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Gravitational contraction of charged fluid spheres

by

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ABSTRACT. — A model of a charged fluid sphere undergoing gravitational contraction is constructed on the lines of the Oppenheimer-Snyder model of uncharged collapsing spheres.

I. — INTRODUCTION

Oppenheimer and Snyder (1939) have discussed the gravitational contraction of massive spheres under the influence of their own gravitational forces and have found that the contraction will continue until a space-time singularity is obtained. Several attempts have been made to get a model which avoids this collapse to a singularity (Hoyle and Narlikar (1964), McVittie (1965), Bonner (1965)). In particular Bonner has discussed the equilibrium of a sphere filled with charged dust under its own gravitational attraction and the electrical repulsion.

In the following we consider the contraction of a charged fluid sphere under the gravitational pull and under the opposing force of electrical repulsion.

II. — FIELD EQUATIONS

Choosing co-moving coordinates (r, θ, Φ) , the most general line-element exhibiting spherical symmetry can be put into the form

$$ds^2 = e^\alpha dt^2 - e^\beta [e^\lambda dr^2 + r^2 d\Omega^2] \quad (2.1)$$

where

$$\begin{aligned} d\Omega^2 &= d\theta^2 + \sin^2 \theta d\varphi^2 \\ \alpha &= \alpha(r, t), \quad \beta = \beta(r, t), \quad \lambda = \lambda(r, t). \end{aligned}$$

The field equations

$$R_i^k - \frac{1}{2} \delta_i^k R = -8\pi T_i^k \quad (2.2)$$

take the form

$$\begin{aligned} -8\pi T_1^1 &= \frac{1}{2} e^{-\beta-\lambda} \left[\frac{1}{2} \beta'^2 + \frac{2}{r^2} + \frac{2\beta'}{r} + \beta'\alpha' + \frac{2\alpha'}{r} \right] \\ &\quad - e^{-\alpha} \left[\ddot{\beta} - \frac{1}{2} \dot{\beta}\dot{\alpha} + \frac{3}{4} \dot{\beta}^2 \right] - \frac{e^{-\beta}}{r^2} \end{aligned} \quad (2.3)$$

$$\begin{aligned} -8\pi T_2^2 &= -8\pi T_3^3 \\ &= \frac{1}{4} e^{-\beta-\lambda} \left[2\alpha'' + \alpha'^2 + 2\beta'' + \frac{2\beta'}{r} + \frac{2\alpha'}{r} - \lambda' \left(\beta' + \alpha' + \frac{2}{r} \right) \right] \\ &\quad + \frac{1}{4} e^{-\alpha} [2\dot{\beta}\dot{\alpha} - 3\dot{\beta}^2 - 4\ddot{\beta} + \dot{\lambda}\dot{\alpha} - 3\dot{\beta}\dot{\lambda} - 2\ddot{\lambda} - \dot{\lambda}^2] \end{aligned} \quad (2.4)$$

$$\begin{aligned} -8\pi T_4^4 &= e^{-\beta-\lambda} \left[\beta'' + \frac{1}{4} \beta'^2 + \frac{2\beta'}{r} + \frac{1}{r^2} - \lambda' \left(\frac{\beta'}{2} + \frac{1}{r} \right) \right] \\ &\quad - \frac{1}{2} e^{-\alpha} \left[\frac{3}{2} \dot{\beta}^2 + \dot{\beta}\dot{\lambda} \right] - \frac{e^{-\beta}}{r^2} \end{aligned} \quad (2.5)$$

$$\begin{aligned} -8\pi T_4^1 &= -8\pi T_1^4 \\ &= \frac{1}{2} e^{-\beta-\lambda} \left[-2\dot{\beta}' + \dot{\lambda} \left(\beta' + \frac{2}{r} \right) + \dot{\alpha}\dot{\beta} \right] \end{aligned} \quad (2.6)$$

Here and in what follows a prime indicates differentiation with regard to r and an overhead dot that with regard to t .

For a distribution of charged fluid we take

$$T_i^k = M_i^k + E_i^k \quad (2.7)$$

with

$$M_i^k = (p + \rho) V_i V^k - p \delta_i^k \quad (2.8)$$

$$V^1 = V^2 = V^3 = 0, \quad V_i V^i = 1 \quad (2.9)$$

and

$$4\pi E_i^{\kappa} = -F^{\kappa a} F_{ia} + \frac{1}{4} \delta_i^{\kappa} F^{ab} F_{ab} \quad (2.10)$$

with

$$F_{[ij;\kappa]} = 0 \quad (2.11)$$

and

$$F_{;\kappa}^{i\kappa} = 4\pi J^i \quad (2.12)$$

For a spherically symmetric charge distribution we shall have F_{14} as the only surviving component of $F_{i\kappa}$. Also since we are using co-moving coordinates, the charge current vector J^i will have components $(0, 0, 0, J^4)$. Equation (2.12) will then require that

$$F_{;\kappa}^{1\kappa} = 4\pi J^1 = 0 \quad (2.13)$$

With the form of T_i^{κ} given by (2.7) above we find that

$$T_1^1 = -p - \frac{1}{8\pi} F^{14} F_{14} \quad (2.14)$$

$$\begin{aligned} T_2^2 &= T_3^3 \\ &= -p + \frac{1}{8\pi} F^{14} F_{14} \end{aligned} \quad (2.15)$$

$$T_4^4 = \rho - \frac{1}{8\pi} F^{14} F_{14} \quad (2.16)$$

$$T_4^1 = T_1^4 = 0. \quad (2.17)$$

Thus an elimination of the physical quantities p , ρ and F^{14} from these equations we find

$$T_4^1 = 0, \quad \text{and} \quad T_1^1 - T_2^2 = \frac{1}{4\pi} e^{-2\beta} f^2(r) \quad (2.18)$$

III. — A SOLUTION OF THE FIELD EQUATIONS

When the form of T_i^{κ} in terms of $g_{i\kappa}$ and their derivatives as given by (2.2) above are substituted in (2.18), we have two equations for three functions λ , α and β . The third equation is supplied by the equation of state of the fluid. In order to obtain an analytic solution which will in some sense be a generalization of Oppenheimer and Snyder's solution to the case of a

charged sphere, we replace the equation of state by the following simplifying assumptions

$$\frac{\partial \lambda}{\partial t} = \bar{\lambda} = 0 \quad (3.1)$$

and

$$\alpha + \beta = 0 \quad (3.2)$$

The latter assumption (3.2) is suggested by Bonner's work (1965).

The two equations (2.18) now became

$$-2\dot{\beta}' - \beta'\dot{\beta} = 0 \quad (3.3)$$

and

$$f^2(r) = \frac{1}{2} e^\beta \left[e^{-\lambda} \left\{ \frac{1}{2} \beta'^2 - \frac{1}{2} \frac{\lambda'}{r} - \frac{1}{r^2} \right\} + \frac{1}{r^2} \right] \quad (3.4)$$

The equation (3.3) gives

$$e^{-\alpha} = e^\beta = (F + G)^2$$

where

$$F = F(r)$$

$$G = G(t)$$

are undetermined functions of their arguments. Therefore (3.4) becomes

$$f^2(r) = F'^2 e^{-\lambda} - \frac{1}{2} (F + G)^2 \left[e^{-\lambda} \left\{ \frac{\lambda'}{2r} + \frac{1}{r^2} \right\} - \frac{1}{r^2} \right] \quad (3.4')$$

If $G = G(t)$ is not a constant, (3.4') will split up into two equations

$$f^2(r) = F'^2 e^{-\lambda}$$

and

$$e^{-\lambda} \left\{ \frac{\lambda'}{2r} + \frac{1}{r^2} \right\} - \frac{1}{r^2} = 0$$

the latter of which gives $e^{-\lambda} = 1 - kr^2$, k being an undetermined constant and the former of which now reduces to

$$f^2(r) = F'^2 (1 - kr^2) \quad (3.5)$$

Thus the solution of the field equations is given by the line-element

$$ds^2 = (F + G)^{-2} dt^2 - (F + G)^2 \left[\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right] \quad (3.6)$$

Using equations (2.2) and (2.13) to (2.17) we find that

$$8\pi p = -\frac{k}{(F+G)^2} - 2\ddot{G}(F+G) - 3\dot{G}^2 \quad (3.7)$$

$$8\pi\rho = -2(F+G)^{-3}\left[F''(1-kr^2) + (2-3kr^2)\frac{F'}{r}\right] \\ + 3k(F+G)^{-2} + 3\ddot{G}^2 \quad (3.8)$$

$$F_{14} = \frac{F'}{(F+G)^2} \quad (3.9)$$

The charge density σ is defined by $\sigma = s | \{ J^a J_a \}^{1/2} |$ where s is the sign of J^4 .

Therefore,

$$4\pi\sigma = (F+G)^{-3}\left[F''(1-kr^2) + (2-3kr^2)\frac{F'}{r}\right] \quad (3.10)$$

IV. — THE BOUNDARY CONDITIONS

The boundary of the charged fluid sphere is given by $r = a$, a being the constant radius in co-moving coordinates. At this boundary the fluid pressure must vanish. Again if the total mass and charge of the distribution are m and e respectively, the line-element (3.6), when couched in appropriate coordinates R and T must go over continuously to Nordstrom's line-element.

$$dS^2 = \left(1 - \frac{2m}{R} + \frac{e^2}{R^2}\right)dT^2 - \left(1 - \frac{2m}{R} + \frac{e^2}{R^2}\right)^{-1}dR^2 - R^2d\Omega^2 \quad (4.1)$$

across the fluid boundary.

The condition that $p = 0$ at $r = a$ gives an equation to determine the arbitrary function $G(t)$. Putting $r = a$, $p = 0$ in (3.7) and integrating once, we find that

$$\dot{s}^2 = k(1-s)/s^3 \quad (4.2)$$

where $s = s(t) \equiv b + G(t)$, $F(a) = b$ and it is assumed that $\dot{s} = 0$ when $s = 1$.

Transforming the co-moving coordinates (r, t) of the line-element (3.6) of the interior solution to the coordinates (R, T) of the exterior solution

and then establishing continuity of $g_{\mu\nu}$ and F_{14} across the boundary of the fluid sphere, we find that

$$m = \frac{ka^3}{2} - a^2(1 - ka^2)c, \quad c = F'(a) \quad (4.3)$$

$$e = ca^2\sqrt{1 - ka^2} \quad (4.4)$$

The boundary $R = R_0(T)$ of the fluid sphere is observed by a distant observer as contracting at the rate

$$\frac{dR_0}{dT} = a\dot{S}/\dot{T} \quad (4.5)$$

with

$$\dot{T} = \frac{(S + ac)(1 - ka^2)^{1/2}}{S^2\left(1 - \frac{2m}{aS} + \frac{e^2}{a^2S^2}\right)} \quad (4.6)$$

V. — CONCLUSION

The above is a model of a charged fluid sphere which contracts under its gravitational field. The general process of contraction to a singularity is of the same nature as in the Oppenheimer and Snyder's model. This is not quite unexpected because once the gravitational pull dominates over electrical repulsion and the contraction starts, there is no reason why, as contraction proceeds, the balance between the two opposing forces should be disturbed. Thus the effect of an electrostatic repulsion will only be on the rate of contraction and that it is not expected to halt the contraction and reverse the process.

REFERENCES

- [1] F. HOYLE and J. NARLIKAR, *Proc. Roy. Soc.*, t. A 278, 1964, p. 465.
- [2] G. C. MCVITTIE, on *Newtonian Gravitational Collapse*, pre-print, 1965.
- [3] W. B. BONNER, *Monthly Not. Roy. Astron. Soc.*, t. 129, 1965, p. 443-446.