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by

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ABSTRACT. — A number of results are derived which are pertinent to the description of physical systems by states on C*-algebras invariant under a symmetry group. In particular an integral decomposition of a state into states of lower symmetry is obtained which is relevant to the study of spontaneously broken symmetries which occur in equilibrium statistical mechanics as existence of crystals, ferromagnetic states, etc. A characterization is given of strongly clustering euclidean invariant states, and it is shown that they cannot be decomposed into states of lower symmetry.

1. INTRODUCTION

In recent papers [1] [2] [3] we began the analysis of the structure of invariant states over C*-algebras and the purpose of the present paper is to continue this analysis. The principal physical motivation for this programme is provided by statistical mechanics, both classical and quantum, where certain invariant states present themselves as natural candidates for the description of equilibrium [4]. A fuller description of the motivation and the mathematical concepts which we use is given in the references cited above. We proceed immediately to the introduction of various definitions and notations, which will then be used throughout the paper, and which allow us to describe more precisely the aims of the sequel.

We consider a C*-algebra $\mathcal{A}$ with identity, a topological group $G$ with

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identity $e$, and a representation $\tau$ of $G$ as automorphisms of $A$ i.e. for every $g \in G$ there is an automorphism $\tau_g$ of $A$, 

$$A \in A \to \tau_g A \in A.$$ 

We almost always assume strong continuity of the automorphisms i.e.

$$\|\tau_g A - A\| \to 0, \quad A \in A$$

(1)

where $\|\|$ is the algebraic norm. We denote by $A^*$ the dual of $A$ and by $E$ the set of states (positive linear forms of norm 1) over $A$ so $E \subseteq A^*$. If $f \in A^*$ we define $\tau_g f$ by

$$\tau_g f(A) = f(\tau_g^{-1} A) \quad \text{for all} \quad A \in A$$

and if $\tau_g f = f$ for all $g$ we say that $f$ is $G$-invariant. Denoting by $L_0$ the subspace of $A$ generated by elements of the form $A - \tau_g A$ and by $L_0^+$ the weakly closed subspace of $A^*$ defined by

$$L_0^+ = \{ f \in A^* : A \in L_0 \Rightarrow f(A) = 0 \}$$

it is immediately clear that $f \in A^*$ is $G$-invariant if, and only if, $f \in L_0^+$. Further, the $G$-invariant states over $A$ are the elements of the convex (weakly) compact set $E \cap L_0^+$. We denote the extremal points of a subset $K$ of $A^*$ by $\partial(K)$.

Now, using a well known method due to Gelfand, Naimark and Segal, it is possible to construct from a state $\rho \in E$ a representation of $A$ by bounded operators $\pi_\rho(A)$, acting on a Hilbert space $\mathcal{H}_\rho$, with a normalized vector $\Omega_\rho \in \mathcal{H}_\rho$ which is cyclic for $\pi_\rho(A)$ in $\mathcal{H}_\rho$. The explicit connection of these various quantities is given by

$$\rho(A) = (\Omega_\rho, \pi_\rho(A)\Omega_\rho)$$

If $\rho \in E \cap L_0^+$ the above construction also yields a unitary representation $U_\rho$ of $G$ acting on $\mathcal{H}_\rho$ which is strongly continuous if (1) is satisfied and such that for all $g \in G$ and $A \in A$

$$U_\rho(g)\Omega_\rho = \Omega_\rho \quad \text{and} \quad U_\rho(g)\pi_\rho(A)U_\rho(g)^{-1} = \pi_\rho(\tau_g(A)).$$

Thus $\Omega_\rho$ is a $G$-invariant vector in $\mathcal{H}_\rho$.

In this paper we will be principally interested in studying extremal $G$-invariant states i.e. states $\rho \in \mathcal{E}(E \cap L_0^+)$, a property that is equivalent to the property that the set of operators $\pi_\rho(A) \cup U_\rho(G)$ is irreducible on $\mathcal{H}_\rho$. It
was demonstrated in [2] [3] that for C*-algebras which are « asymptotically abelian » with respect to G, i.e. algebras with the property
\[ \| [A_1, \tau_g A_2] \| \xrightarrow{g \to \infty} 0, \quad A_1, A_2 \in \mathcal{A} \quad \text{and} \quad g \in G, \]
and if certain separability conditions are satisfied, a state \( \rho \in E \cap L^1_0 \) can be uniquely expressed as an integral over states \( \rho_k \in \mathcal{E}(E \cap L^1_0) \). Thus in such cases the study of states \( \rho \in E \cap L^1_0 \) is effectively reduced to a study of the extremal states.

Actually the above mentioned result was presented in [2] [3] for \( G = \mathbb{R}^n \) but the generalization to a large class of locally compact groups can be obtained by merely making appropriate notational changes in the proofs.

Our first new result, presented in Section 2, is an integral representation of a state \( \rho \in \mathcal{E}(E \cap L^1_0) \) in terms of states in \( \mathcal{E}(E \cap L^1_n) \) where \( H \) is a closed invariant subgroup of \( G \) such that \( G/H \) is compact. If the algebra \( \mathcal{A} \) has the asymptotically abelian property with respect to \( H \) the decomposition is unique. This theorem is of interest in statistical mechanics in connection with the occurrence of spontaneously broken symmetries; it is a generalization of a theorem obtained in [3] for \( G = \mathbb{R}^n \). In Section 3 we specialize to the case where \( G \) is the Euclidean group in \( n \) dimensions and discuss strongly clustering euclidean invariant states. In Section 4 we establish properties of the spectrum of the \( n \)-dimensional translation groups for various cases of physical interest.

## 2. DECOMPOSITION THEOREM

In the following decomposition theorem we use the measure theoretic techniques introduced in [2]. As explained in [2] (for fuller details see [5]) a partial order \( \succ \) may be introduced among the positive measures on the convex compact set \( E \cap L^1_0 \) such that \( \mu_1 \succ \mu_2 \) is equivalent to \( \mu_1(\varphi) \geq \mu_2(\varphi) \) for all convex continuous functions \( \varphi \) on \( E \cap L^1_0 \) and thus \( \mu_2(\psi) = \mu_2(\psi) \) for all continuous linear functions \( \psi \) on \( E \cap L^1_0 \). Thus if for \( A \in \mathcal{A} \) we define the complex continuous linear function \( \hat{A} \) on \( E \cap L^1_0 \) by
\[ \hat{A}(\sigma) = \sigma(A), \quad \sigma \in E \cap L^1_0, \]
then a measure \( \mu_\rho \) with the property \( \mu_\rho \succ \delta_\rho \), where \( \delta_\rho \) is the unit mass at \( \rho \), provides the decomposition
\[ \rho(A) = \hat{A}(\rho) = \delta_\rho(\hat{A}) = \mu_\rho(\hat{A}). \]
THEOREM 1 (I). — Let $G$ be a topological group and $H$ a closed invariant subgroup of $G$ such that the quotient space $G/H$ is compact and has a $G$-invariant measure $dg$ normalized to 1 i.e.

$$\int_{G/H} dg = 1$$

Further, let $\mathcal{A}$ be a C*-algebra with identity, $\tau$ a strongly continuous representation of $G$ as automorphisms of $\mathcal{A}$, and take $\rho \in \mathcal{E}(E \cap L^1_H)$. Then there exists a measure $\mu_\rho$ concentrated on $\mathcal{E}(E \cap L^1_H)$ (and thus maximal for the order $\succ$) which majorizes the unit mass $\delta_\rho$ at $\rho$. Furthermore there exists a $\tilde{\rho} \in \mathcal{E}(E \cap L^1_H)$ such that for every continuous function $\varphi$ on $E \cap L^1_H$

$$\mu_\rho(\varphi) = \int_{G/H} \varphi(\tau_g \tilde{\rho})$$

(2)

In particular for all $\Lambda \in \mathcal{A}$

$$\rho(\Lambda) = \int_{G/H} \tilde{\rho}(\tau_g^{-1}\Lambda)$$

We remark firstly that the condition that the space $G/H$ should have a $G$-invariant measure $dg$ would automatically be satisfied if $G$ and $H$ were unimodular locally compact groups (see for instance [6]). More generally if $G$ is locally compact and $\Delta_G, \Delta_H$ are the modular functions of $G$ and $H$ respectively then a necessary and sufficient condition for the existence of $dg$ is that

$$\Delta_G(h) = \Delta_H(h) \quad \text{for all} \quad h \in H$$

PROOF. — We begin by noting that for $\rho \in E \cap L^1_H$ fixed, $\hat{\Lambda}(\tau_g \rho)$ is a continuous function on $G$ and is invariant under right translations by $H$. Therefore it defines a continuous function on $G/H$. Next let us define the average $\langle \hat{\Lambda} \rangle$ of $\hat{\Lambda}$ by

$$\langle \hat{\Lambda} \rangle(\rho) = \int_{G/H} \hat{\Lambda}(\tau_g \rho)$$

and we now demonstrate that $\langle \hat{\Lambda} \rangle$ is a continuous function on $E \cap L^1_H$. We denote by $\psi_\rho$ the image of $g \in G$ under the canonical mapping $p$ of $G$ onto $G/H$ and write

$$\psi_\rho(\hat{\varphi}) = \hat{\Lambda}(\tau_g \rho)$$

(1) We are very indebted to J. Ginibre for discussions of this theorem; the proof given here was, up to minor changes, written by him, and is reproduced with his permission.
Now for any \( \varepsilon > 0 \) there exists an open neighbourhood \( N(g_i) \) of \( g_i \in G \) such that
\[
\| \tau_{g_i}^{-1}A - \tau_{g_i}^{-1}A \| \leq \varepsilon/4 \quad \text{for all} \quad g \in N(g_i).
\]
As \( \rho \) is open the image \( \hat{N}(\hat{g}_i) = pN(g_i) \) is an open neighbourhood of \( \hat{g}_i \) in \( G/H \) and as for any \( \hat{g} \in \hat{N}(\hat{g}_i) \) there exists \( g \in p^{-1}(\hat{g}) \cap N(g_i) \) we have
\[
| \psi_\rho(\hat{g}) - \psi_\rho(\hat{g}_i) | = | \hat{A}(\tau_{\hat{g}_i}\rho) - \hat{A}(\tau_{\hat{g}_i}\rho) | \leq \varepsilon/4
\]
Now let \( (\hat{N}(\hat{g}_i))_{i=1}^n \) be a finite covering of the compact space \( G/H \) by such neighbourhoods (i.e. \( \hat{N}(\hat{g}_i) = pN(g_i) \)). Consider \( \rho' \in E \cap L^1_H \) such that
\[
| \hat{A}(\tau_{\hat{g}_i}\rho) - \hat{A}(\tau_{\hat{g}_i}\rho') | \leq \varepsilon/2 \quad \text{for} \quad i = 1 \ldots n
\]
Since any \( \hat{g} \in G/H \) must lie in \( \hat{N}(\hat{g}_i) \) for some value of \( i \), we have
\[
| \psi_\rho(\hat{g}) - \psi_\rho(\hat{g}_i) | \leq | \psi_\rho(\hat{g}) - \psi_\rho(\hat{g}_i) | + | \psi_\rho(\hat{g}_i) - \psi_\rho(\hat{g}_i) | \\
+ | \psi_\rho'(\hat{g}) - \psi_\rho(\hat{g}_i) | \leq \varepsilon
\]
Therefore \( \psi_\rho(\hat{g}) \) tends uniformly to \( \psi_\rho(\hat{g}_i) \) as \( \rho' \) tends to \( \rho \), this proves the continuity of \( \langle \hat{A} \rangle \) as a function of \( \rho \).

We next define, for any \( \rho \in E \cap L^1_H \), the average of \( \rho \) by
\[
\langle \rho \rangle (A) = \langle \hat{A} \rangle (\rho)
\]
Clearly \( \langle \rho \rangle \in E \cap L^1_0 \) and if \( \rho \in E \cap L^1_0 \) then \( \langle \rho \rangle = \rho \). We now consider a fixed \( \rho \in E \cap L^1_0 \) and define the set \( K_\rho \) by
\[
K_\rho = \{ \sigma \quad ; \quad \sigma \in E \cap L^1_H , \quad \langle \sigma \rangle = \rho \}
\]
or, more explicitly,
\[
K_\rho = \{ \sigma \quad ; \quad \sigma \in E \cap L^1_H , \quad \langle \hat{A} \rangle (\sigma) = \hat{A}(\rho) \quad \text{for all} \quad A \in \mathcal{A} \}
\]
For every \( A \in \mathcal{A} \) the set
\[
\{ \sigma \quad ; \quad \sigma \in E \cap L^1_H , \quad \langle \hat{A} \rangle (\sigma) = \hat{A}(\rho) \}
\]
is closed, because of the continuity of \( \langle \hat{A} \rangle \). Therefore \( K_\rho \) is a closed subset of \( E \cap L^1_H \) and hence compact. On the other hand \( K_\rho \) is convex and not empty so it has extremal points. We next show that if \( \tilde{\rho} \in \mathcal{E}(K_\rho) \)
then $\tilde{\rho} \in \mathcal{E}(E \cap L_+^1)$. In fact suppose that $\tilde{\rho} \notin \mathcal{E}(E \cap L_+^1)$ then there exist $\tilde{\rho}_1$, $\tilde{\rho}_2 \in E \cap L_+^1$ and $\lambda$ real $(0 < \lambda < 1)$ such that $\tilde{\rho} \neq \tilde{\rho}_1$ and

$$\tilde{\rho} = \lambda \tilde{\rho}_1 + (1 - \lambda)\tilde{\rho}_2, \quad \tilde{\rho} \in \mathcal{E}(K_p).$$

At least one of $\tilde{\rho}_1$, $\tilde{\rho}_2$ cannot lie in $K_p$ and we suppose $\tilde{\rho}_1 \notin K_p$. Hence $\rho \neq \langle \tilde{\rho}_1 \rangle$. On the other hand

$$\rho = \langle \tilde{\rho} \rangle = \lambda \langle \tilde{\rho}_1 \rangle + (1 - \lambda)\langle \tilde{\rho}_2 \rangle$$

which is in contradiction with the assumption that $\rho \in \mathcal{E}(E \cap L_+^1)$. Therefore $\tilde{\rho} \in \mathcal{E}(E \cap L_+^1)$.

Now let $\varphi$ be a continuous function on $E \cap L_+^1$, i.e., $\varphi \in C(E \cap L_+^1)$. Then $\varphi(\tau_{g}\tilde{\rho})$ is a continuous function on $G$ and is invariant under right translations by $H$. We can therefore define

$$\mu_{\varphi}(\varphi) = \int_{G/H} dg \varphi(\tau_{g}\tilde{\rho})$$

With this definition $\mu_{\varphi}$ is a positive linear functional on $C(E \cap L_+^1)$ and hence a positive measure. If $\varphi \in C(E \cap L_+^1)$ vanishes on $\mathcal{E}(E \cap L_+^1)$ then $\varphi(\tau_{g}\tilde{\rho}) = 0$ for all $g \in G$ and $\mu_{\varphi}(\varphi) = 0$. Therefore $\mu_{\varphi}$ is concentrated on $\mathcal{E}(E \cap L_+^1)$.

Finally for any $A \in \mathcal{A}$ we have $\mu_{\varphi}(\hat{A}) = \langle \tilde{\rho} \rangle(A) = \rho(A)$ which concludes the proof.

Thus we have now established the existence of a measure $\mu_{\varphi} \geq \delta_{v}$ which is not only maximal with respect to the order relation but is also concentrated on the extremal points $\mathcal{E}(E \cap L_+^1)$. If $\mathcal{A}$ is asymptotically abelian with respect to $H$, the maximal measure $\mu_{\varphi}$ is unique due to the results of [2] [3] and hence the decomposition (2) is unique.

We now turn our attention to properties of extremal invariant states for more specific cases of physical interest.

3. EUCLIDEAN INVARIANT STATES

The first result we derive is independent of algebraic structure.

**Lemma (2).** — Let $U : (a, R) \to U(a, R)$ be a strongly continuous unitary representation in the Hilbert space $\mathcal{H}$ of the euclidean group of $R^{v}$, $v \geq 2$.

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(2) This lemma is related to some results obtained in quantum field theory for the Lorentz group, see for instance Borchers [7], lemma 4.
If \( \Phi, \Psi \in \mathcal{K} \) and if \( P_0 \) is the projection on the subspace of \( \mathcal{K} \) formed by the vectors invariant under \( U(R^v, 1) \), then

\[
\lim_{a \to \infty} (\Psi, U(a, 1)\Phi) = (\Psi, P_0\Phi) \tag{1}
\]

**Proof.** — For clarity we shall use in this proof the functional notation \( dp_\mu(p) \) instead of \( d\mu(p) \) for a measure \( \mu \).

By Stone's theorem, there exists a projection-valued measure \( E \) on \( R^v \) such that

\[
U(a, 1) = \int dpe^{-ipa}E(p) \tag{2}
\]

The measure \( \mu = (\Psi, E\Phi) \) then satisfies

\[
\int dp \left| \mu(R^{-1}p) - \mu(p) \right| = \int dp \left| (\Psi, [E(R^{-1}p) - E(p)]\Phi) \right|
\]

\[
= \int dp \left| (U(o, R)\Psi', E(p)U(o, R)\Phi) - (\Psi, E(p)\Phi) \right|
\]

\[
\leq \int dp \left| ([U(o, R) - 1]\Psi', E(p)U(o, R)\Phi) \right|
\]

\[
+ \int dp \left| (\Psi, E(p)[U(o, R) - 1])\Phi) \right|
\]

\[
\leq \|U(o, R)\Psi - \Psi\| \cdot \|\Phi\| + \|U(o, R)\Phi - \Phi\| \cdot \|\Psi\| \tag{3}
\]

Thus, given \( \epsilon > 0 \) there exists a neighbourhood \( \mathcal{N} \) of 1 in the orthogonal group in \( v \) dimensions such that if \( R \in \mathcal{N} \), then

\[
\int dp \left| \mu(R^{-1}p) - \mu(p) \right| < \epsilon \tag{4}
\]

Because the orthogonal group is compact, we may choose \( \mathcal{N} \) invariant under inner automorphisms.

We may choose a \( C^\infty \) function \( \varphi \geq 0 \) with support in \( \mathcal{N} \), invariant under inner automorphisms of its argument and such that \( \int \varphi(R)dR = 1 \). We have then

\[
\int dp \left| \int dR\varphi(R)\mu(R^{-1}p) - \mu(p) \right| < \epsilon \tag{5}
\]

If we can prove that

\[
\lim_{a \to \infty} \int dpe^{-ipa}\int dR\varphi(R)\mu(R^{-1}p) = (\Psi, P_0\Phi) \tag{6}
\]
then, (1) will result from (5), (6) and

\[(\Psi', U(a, 1)\Phi) = \int dpe^{-ipa}\mu(p)\]  

(7)

We have

\[\int dpe^{-ipa} \int dR\varphi(R)\mu(R^{-1}p) = \int d\mu(p) \int dR\varphi(R)e^{-i(Rp)a}\]  

(8)

where

\[\int dR\varphi(R)e^{-i(Rp)a} = f(pa)\]  

(9)

depends only on the scalar product \(pa\) because of the assumed invariance of \(\varphi\) under inner automorphisms of its argument. We may write

\[f(pa) = \int dp' e^{-ip'a} \int dR\varphi(R)\delta(p' - Rp)\]  

(10)

and for \(p \neq 0\) fixed

\[\int dR\varphi(R)\delta(p' - Rp) = \delta(\mid p' \mid - \mid p \mid)\psi(p')\]  

(11)

where \(\psi \in \mathcal{D}(R^\alpha)\). Since the Fourier transform of \(\delta(\mid \cdot \mid - \mid p \mid)\) is continuous and tends to zero at infinity and the Fourier transform of \(\psi\) is absolutely integrable, it follows from (10) that \(f\) tends to zero at infinity. Since \(f(0) = 1\), we see from (8), (9) that

\[\lim_{a \to \infty} \int dpe^{-ipa} \int dR\varphi(R)\mu(R^{-1}p) = \lim_{a \to \infty} \int d\mu(p) f(pa) = \mu(\{0\}) = (\Psi', P_0\Phi)\]  

(12)

which proves (6) and therefore the lemma.

Now, with the aid of the above lemma, we can deduce the following theorem concerning the structure of euclidean invariant states.

**Theorem 2.** — Let \(\mathcal{A}\) be a C*-algebra with identity, \(G\) the euclidean group of \(R^\alpha\), \(\varphi \in \mathcal{E} \cap L_0^\alpha\), \(\tau\) a representation of \(G\) as automorphisms of \(\mathcal{A}\) such that the corresponding unitary representation \(U_\varphi\) of \(G\) in \(\mathcal{K}_\varphi\) is strongly continuous. The following conditions are equivalent.

1. \(\lim_{a \to \infty} \varphi(A_1 \tau(a, 1)A_2) = \varphi(A_1) \varphi(A_2)\) for \(A_1, A_2 \in \mathcal{A}\)

2. \(\Omega_\varphi\) is the unique vector in \(\mathcal{K}_\varphi\) invariant under \(U_\varphi(R^\alpha, 1)\)

and imply the following equivalent conditions.
3. $\rho \in \mathcal{E}(E \cap L^1_n)$.
4. $\pi_\rho(\mathcal{A}) \cap U_{\rho}(H)$ is irreducible

for any closed non compact subgroup $H$ of $G$. Conversely, if $\mathcal{A}$ is asymptotically abelian with respect to $R^\nu$ and $H \subset R^\nu$, then 3. (or 4.) implies 1. (or 2.).

The equivalence 1. $\Leftrightarrow$ 2. follows directly from the Lemma and [3] and the equivalence 3. $\Leftrightarrow$ 4. is of general nature (see [2] and [3] for characterizations of extremal invariant states). The irreducibility of $\pi_\rho(\mathcal{A}) \cup U_{\rho}(H)$ for $H \subset R^\nu$ implies the irreducibility of $\pi_\rho(\mathcal{A}) \cup U_{\rho}(R^\nu)$ and therefore if $\mathcal{A}$ is asymptotically abelian 4. $\Rightarrow$ 2.

We conclude the proof of the theorem by showing that 1. $\Rightarrow$ 4. Since $H$ is closed non compact we can choose a sequence $(a_i, R_i)$ of elements of $H$ such that $a_i \to \infty$, and since the orthogonal group is compact we may assume (possibly going to a subsequence) that $R_i \to R_0$. If $\Phi \in \mathcal{K}_\rho$ is invariant under $U_{\rho}(H)$, we have then $U_{\rho}(a_i, R_i)\Phi = \Phi$ or $U_{\rho}(a_i, R_i)\Phi = U_{\rho}(-a_i, 1)\Phi$. By construction

$$\lim_{i \to \infty} U_{\rho}(a_i, R_i)\Phi = U_{\rho}(a, R)\Phi \text{ strongly}$$

On the other hand the assumption 1 and the lemma yield

$$\lim_{i \to \infty} U_{\rho}(-a_i, 1)\Phi = P_0\Phi \text{ weakly}$$

Therefore

$$\Phi = U_{\rho}(a, R^{-1})P_0\Phi = P_0\Phi'$$

and the subspace of vectors $\Phi \in \mathcal{K}_\rho$ invariant under $U_{\rho}(H)$ reduces to the scalar multiples of $\Omega_\rho$. It follows then by standard arguments that

$$[\pi_\rho(\mathcal{A}) \cup U_{\rho}(H)]' = [\pi_\rho(\mathcal{A}) \cup U_{\rho}(R^\nu, 1)]' = \{ \lambda I \}$$

so that 1. implies 4.

The above theorem is similar to theorems given in [1], [2] and [3] for invariant states with $G = R^\nu$. The major difference between the above theorem and the previous theorems is that the strong cluster property 1. has replaced the weak cluster property which characterizes the states $\rho \in \mathcal{E}(E \cap L^1_n)$. It is interesting to note that as Theorem 2 establishes that a euclidean invariant state $\rho$ with the strong cluster property is such that $\rho \in \mathcal{E}(E \cap L^1_n)$ for all non-compact subgroups $H \subset G$ there can be no (non-trivial) decomposition of $\rho$ of the type derived in theorem 1. Thus one might say that the «natural» invariance of such a state is the full euclidean invariance. Conversely, if $\mathcal{A}$ is asymptotically abelian, $\rho \in \mathcal{E}(E \cap L^1_n)$
but is not strongly clustering, then there must exist a non-trivial decomposition of \( \rho \) into states with a lower invariance.

In [3] extremal invariant states over an asymptotically abelian algebra were studied with \( G = \mathbb{R}^n \) and a classifications \( E_1, E_2, \) and \( E_3 \) of such states was introduced. In the light of the above discussion this classification can be understood as follows. An \( E_1 \)-state has « natural » invariance under all translations \( G = \mathbb{R}^n \); an \( E_2 \)-state can be decomposed uniquely into states with « natural » invariance under a subgroup of translations, \( H = \mathbb{R}^{n-n_1} \times \mathbb{Z}^{n_1} \) with \( 0 < n_1 \leq n \); an \( E_3 \)-state can be decomposed with respect to many different subgroups of \( G \) but no decomposition leads to states with a « natural » invariance. Actually this classification was introduced in [3] through consideration of the spectral properties of the unitary operators \( U_\rho(\mathbb{R}^n) \) associated with a state \( \rho \in \mathcal{E}(E \cap L^1_{\mathbb{R}^n}) \). In the next section we derive further properties of this type.

4. SPECTRUM PROPERTIES

**Theorem 3 a.** — Let \( G = \mathbb{R}^{n_1} \times \mathbb{Z}^{n_1} \) and \( \rho \in \mathcal{E} \cap L^1_{\mathbb{R}^n} \). We assume that the representation \( U_\rho \) is strongly continuous. Let \( \mathcal{E} \) be the projection-valued measure on \( \mathbb{R}^{n_1} \times \mathbb{T}^{n_1} \) such that

\[
U_\rho(g) = \int d\mathcal{E}(p) e^{-ip \cdot g}.
\]

1. Let \( \mathcal{A} \) be abelian. Then, \( \text{supp} \, \mathcal{E} = - \text{supp} \, \mathcal{E} \).

2. Let \( \mathcal{A} \) contain a dense subset \( \tilde{\mathcal{A}} \) such that, if \( A_1, A_2 \in \tilde{\mathcal{A}} \), then \( [A_1, \tau_g A_2] = 0 \) for \( g \) outside of some compact.

If \( S \subset \mathbb{R}^n \) has Lebesgue measure zero and \( \mathcal{E}(S) \neq 0 \), then \( \mathcal{E}(-S) \neq 0 \).

3. Let \( \mathcal{A} \) be asymptotically abelian and \( S_d = \{ p \in \mathbb{R}^n : \mathcal{E}(\{ p \}) \neq 0 \} \). Then \( S_d = - S_d \).

4. Let \( \mathcal{A} \) be asymptotically abelian and \( \rho \in \mathcal{E}(E \cap L^1_{\mathbb{R}^n}) \), then \( S_d + S_d \subset S_d \) and if \( p \in S_d \), then \( \mathcal{E}(\{ p \}) \) is one-dimensional.

To prove 1. and 2. we use the fact that, if \( A \in \mathcal{A} \), the Fourier transform of the measure

\[
\mu(p) = (\pi_\rho(A)\Omega_\rho, E(p)\pi_\rho(A)\Omega_\rho) - (\pi_\rho(A)^*\Omega_\rho, E(-p)\pi_\rho(A)^*\Omega_\rho)
\]

is \( \rho([A^*, \tau_g A]) \). In case 1. we have thus \( \mu = 0 \). In case 2., for \( A \in \tilde{\mathcal{A}} \), \( \mu \) is an analytic function.
To prove 3. we use lemma 1 of [3] which establishes, for a certain filter of functions $f_x$, that

$$0 = \lim_{x \to \infty} \int d\gamma f_x(g) e^{-ipg} \varphi([A_x, \tau_x A])$$

$$= (\pi_x(A)\Omega_\varphi, E(\{p\})\pi_x(A)\Omega_\varphi) - (\pi_x(A)^*\Omega_\varphi, E(\{-p\})\pi_x(A)^*\Omega_\varphi)$$

Statement 4. has been included for completeness: it was already proved in [3] (Theorem 4).

With slightly stronger continuity assumptions we have

**THEOREM 3 b (?)** — Let $G = \mathbb{R}^n \times \mathbb{Z}^n$ and $\varphi \in \mathcal{E}(E \cap L^1_0)$. We assume that for all $A \in \mathcal{A}$, $\lim_{g \to 0} \| \tau_g A - A \| = 0$. Let $E$ be the projection-valued measure on $\mathbb{R}^n \times T^*$ such that

$$U_\varphi(g) = \int dE(p) e^{-ipg}$$

If $\mathcal{A}$ is asymptotically abelian, then $\text{supp } E + \text{supp } E \subset \text{supp } E$.

If $A \in \mathcal{A}$, let $p \to \hat{A}(p)$ be the Fourier transform of $g \to \tau_g A$. If $A_\varphi = \int dg \varphi(g)^* \tau_g A$, then $\text{supp } \hat{A}_\varphi \subset \text{supp } \hat{\varphi}$.

Let now $p_1, p_2 \in \text{supp } E$, and $\mathcal{N}_i$ be a neighbourhood of $p_i$. One may choose $A_i$ of the form $A_\varphi$ such that $\text{supp } \hat{A}_i \subset \mathcal{N}_i$ and $\pi_x(A_i)\Omega_\varphi \neq 0$ (because $\pi_x(A_i)\Omega_\varphi$ is of the form $E(\hat{\varphi}^* \pi_x(A)\Omega_\varphi)$).

If $A = A_1 \tau_h A_2$ then, for every $h \in \mathbb{R}^*$, $\text{supp } \hat{A} \subset \mathcal{N}_1 + \mathcal{N}_2$ hence

$$E(\mathcal{N}_1 + \mathcal{N}_2)\pi_\varphi(A_1 \tau_h A_2)\Omega_\varphi = \pi_\varphi(A_1 \tau_h A_2)\Omega_\varphi.$$

(1)

We have on the other hand from [3] (lemma 1) that $f_x \geq 0$ exists such that

$$\lim_{x \to \infty} \int dh f_x(h) \| \pi_x(A_1 \tau_h A_2)\Omega_\varphi \|^* =$$

$$= \lim_{x \to \infty} \int dh f_x(h) \varphi(A_1^* \tau_h A_1, A_2^*)$$

$$= \lim_{x \to \infty} \int dh f_x(h) \varphi(A_1^* A_2, \tau_h (A_1^* A_1))$$

$$= \varphi(A_1^* A_1) \varphi(A_2^* A_2) \neq 0$$

which shows that, for some $h$, $\pi_\varphi(A_1 \tau_h A_2)\Omega_\varphi \neq 0$.

(1) implies then that $E(\mathcal{N}_1 + \mathcal{N}_2) \neq 0$, hence that $p_1 + p_2 \in \text{supp } E$, which proves the proposition.

(?) The proof of this lemma is based on a technique used in relativistic field theory. See Wightman [8], p. 30.
The above theorems have not established that \( \text{supp } E \) is in general symmetric and, indeed, asymmetry can arise but then the following result is valid.

**Theorem 3 c.** — Let \( G = \mathbb{R}^v \), \( \rho \in E \cap L^1_G \) and \( U_\rho \) be strongly continuous. Let \( E \) be the projection-valued measure on \( \mathbb{R}^v \) such that

\[
U_\rho(g) = \int dE(p) e^{-ipg}
\]

If \( E(\{0\}) \) is one-dimensional and \( E \) is concentrated on \( \{0\} \cup S \), where \( E(-S) = 0 \), then \( \tau_\rho(\mathcal{A}) \) is irreducible, i.e. \( \rho \in \mathcal{E}(E) \).

A simple proof is given in [9], p. 65.

Thus we see that the asymmetry of the spectrum implies that \( \rho \) is extremal among all states (a pure state), a situation which is typical of quantum field theory, but which would arise only exceptionally in statistical mechanics.

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