a coordinate time-interval dt, for fixed values of r, θ, φ, corresponds to a proper-time interval on Σ of amount

$$ds = y dt.$$

Hence a second definition of the speed could be $V = du/ds$ where

$$V = R_0 f S_t e^{\eta/2}. \tag{12}$$

In either case, the speed is proportional to S_t.

The last stage in the analysis consists in determining the hitherto unspecified function $S(t)$. The Einstein equations (2) are now reduced to two which are written out in full in (A.33) and (A.34). The first of these gives the component T_4^4 of the energy-tensor which will be called the density, ρ. The second equation gives the three stress components T_1^1, T_2^2, T_3^3, now equal to one another. Any one of them will be denoted by $-p/c^2$, where p is the pressure. It will be seen that ρ and p involve η, y, f and Q and also the still-unknown function S. Moreover ρ and p are found separately, which always happens when Einstein's equations are used « in reverse » to calculate the energy-tensor which corresponds to a metric of given form. An analogous situation arises in classical gas-dynamics when the method of indeterminate functions is employed (McVittie, 1953).

The simplest way of finding $S(t)$ is to impose a boundary condition. If the region outside the spherical mass is a vaccum, whose metric is that of the Schwarzschild space-time, then the pressure at the outer surface must be zero. Let the outer boundary be the psuedo-spherical surface Σ_b on which $r = r_b$. Then r_b is constant throughout the motion since r is co-moving. Thus f, f_r, Q, Q_r are constants on Σ_b whereas z becomes a function of S alone. Hence the condition $p = 0$ on Σ_b will produce from (A.34) an ordinary second-order differential equation for S as a function of t. From this equation, S could be found, numerically if necessary, and analytically in certain cases as will presently be shown.

In classical gas-dynamics attention is often fixed on adiabatic or isentropic motions in which p is proportional to a constant power of ρ and the entropy of a fluid elements is constant during its motion. Suppose that the present theory were to apply to a gas. Then it is clear that, in principle at least, either t or r could be eliminated from (A.33) and (A.34) to give relations of the form

$$p = P_1(\rho\,; r) \qquad \text{or} \qquad p = P_2(\rho\,; t). \tag{13}$$

The probability that r and t could be eliminated simultaneously from (A. 33) and (A.34) appears to be negligible. But this would be necessary if it was desired to extract from these formulæ a relation of the form $p \propto \rho^\gamma$. In general therefore it would appear that the present theory gives the analogues of the non-adiabatic motions of classical theory. Or, at least, of those in which the equation of state varies from point to point in the gas. However, these questions need further investigation in the light of a general relativity definition of entropy.

In order to examine some features of the motions that are possible, the case of implosion ($S_t < 0$) will be considered with the aid of two examples. Consider first the 1933 solution. If $C = 1$ in equation (8) then

$$f = \sin r,\ r,\ \sinh r, \tag{14}$$

and if in (A.21), which implies $b = 0$, one puts

$$a = 3, \qquad C_{+1} = C_{-1} = 1, \qquad C_0 = 0, \qquad A = -\frac{1}{4}\mu_0^2,$$

where μ_0 is a positive constant, then

$$Q = \frac{\mu_0}{4}\frac{1}{\sin (r/2)}, \qquad \frac{\mu_0}{4}\frac{1}{(r/2)}, \qquad \frac{\mu_0}{4}\frac{1}{\sinh (r/2)}. \tag{15}$$

Again, if in (A.31), $\Gamma = 1$ and $n = 1$ because $b = 0$, it follows, with $e^z = Q/S$, that

$$y^2 = e^\lambda = \left(\frac{1 - Q/S}{1 + Q/S}\right)^2, \qquad e^\eta = (1 + Q/S)^4.$$

But these are the coefficients of the metric of the 1933 solution in the form given in McVittie (1966) (1). This solution is therefore the particular case of the present theory obtained by taking $a = 3$ and $b = 0$. The detailed examination will be found in McVittie (1966) where it is shown that the boundary condition $p = 0$ gives S in terms of elementary functions. Initially $S = 1$ and the final value is $S = 0$ for implosions. The motions can be interpreted as those of a spherical mass of gas falling onto a central particle. The particle is defined by a Schwarzschild type singular region of radius r_s around the origin $r = 0$. The constant μ_0 plays the part of the length-equivalent of the mass of the particle at the initial instant. The speed of the gas, V in equation (12), is, for $f = \sin r$ for example,

$$V = 2R_0S_t\left(\sin\frac{r}{2} + \frac{\mu_0}{4S}\right)^2 \cot\frac{r}{2}. \tag{16}$$

Thus V fails to vanish at the centre $r = 0$ and the same is true for v in (11). But the solution is hardly applicable at the centre; for at time t the coordinate-radius r_s of the singular region is defined by $Q(r_s)/S = 1$, which means that

$$\sin\,(r_s/2), \qquad r_s/2, \qquad \sinh\,(r_s/2) = \frac{\mu_0}{4}\frac{1}{S}.$$

Hence r_s increases as S decreases from 1 to 0. Therefore by (16) the material on the singular surface Σ_s, of radius $r = r_s$, is moving inwards, whereas the surface itself is moving outwards. The pressure given by (A.34) contains, in this case, a factor $(Q - Q_b)/(S - Q)$, where Q_b denotes the boundary value of Q (McVittie, 1966). Hence, since $S - Q(r_s) = 0$, the pressure becomes positive infinite as Σ_s is approached from $r > r_s$. The pressure would be negative on the side $r < r_s$ of Σ_s, if it were legitimate to extend the solution to that side. In some ways therefore Σ_s has the properties of a shock front. At any rate, the motion continues until Σ_s and the outer boundary coincide. Since, at this moment $Q(r_s) - Q_b = 0$ and $S - Q(r_s) = 0$ independently of one another, the boundary value of the pressure is indeterminate. One may say that all the material has then been swallowed up by the singular region.

These peculiar features of the motions defined by the 1933 solution are absent in other cases. Consider the solution discussed in Sec. V of the

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(1) The coordinate ω is written for r and S_k stands for the three possible values of f in (14), according as $k = +1, 0,$ or -1. The derivatives of S are denoted by primes.

Appendix in which a spherical mass collapses from a state of rest. The configuration has an outer boundary $r = r_b$ and the motion is specified by

$$a = 1/2, \qquad b = 0\,;$$

$$f = (2r_b/\pi)\,\sin\left(\frac{\pi}{2}\,r/r_b\right), \qquad Q = \cos^2\left(\frac{\pi}{2}\,r/r_b\right)$$

$$A = -\,2r_b^2/\pi^2\,;$$

$$y^2 = e^{\lambda} = \{\,1 + (Q/S)^{1/2}\,\}^{-2}, \qquad e^{\eta} = \{\,1 + (Q/S)^{1/2}\,\}^4,$$

$$e^{z/2} = S^{-1/2}\cos\left(\frac{\pi}{2}\,r/r_b\right).$$

Since the outer boundary corresponds to $Q = 0$ and the centre to $Q = 1$, there are no singularities or zero values of y or of e^{η}. The function S is found from $p = 0$ at $r = r_b$ and it satisfies exactly the same differential equation as it does in uniform collapse (ρ and p functions of t alone). The time T of collapse to zero volume depends on the initial value of the boundary density and the formula (A.44) for T is similar to that found in uniform collapse (McVittie, 1964 *c*). The time T is short in spite of the presence of density and pressure gradients within the material. If $n > 0$ and $(\rho_b)_0 = 10^{-n}$ gr.cm.$^{-3}$, then (A.44) yields

$$T = 6.66 \times 10^{(n-10)/2} yr.$$

Thus $n = 20$ would give less than 700,000 years for the collapse, whereas $n = 10$ would produce an, astronomically speaking, instantaneous collapse.

The formula (A.47) shows that the speed of the material has no singular values and is always zero at the centre.

In conclusion, it should be stated that an exhaustive search of the literature has not been made in order to discover if any of the results of this paper have appeared in some other form elsewhere. One of the difficulties encountered in such a search is that nearly every author has his own notation and method of procedure. Two spherically symmetric metrics with orthogonal coordinates may have a very different appearance until it is found that some transformation of the t and r coordinates turns one metric into the other. But certain investigations should be mentioned. Bondi (1947) and later Omer (1965) discuss metrics with orthogonal coordinates in which the coefficient of dt^2 is unity and the coefficients of dr^2 and $d\omega^2$ are general functions of r and t. Attention is concentrated on cases in which the pressure is either zero or spatially uniform. In the investigations of Misner (1964, 1965, 1966) and his co-workers the coefficients of dt^2, dr^2

and $d\omega^2$ are more general functions of r and t than are those found in (1). However, attention is concentrated less on the determination of the coefficients than on conclusions about the nature of the motion. May and White (1965) employ the Misner form for the metric but their aim is to compute solutions numerically. Some of the 3-spaces found in Sec. II of the Appendix, notably (A.22) and (A.24), also occur in the work of Takeno (1963).

APPENDIX

1. — The Einstein equation (2) for $ij = 14$ is

$$-\frac{8\pi G R_0^2}{c^2} S^2 e^{\lambda+\zeta} T^{14} = (S_t/S)\Big[(Q_r/Q)\Big\{\eta_{zz} + \lambda_z\Big(1-\frac{1}{2}\eta_z\Big) + \frac{1}{2}\eta_z(\eta-\zeta)_z\Big\} + (f_r/f)(\eta-\zeta)_z\Big]. \quad (A.1)$$

Hence $T^{14} = 0$ is satisfied if $\eta = \zeta$ and

$$\eta_{zz} + \lambda_z\Big(1-\frac{1}{2}\eta_z\Big) = 0,$$

the solution of which is given by (3).

The remaining equations (2) are then, with the form (4) of the metric,

$$8\pi G T_4^4 = 3(S_t/S)^2 - \frac{c^2 e^{-\eta}}{R_0^2 S^2}\big[2f_{rr}/f + f_r^2/f^2 - 1/f^2 + 2(1-y)\{Q_{rr}/Q - Q_r^2/Q^2 + 2f_r Q_r/(fQ)\} + \{(1-y)^2 - 2y_z\}(Q_r/Q)^2\big], \quad (A.2)$$

$$-8\pi G T_1^1 = -(2/y)(S_{tt}/S) - (3y-2)y^{-1}(S_t/S)^2 - \frac{c^2 e^{-\eta}}{R_0^2 S^2 y}\{y(1-f_r^2)/f^2 + 2(y^2-y-y_z)f_r Q_r/(fQ) + (1-y)(y^2-y-2y_z)(Q_r/Q)^2\}, \quad (A.3)$$

$$-8\pi G T_2^2 = -8\pi G T_3^3 = -(2/y)(S_{tt}/S) - (3y-2)y^{-1}(S_t/S)^2 - \frac{c^2 e^{-\eta}}{R_0^2 S^2 y}[-yf_{rr}/f + (y^2-y-y_z)\{f_r Q_r/(fQ) + Q_{rr}/Q\} + \{y(1-y) + (1+y)y_z - y_{zz}\}(Q_r/Q)^2]. \quad (A.4)$$

The isotropy of stress is expressed by (5) and means that

$$(f_{rr}/f - f_r^2/f^2 + 1/f^2)y + \{Q_{rr}/Q - Q_r f_r/(fQ)\}(y - y^2 + y_z) + \{y(1-y)(y-2) - (3-y)y_z + y_{zz}\}(Q_r/Q)^2 = 0. \quad (A.5)$$

The independence of r and z implies that constants a and b exist such that

$$Q_{rr}/Q - Q_r f_r/(fQ) = a(Q_r/Q)^2, \quad (A.6)$$

$$f_{rr}/f - f_r^2/f^2 + 1/f^2 = b(Q_r/Q)^2. \quad (A.7)$$

Then (A.5) becomes

$$y_{zz} + (a-3+y)y_z + y\{a+b-2-(a-3)y-y^2\} = 0. \quad (A.8)$$

2. — *Integration of equations (A.6) and (A.7).* — The integral of (A.6) is

$$f = AQ_r/Q^a, \quad (A.9)$$

where A is the constant of integration. Introduce the new radial coordinate q by

$$\left.\begin{aligned} q &= A(1-a)^{-1}Q^{1-a}, & a &\neq 1 \\ q &= A \ln Q, & a &= 1 \end{aligned}\right\} \tag{A.10}$$

and then

$$f = \frac{dq}{dr}. \tag{A.11}$$

The equation (A.7) becomes

$$\left.\begin{aligned} f_{qq} + 1/f^3 &= b(1-a)^{-2} f/q^2, & a &\neq 1 \\ &= (b/A^2)f, & a &= 1 \end{aligned}\right\} \tag{A.12}$$

Alternative forms of the metric (7) are :

If f is determined as a function of q by (A.12) then, by (A.11) also,

$$d\sigma^2 = \frac{dq^2}{f^2(q)} + f^2(q)d\omega^2, \tag{A.13}$$

and the associated expressions for z are

$$\left.\begin{aligned} z &= (1-a)^{-1} \ln \{ (1-a)q/A \} - \ln S, & a &\neq 1 \\ &= q/A - \ln S ; & a &= 1 \end{aligned}\right\} \tag{A.14}$$

If f itself is used as the radial coordinate and the first integral of (A.12) is written as

$$\frac{fdf}{F(f)} = dq = fdr, \tag{A.15}$$

then

$$d\sigma^2 = \frac{df^2}{F^2(f)} + f^2 d\omega^2. \tag{A.16}$$

The associated values of z are again given by (A.14) with $q = \int fdf/F(f)$.

The curvature of the 3-space with metric $d\sigma^2$ is obtained from the non-zero components of the Riemann-Christoffel tensor. These are expressed as follows:

When 1, 2, 3 represent r, θ, φ, then (7) yields

$$\left.\begin{aligned} R_{1212} &= \sin^{-2}\theta\, R_{1313} = -ff_{rr}, \\ R_{2323} &= f^2(1-f_r^2)\sin^2\theta ; \end{aligned}\right\} \tag{A.17}$$

When 1, 2, 3 represent q, θ, φ, then (A.13) yields

$$\left.\begin{aligned} R_{1212} &= \sin^{-2}\theta\, R_{1313} = -\left(ff_{qq} + f_q^2\right), \\ R_{2323} &= f^2\left(1 - f^2 f_q^2\right)\sin^2\theta ; \end{aligned}\right\} \tag{A.18}$$

When 1, 2, 3 represent f, θ, φ, then (A.16) yields

$$\left.\begin{aligned} R_{1212} &= \sin^{-2}\theta\, R_{1313} = -fF_f/F, \\ R_{2323} &= f^2(1-F^2)\sin^2\theta. \end{aligned}\right\} \tag{A. 19}$$

Constant curvature occurs when

$$R_{hijk} = kC^2(g_{hj}g_{ik} - g_{hk}g_{ij}), \tag{A.20}$$

where C is a positive real constant and

$$k = +1, 0 \quad \text{or} -1.$$

(i) *Case* $b = 0$, *any* a. The equation (A.7) yields the three alternatives shown in equation (8), where it has also been assumed that $f = 0$ when $r = 0$. Equation (A.17) then shows that the 3-space (7) is of constant curvature. Also by (A.9),

$$\left.\begin{array}{l}(a \neq 1),\ A(1-a)^{-1}\,Q^{1-a}\\(a = 1),\ A \ln Q\end{array}\right\} = C_{+1} - \frac{\cos Cr}{C^2},\ \frac{r^2 + C_0}{2},\ \frac{\cosh Cr}{C^2} - C_{-1}, \qquad (A.21)$$

where C_{+1}, C_0, C_{-1} are constants of integration.

(ii) *Case* $a = 1$, $b \neq 0$. If $b > 0$ and $b_0^2 = b/A^2 > 0$, the second of equations (A.12) gives

$$dq = fdr = \frac{1}{b_0}\left\{(f^2 + \gamma)^2 + b_0^{-2} - \gamma^2\right\}^{-1/2} fdf,$$

where γ is a constant of integration. Thus f^2 as a function of q may be obtained by quadratures. The form (A.16) for the metric becomes

$$d\sigma^2 = \left\{ 1 + b_0^2(2\gamma f^2 + f^4) \right\}^{-1} df^2 + f^2 d\omega^2, \qquad (A.22)$$

and it is easy to show from (A.19) and (A.20) that the curvature is not constant. If $b < 0$, a similar method applies.

(iii) *Case* $a \neq 1$, $b \neq 0$. The first of equations (A.12) applies and the integration is performed by means of the substitutions

$$q = e^w, \qquad f = e^{w/2} v(w). \qquad (A.23)$$

Suppose that

$$b_1^2 = \frac{1}{4} + \frac{b}{(1-a)^2}$$

is positive. Then by a suitable choice of the constant of integration γ, the equation (A.12) reduces to

$$2b_1 dw = \left\{ (v^2 - \gamma)^2 + b_1^{-2} - \gamma^2 \right\}^{-1/2} d(v^2 - \gamma)$$

and is thus integrable by quadratures. For example, if $\gamma = 1/b_1$, then

$$f^2 = \frac{1}{b_1} q(1 + \delta q^{2b_1}),$$

where $\gamma\delta$ is the second constant of integration. The condition that f shall vanish at $q = 0$ only, means that $\delta \geq 0$. The metric of the 3-space is given by the form (A.13). When $\delta = 0$, then the function F of (A.15) is equal to $(2b_1)^{-1}$ and the metric (A.16) is

$$d\sigma^2 = 4b_1^2 df^2 + f^2 d\omega^2. \qquad (A.24)$$

By (A.19) this is a space of variable curvature.

The substitutions (A.23) also reduce the integration of (A.12) to quadratures in the cases $b_1^2 = 0$ and $b_1^2 < 0$.

3. — *The equation (A.8).* — If $Y^{-1} = y_z$, then (A.8) is

$$\frac{dY}{dy} = (a - 3 + y)Y^2 + y\left\{ b + a - 2 - (a-3)y - y^2 \right\} Y^3, \qquad (A.25)$$

which is a form of Abel's equation. When one of the two constants of integration of (A.8) is zero, particular integrals of the equation arise as follows:

(*i*) If

$$b = -(6a^2 - 11a + 4)/25,$$

then

$$\frac{dy}{dz} = \frac{a-3}{5}y + \frac{1}{2}y^2 \; ; \qquad (A.26)$$

(*ii*) If

$$b = 2 - a,$$

then

$$\frac{dy}{dz} = -(a-3)y - y^2 \; ; \qquad (A.27)$$

(*iii*) If

$$a = 3, \text{ any } b,$$

then

$$\frac{dy}{dz} = -\frac{1}{2}(b+1) + \frac{1}{2}y^2 \; ; \qquad (A.28)$$

(*iv*) For any a and b there is always the particular integral

$$\frac{dy}{dz} = (a + b - 2) - (a - 3)y - y^2. \qquad (A.29)$$

In all four cases y can be found by quadratures.

For example, if in (A.26), $a = 1/2$, so that $b = 0$, then

$$y^2 = e^{\lambda} = \Gamma^2(e^{z/2} + \Gamma)^{-2}, \qquad (A.30)$$
$$e^{\eta} = (e^{z/2} + \Gamma)^4,$$

where Γ is the constant of integration and η is found from (3), a multiplicative constant of integration being taken equal to one. Again if $a = 3$ and $b + 1 = n^2 > 0$, then (A.28) yields

$$y^2 = e^{\lambda} = n^2(1 - \Gamma e^{nz})^2 (1 + \Gamma e^{nz})^{-2}, \qquad (A.31)$$
$$e^{\eta} = (1 + \Gamma e^{nz})^4 e^{2(1-n)z},$$

by a suitable choice of the constant of integration in the determination of η.

4. — *The energy-tensor.* — Write

$$T_4^4 = \rho, \qquad T_1^1 = T_2^2 = T_3^3 = -p/c^2, \qquad (A.32)$$

and use (A.6) (A.7) in (A.2) and in either (A.3) or (A.4) to give

$$8\pi G\rho = 3(S_t/S)^2 + \frac{c^2 e^{-\eta}}{R_0^2 S^2}[3(1 - f_r^2)f^{-2} - 6(1 - y)Q_r f_r/(Qf)$$
$$- \{ 2b - 2y_z + (1 - y)(2a - 1 - y) \} (Q_r/Q)^2], \qquad (A.33)$$

$$8\pi G\frac{p}{c^2} = \frac{1}{y}[- 2S_{tt}/S - (3y - 2)(S_t/S)^2$$
$$- \frac{c^2 e^{-\eta}}{R_0^2 S^2}\{ y(1 - f_r^2)f^{-2} + 2(y^2 - y - y_z)f_r Q_r/(fQ)$$
$$+ (1 - y)(y^2 - y - 2y_z)(Q_r/Q)^2 \}]. \qquad (A.34)$$

In these expressions, of course, f and Q stand for solutions of (A.6) and (A.7) and y, η for those of (A.8) and (3). Thus ρ, p are known to within one undetermined function S(t). If q is used as the radial coordinate, the substitutions

$$f_r = ff_q, \qquad Q_r/Q = fQ_q/Q,$$

must be made in (A.33) and (A.34).

5. — *A Special case.* — This is defined by

$$a = 1/2, \qquad b = 0, \qquad f = C^{-1}\sin Cr.$$

Then by (A.21) with $C_{+1} = 0$ and $A = -1/(2C^2)$

$$Q = \cos^2 Cr, \qquad e^z = S^{-1}\cos^2 Cr. \tag{A.35}$$

Again in (A.30) let $\Gamma = 1$ and then

$$\left.\begin{aligned} y^2 = e^\lambda &= (1 + e^{z/2})^{-2} \\ e^\eta &= (1 + e^{z/2})^4. \end{aligned}\right\} \tag{A.36}$$

When these results are used in (A.33) (A.34), there is obtained

$$8\pi G\rho = 3(S_t/S)^2 + \frac{3c^2C^2}{R_0^2S^2}\frac{1 + 5e^{z/2}}{(1 + e^{z/2})^5}, \tag{A.37}$$

$$8\pi G\frac{p}{c^2} = -2(1 + e^{z/2})(S_{tt}/S) - (1 - 2e^{z/2})(S_t/S)^2 - \frac{c^2C^2}{R_0^2S^2}\frac{1 + 3e^{z/2}}{(1 + e^{z/2})^5}. \tag{A.38}$$

If the outer boundary is $r = r_b$, then C may be chosen so that

$$C = \frac{\pi}{2}(1/r_b), \tag{A.39}$$

which means that $e^z = 0$ at the outer boundary while at the centre $r = 0$, we have $e^z = 1/S$. Moreover

$$T = \frac{\pi}{2}R_0/(cC) = R_0r_b/c, \tag{A.40}$$

is a fixed time-interval. To fix ideas, consider a motion of collapse from rest, so that, at $t = 0$, $S = 1$, $S_t = 0$, $S_{tt} < 0$. Let $(\rho_b)_0$ be the initial boundary value of the density. Then by (A.37) (A.39) and (A.40)

$$\alpha = \frac{8\pi G}{3}(\rho_b)_0 = \pi^2/(2T)^2. \tag{A.41}$$

The boundary is given by $p = 0$ at $r = r_b$ and, since $e^{z/2}$ is zero there, S satisfies

$$2(S_{tt}/S) + (S_t/S)^2 + \alpha/S^2 = 0, \tag{A.42}$$

whence

$$S_t^2 = \alpha(1 - S)/S,$$

$$S = \cos^2 x, \qquad \alpha^{1/2}t = x + \sin x\cos x, \tag{A.43}$$