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Section A :
Physique théorique.

On the Regge symmetries of the 3J symbols of SU (2)

by

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ABSTRACT. — The symmetries of the $3j$ symbols of SU (2) are fitted into a broader group theory framework. The $3j$ symbols of SU (2) are identified to certain $3j$ symbols of SU (3). Viewed as SU (3) properties, their Regge symmetries are nothing but ordinary permutation symmetries.

1. — INTRODUCTION

It is known from Regge's work [*1*] that the symmetry group of the $3j$ symbols (or Clebsch-Gordan, or Wigner coefficients) of SU (2) is not the usual twelve element group of permutations and reflection but rather a larger seventy-two element group the deeper significance of which is not clear as yet. In this paper we give these new symmetries a group theory interpretation within a framework which, we hope, could be generalized to other groups.

We shall be guided by two remarks:

First: it makes more sense to permute like objects than different ones as in the former case the permutation group is a group of *invariance* while in the latter it is a mere group of operators acting on the set of those objects. This is well known in quantum mechanics. Thus, instead of dealing separately with the irreducible representations of SU (2), we embed them in a larger completely reducible representation space which contains each of them once, and only once, and we consider a collection of replicae of such identical spaces. Such an object has been constructed already by

Schwinger [2], and Bargmann [2] has given it a good status as a function space; he called it $\mathcal{F}_2$ (we recall their construction at the beginning of section 2). Now we consider the tensor product $\mathcal{F}_2 \otimes \ldots \mathcal{F}_2$ n times, the closure of it is $\mathcal{F}_{2n}$ in Bargmann's notation. It is now natural to permute the subspaces $\mathcal{F}_2$ and indeed we shall realize the group of the permutations of these n objects by explicit unitary operators in $\mathcal{F}_{2n}$ which will generate the usual permutation symmetries of the $3j$ symbols in the case of $\mathcal{F}_6$.

Second: Problems about $3j$ symbols are more properly to be viewed as problems in the theory of invariants. Indeed the $3j$ symbols for an arbitrary group G can be considered as coupling three irreducible representations of G to the trivial one-dimensional representation. A general notation which is well-suited to this viewpoint is $\begin{pmatrix} m_1 & m_2 & m_3 \\ j_1 & j_2 & j_3 \end{pmatrix}$; it is patterned after the notation for the $3j$'s of SU (2) (we forget about multiplicity indices which are necessary in the general case). Let us call $\begin{pmatrix} j_1 \\ m_1 \end{pmatrix}\begin{pmatrix} j_2 \\ m_2 \end{pmatrix}\begin{pmatrix} j_3 \\ m_3 \end{pmatrix}$ the corresponding basis vectors of the three representations. Then every vector $\begin{pmatrix} m_1 & m_2 & m_3 \\ j_1 & j_2 & j_3 \end{pmatrix}\begin{pmatrix} j_1 \\ m_1 \end{pmatrix}\begin{pmatrix} j_2 \\ m_2 \end{pmatrix}\begin{pmatrix} j_3 \\ m_3 \end{pmatrix}$ (the summation convention on the indices m_i is to be understood in this type of formulae) is invariant under the diagonal of $G \otimes G \otimes G$. Similar remarks could be made about the generalized Wigner coefficients which generalize $3j$ symbols for the case of n representations of G.

Let us then consider the direct product $G = SU(2) \otimes \ldots \otimes SU(2)$ n times. It acts in a canonical way in $\mathcal{F}_{2n}$. The preceding remark leads us into studying the subspace I_n of $\mathcal{F}_{2n}$ which is pointwise invariant under the diagonal K_n of G. We consider now the algebra of the linear operators in I_n. Clearly these operators are scalars i. e. they commute with the elements of K_n. It is natural to inquire about the existence of a subalgebra L_n of linear operators in I_n with the following properties: it is a finite Lie algebra and the space I_n is irreducible under L_n. We find that this is indeed the case, L_n being $\underline{SO}^*(2n)$ a special non compact form of $\underline{SO}(2n)$. $\underline{U}(n)$ is a maximal compact subalgebra. It will play a crucial role in our developments. The most remarkable fact is that it is the diagonal of the direct sum of two commuting $\underline{U}(n)$ algebras. We take up these matters in section 2.

Before proceeding further, we introduce a classification of the symmetry properties [3] of the $3j$ symbols of an arbitrary group into two classes. This will help us state our results.

CLASS I: *the permutation symmetries,* i. e., the ordinary properties under any permutation of the three representations labeled by j_1, j_2, j_3. In most cases it is possible to choose the bases in the representation spaces so that the effect of a permutation is a mere phase factor. Let us remark however that the problem is an open one for the special case wherein $j_1 j_2 j_3$ are equivalent to one another [4]. For SU (2) the usual choice leads to the invariance of the Wigner coefficients under even permutations and to its being multiplied by $(-1)^J$ for odd ones where $J = j_1 + j_2 + j_3$.

CLASS II comprehends all other symmetries the symbol may have, i. e., symmetries which cannot be accounted for by a permutation of the representations. In particular they may relate symbols involving different triples of representations. For SU (2) the new Regge symmetries and the symmetry under « time reversal » (m_α goes into $-m_\alpha$, $\alpha = 1, 2, 3$ together with multiplication by $(-1)^J$ belong to class II.

Regge's notation for the 3j symbols of SU (2) exhibits all symmetries at once although it is highly redundant

$$\begin{pmatrix} m_1 & m_2 & m_3 \\ j_1 & j_2 & j_3 \end{pmatrix} = \begin{pmatrix} k_1 & k_2 & k_3 \\ K_1 & K_2 & K_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix} \qquad \textit{where} \qquad \begin{aligned} k_\alpha &= J - 2j_\alpha \\ K_\alpha &= j_\alpha + m_\alpha \\ \lambda_\alpha &= j_\alpha - m_\alpha \end{aligned}$$

The class I symmetries consist of the permutations of the columns. The time reversal symmetry is associated with the permutation of the second and third rows. The other permutations of rows and the transposition of the matrix yield the new Regge symmetries (invariance under the even permutations of rows, and transposition, multiplication by $(-1)^J$ under the odd permutations). The symmetry group so obtained is made up of $3! \times 3! \times 2 = 72$ elements as announced at the beginning.

Now we can state our results on the 3j symbols of SU (2): The known class II symmetries of the general 3j symbols of SU (2) are identical to the class I symmetries of the 3j symbols of SU (3) (the maximal compact subalgebra of $L_3 = \underline{SO}^*(6)$) involving three *identical* representations of SU (3) of the completely symmetric type (here we reap the dividends of remark 1 above), with the exception of the transposition symmetry. The latter is of a different nature albeit also a SU (3) property; it is linked to the existence of an outer automorphism of SU (3). To prove these statements (in section III) we shall identify the general 3j for SU (2) with the above special 3j symbols of SU (3), up to a normalization factor. We shall also prove that the SU (3) algebra we have introduced can be used to generate a class of recursion relations between the $3j$ symbols, those among coefficients with the same value of J. The other recursion relations require either going outside SU (3) to $\underline{\mathrm{SO}^*}$ (6) or for those involving the same triple $j_1 j_2 j_3$ simply using the diagonal of SU (2) $\otimes$ SU (2) $\otimes$ SU (2).

Let us note that our technique does not seem to provide us with any new property of the Wigner coefficient of SU (3).

2. — THE CONSTRUCTION OF THE INVARIANT SUBSPACE I_n AND OF THE LIE ALGEBRA L_n FOR SU (2)

We rely upon Schwinger's powerful description of SU (2) in terms of spin one half boson creation and annihilation operators [2]. We shall use either Schwinger's abstract operator formalism or Bargmann's Hilbert function space realization of Schwinger's bosons. In the former description we write :

$$J_+ = a^*b \qquad J_0 = \frac{1}{2}(a^*a - b^*b) \qquad J_- = b^*a.$$

Naturally $[a, a^*] = 1 = [b, b^*]$, $[a, b] = 0 = [a^*, b^*]$ and the state space is Fock space. The bracket [A, B] is of course AB − BA. The casimir operator is $J^2 = j(j + 1)$ where $j = \frac{1}{2}(a^*a + b^*b)$.

In the latter, the elements of Barmann's space $\mathcal{F}_2$ are entire analytic functions of two complex variables ξ, η; the inner product is so chosen as to make ξ and $\frac{\partial}{\partial\xi}$, respectively η and $\frac{\partial}{\partial\eta}$, *adjoint to each other* with respect to this inner product. All polynomials in ξ and η belong to $\mathcal{F}_2$ and the set of all the monomials is a basis for $\mathcal{F}_2$.

The isomorphism between Schwinger's construction and Bargmann's is indicated by the following arrows $a \leftrightarrow \frac{\partial}{\partial\xi}$ $a^* \leftrightarrow \xi$ $b \leftrightarrow \frac{\partial}{\partial\eta}$ $b^* \leftrightarrow \eta$; the vacuum $|\, 0 >\ \leftrightarrow$ function 1.

The standard orthonormal basis for the irreducible representations of SU (2) can be written as follows:

$$|\,jm\,\rangle = \frac{(a^*)^{j+m}(b^*)^{j-m}}{\sqrt{(j+m)\,!\,(j-m)\,!}}\,|\,0\,\rangle \leftrightarrow \binom{j}{m} = \frac{\xi^{j+m}\,\eta^{j-m}}{\sqrt{(j+m)\,!\,(j-m)\,!}}\,.$$

Now let us consider $\mathcal{F}_{2n}$. The operators a, b, etc. now have an index $i = 1, \ldots, n$. Since the vacuum is a cyclic vector it is possible to construct I_n and L_n both at a time by just building all possible invariant operators out of the a_i, b_i, a_i^*, b_i^*. This is easy enough since the n couples $(a_i\, b_i)$, $i = 1, \ldots, n$, are n SU (2) spinors, the couples $(a_i^*\; b_i^*)$ being the adjoint spinors.

The basic invariant operators in terms of which *any* invariant operator can be expressed are then the scalar products (unitarity)

$$t_{ij} + a_i^* a_j + b_i^* b_j$$

and the determinants (unimodularity)

$$D_{ij} = a_i b_j - a_j b_i \qquad \text{and} \qquad D_{ij}^* = a_i^* b_j^* - a_j^* b_i^*.$$

Instead of t_{ij} it is more convenient to define

$$T_{ij} = t_{ij} + \delta_{ij} \qquad (\delta_{ij} \text{ Kronecker symbol}).$$

It is clear that the set of T_{ij}'s has the structure of the Lie algebra of $\underline{\mathrm{U}}\,(n)$

$$[T_{ij}, T_{kl}] = \delta_{jk} T_{il} - \delta_{il} T_{jk} \quad , \quad T_{ij}^* = T_{ji}.$$

Now the set of the T's, the D's and the D*'s is closed under the bracket operation. It is indeed a realization of a non-compact form of $\underline{\mathrm{SO}}\,(2n)$. (It is $\underline{\mathrm{SO}}^*\,(2n)$ as proved in the appendix.)

$$[D_{ij}, D_{kl}] = 0 \qquad [D_{ij}, D_{kl}^*] = \delta_{ik} T_{lj} + \delta_{jl} T_{ki} - \delta_{il} T_{kj} - \delta_{jk} T_{li}$$
$$[D_{ij}, T_{kl}] = \delta_{ik} D_{lj} + \delta_{jk} D_{il}.$$

This is our sought-for Lie algebra L_n. The invariant subspace I_n is easy to construct: it is spanned by the vectors

$$|\{k_{ij}\}\rangle = \prod_{1=i<j}^{n} [D_{ij}^*]^{k_{ij}} |0\rangle \qquad k_{ij} = \text{non negative integers.}$$

Let us note the fact that these vectors are not linearly independent for $n > 4$ because of the following set of algebraic relations (these actually are the only relations):

$$D_{ij} D_{kl} + D_{jk} D_{il} + D_{ki} D_{jl} = 0.$$

It is clear that I_n is an irreducible representation space for L_n and the vacuum a cyclic vector for it. Let us now focus our attention on the maximal compact subalgebra $\underline{\mathrm{U}}\,(n)$ or equivalently $\underline{\mathrm{SU}}\,(n)$. First we decompose I_n

into a direct sum of irreducible representations of $\underline{SU}(n)$. It is fairly obvious that the set of vectors $| \{ k_{ij} \} >$ with $\sum_{1=i<j}^{n} k_{ij} = J$ a fixed value, span an irreducible representation space for the T algebra (J is the value of the operator $\left.\sum_{i=1}^{n} t_{ii}\right)$. Indeed the T's act *within* every such set of vectors and any vector $| \{ k_{ij} \} >$ belonging to one value of J can be generated from any other one with the *same* J value by repeated commutation with the T's. We give this representation the label $2 \times J$ for reasons which should be clear in a moment.

Then we notice the capital and most unexpected fact that the T algebra is the *diagonal of the direct sum of two commuting* SU (n) *algebras*

$$T_{ij} = A_{ij} + B_{ij}$$

where

$$A_{ij} = a_i^* a_j \qquad \text{and} \qquad B_{ij} = b_i^* b_j .$$

This property seems to be rather general [5]; however it is in want of a mathematical theory which we do not attempt here.

We now concentrate on some implications of this property. It is obvious that the irreducible representations of the algebras A and B which are allowed in the space $\mathcal{F}_{2n}$ are all the completely symmetrical ones. They are characterized by an integer J and their Young diagram $\overset{\leftarrow J \rightarrow}{\boxed{}}$ of length J. We write down their standard orthonormal bases in Bargmann's notations:

$$\text{for A} : \frac{\xi_1^{K_1} \dots \xi_n^{K_n}}{\sqrt{K_1! \dots K_n!}} \; K_1 + \dots + K_n = J \text{ and for B} : \frac{\eta_1^{\lambda_1} \dots \eta_n^{\lambda_n}}{\sqrt{\lambda_1! \dots \lambda_n!}} \; \lambda_1 + \dots + \lambda_n = J.$$

Let us denote these monomials respectively $\binom{J}{[K]}$ and $\binom{J}{[\lambda]}$ in keeping with the notation $\binom{j}{m}$ we considered in the introduction. Now it is clear that the Young diagram of the representation we labeled $2 \times J$ is $\overset{\leftarrow J \rightarrow}{\boxed{\begin{array}{c}\\ \hline \end{array}}}$ of length J: the associated Young symmetrizer applied to our polynomial bases precisely yields the basis $| \{ k_{ij} \} >$ which we denote $\binom{2 \times J}{[k]}$ after orthonormalization. This enables us to express $\binom{2 \times J}{[k]}$ as a superposition of the vectors $\begin{pmatrix} J \\ [K] \end{pmatrix}\begin{pmatrix} J \\ [\lambda] \end{pmatrix}$ through the use of a 3j symbol for SU (n) (we are using unconventional but natural bases for our SU (n) representations).

$$\begin{pmatrix} 2 \times J \\ [k] \end{pmatrix} = \sqrt{\dim 2 \times J}\begin{pmatrix} 2 \times J & [K] & [\lambda] \\ [k] & J & J \end{pmatrix}\begin{pmatrix} J \\ [K] \end{pmatrix}\begin{pmatrix} J \\ [\lambda] \end{pmatrix}.$$

Thus this formula casts a new light on the *generalized* Wigner coefficients for SU (2) which can be identified (up to a normalization factor) with certain 3j symbols of SU (n). However we shall not pursue this line here but rather look at the cases $n = 2$ and 3 in closer detail.

Before leaving the general case let us write down explicit expressions for the operators which realize the permutations of the various subspaces $\mathcal{F}_2$ of $\mathcal{F}_{2n}$.

Since the permutation group of n objects can be generated by transpositions, we need only worry about these simpler operations. Let us consider the unitary operators

$$P_{ij} = \exp\left[i\frac{\pi}{2}(T_{ij} + T_{ji})\right] \qquad i \neq j.$$

It is a simple matter to compute *e.g.*

$$P_{ij}a_k^* P_{ij}^{-1} = a_k^* \qquad \text{for} \qquad k \neq i \text{ and } j$$
$$P_{ij}a_i^* P_{ij}^{-1} = ia_j^* \qquad \text{and} \qquad P_{ij}a_j^* P_{ij}^{-1} = ia_i^*.$$

Thus P_{ij} permutes the subspaces $\mathcal{F}_2^{(i)}$ and $\mathcal{F}_2^{(j)}$ and leaves the other subspaces $\mathcal{F}_2$ untouched. The operators P_{ij} and their products are the operators we are looking for. They represent a finite subgroup of the group SU (n) constructed from the T algebra, which is actually the Weyl group of SU (n).

3. — THE CASES $n = 2$ AND 3; THE REGGE SYMMETRIES AND THE RECURSION RELATIONS OF THE 3j SYMBOLS

3.1. — The case $n = 2$.

The space I_2 is readily constructed; it is spanned by the following orthonormal vectors:

$$\frac{\Delta_{12}^{J}}{J\,!\,\sqrt{J+1}} \qquad \text{where} \qquad \Delta_{12} = \xi_1\eta_2 - \xi_2\eta_1 \text{ in Bargmann's notation.}$$

Let us note that Δ_{12} in this formula is isomorphic to D^*_{12} as we defined it in section 2. Using both Bargmann's and Schwinger's schemes is at the source of minor notational difficulties.

We define what we should call a « 2*j* coefficient » were it not for the fact that it has names already: Herring metric tensor or 1*j* coefficient:

$$\frac{\Delta_{12}^{J}}{J\,!\,\sqrt{J+1}} = \sqrt{J+1}\begin{pmatrix} m_1 & m_2 \\ j_2 & j_2 \end{pmatrix}\begin{pmatrix} j_1 \\ m_1 \end{pmatrix}\begin{pmatrix} j_2 \\ m_2 \end{pmatrix}$$

and we readily find the well known result:

$$\begin{pmatrix} m_1 & m_2 \\ j_1 & j_2 \end{pmatrix} = \delta_{j_1 j_2}\frac{(-1)^{j_2+m_2}}{\sqrt{2j_2+1}}\delta_{m_1,-m_2} \qquad \text{and} \qquad 2j_1 = J.$$

The Lie algebra L_2 has a simple structure: it splits into a direct sum: $\underline{SU}(2) \oplus \underline{SL}(2, R)$ which is in keeping with an identification of L_n with $\underline{SO}^*(2n)$. The generators of SU (2) are the T's and those of $\underline{SL}$ (2, R) are the D's (and $T_{11} + T_{22}$).

Applying the techniques of section two, we readily identify the Herring metric with a SU (2) 3*j* symbol which is of course well known! One verifies that the space I_2 is an irreducible representation space for SL (2, R), of the discrete series type.

Now we note a property which we generalize below. The T algebra $T = A + B$ (in the notation of section 2) and the J algebra $J = J^{(1)} + J^{(2)}$ of the initially given angular moments are related to each other by an explicit outer automorphism which we write as follows

$$a_1^* b_1 + a_2^* b_2 \leftrightarrow a_1^* a_2 + b_1^* b_2, \text{ etc...}$$

where the role of summation indices goes over from (1, 2) to (*a*, *b*). The property we want to note is that the transposition symmetry of the determinant Δ_{12} is an expression of the fact that it is invariant under both the T algebra and the J algebra.

3.2. — The case $n = 3$: the Regge symmetries of the 3*j* symbol.

The coupling to the identity representation of three angular momenta is expressed by the formulae [*6*]

$$H_{[k]} = \begin{pmatrix} m_1 & m_2 & m_3 \\ j_1 & j_2 & j_3 \end{pmatrix}\begin{pmatrix} j_1 \\ m_1 \end{pmatrix}\begin{pmatrix} j_2 \\ m_2 \end{pmatrix}\begin{pmatrix} j_3 \\ m_3 \end{pmatrix}$$

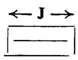
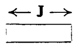
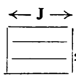
where

$$H_{[k]} = C_{[k]}\Delta_1^{k_1}\Delta_2^{k_2}\Delta_3^{k_3}\ ,\quad C_{[k]} = [k_1\,!\,k_2\,!\,k_3\,!\,(J+1)\,!]^{-1/2}\qquad (\|H_{[k]}\| = 1)$$

$$\Delta_i = \xi_j\eta_k - \xi_k\eta_j \qquad (i\,j\,k = \text{circular permutation of } (1\ 2\ 3)).$$

The polynomials $H_{[k]}$ form a basis for the invariant subspace I_3. The polynomials $H_{[k]}$ with $k_1 + k_2 + k_3 = J$ form a basis for the representation $2 \times J$ of SU (3). Thus we write:

$$H_{[k]} \equiv \begin{pmatrix} 2\times J \\ [k] \end{pmatrix} = \sqrt{\frac{(J+1)(J+2)}{2}}\begin{pmatrix} 2\times J & [k] & [\lambda] \\ [k] & J & J \end{pmatrix}\begin{pmatrix} J \\ [K] \end{pmatrix}\begin{pmatrix} J \\ [\lambda] \end{pmatrix}.$$

We have yet to remark that  and  are contragradient to each other since they can couple to the scalar , or determinant to the power J in standard polynomial bases. We then immediately derive the relation

$$\begin{pmatrix} 2\times J & [K] & [\lambda] \\ [k] & J & J \end{pmatrix} = \begin{pmatrix} [k] & [K] & [\lambda] \\ J & J & J \end{pmatrix}.$$

We can now carry out the identification of the general 3*j* symbol of SU (2) with a special 3*j* symbol of SU (3) as announced in the introduction; this symbol involves three *equivalent* completely symmetric representations of SU (3)

$$\begin{pmatrix} m_1 & m_2 & m_3 \\ j_1 & j_2 & j_3 \end{pmatrix} = \sqrt{\frac{(J+1)(J+2)}{2}}\begin{pmatrix} [k] & [K] & [\lambda] \\ J & J & J \end{pmatrix}$$

We remind the reader of the significance of certain letters:

$$J = j_1 + j_2 + j_3 \qquad k_\alpha = J - 2j_\alpha \qquad K_\alpha = j_\alpha + m_\alpha$$
$$\lambda_\alpha = j_\alpha - m_\alpha \qquad \alpha = 1, 2, 3$$

and, e. g., $[k]$ stands for the triple $(k_1\, k_2\, k_3)$.

It is easy to check that the 3*j* symbol of SU(3) at hand possesses the class I permutation symmetries i. e. it is invariant up to a phase factor under any permutation of the representations. This is also proved for the 3*j* symbols involving three arbitrary equivalent representations in ref. [7]. We verify this at the beginning of the next subsection.

We can now state our interpretation of the symmetries of the 3*j* symbol of SU (2): the classe I symmetries of the coefficient as a SU (3) $3j$ symbol *are* its class II symmetries as a SU (2) symbol with the exception of the transposition symmetry. Conversely its class I symmetries as a SU (2) symbol yield class II symmetries for it as a SU (3) symbol e. g.

$$\begin{pmatrix} (k_1 & k_2 & k_3)(\mathrm{K}_1 & \mathrm{K}_2 & \mathrm{K}_3)(\lambda_1 & \lambda_2 & \lambda_3) \\ & \mathrm{J} & & \mathrm{J} & & \mathrm{J} & \end{pmatrix} = \begin{pmatrix} (k_1 & k_3 & k_2)(\mathrm{K}_1 & \mathrm{K}_3 & \mathrm{K}_2)(\lambda_1 & \lambda_3 & \lambda_2) \\ & \mathrm{J} & & \mathrm{J} & & \mathrm{J} & \end{pmatrix}(-1)^{\mathrm{J}}.$$

Of course these are easily established in a direct way.

3.3. — The transposition symmetry.

We view it as a pure SU (3) problem. Let us consider three completely symmetrical irreducible representations of SU (3) labeled by the same Young diagram $\overset{\leftarrow \mathrm{J} \rightarrow}{\boxed{\quad\quad}}$. Let us call the corresponding polynomial bases $\begin{pmatrix} \mathrm{J} \\ [k] \end{pmatrix} = \dfrac{\tau_1^{k_1}\tau_2^{k_2}\tau_3^{k_3}}{\sqrt{k_1!\ k_2!\ k_3!}}$ and $\begin{pmatrix} \mathrm{J} \\ [k] \end{pmatrix}$ and $\begin{pmatrix} \mathrm{J} \\ [\lambda] \end{pmatrix}$ as before. We are thus working in $\mathcal{F}_9$ in Bargmann's notation. Then we form a scalar S with them and our SU (3) $3j$ symbol.

$$\mathrm{S} \equiv \begin{pmatrix} [k] & [\mathrm{K}] & [\lambda] \\ \mathrm{J} & \mathrm{J} & \mathrm{J} \end{pmatrix}\begin{pmatrix} \mathrm{J} \\ [k] \end{pmatrix}\begin{pmatrix} \mathrm{J} \\ [\mathrm{K}] \end{pmatrix}\begin{pmatrix} \mathrm{J} \\ [\lambda] \end{pmatrix}$$
$$= \frac{1}{\mathrm{J}!\sqrt{(\mathrm{J}+1)}}\sqrt{\frac{2}{(\mathrm{J}+1)(\mathrm{J}+2)}}\left\{\det\begin{vmatrix} \tau_1 & \tau_2 & \tau_3 \\ \xi_1 & \xi_2 & \xi_3 \\ \eta_1 & \eta_2 & \eta_3 \end{vmatrix}\right\}^{\mathrm{J}}.$$

Let us switch to Schwinger's notation $\xi_\alpha \leftrightarrow a_\alpha^*$ $\eta_\alpha \leftrightarrow b_\alpha^*$ $\tau_\alpha \leftrightarrow c_\alpha^*$, etc. By definition S being a scalar commutes with the following SU (3) algebra

$$\mathrm{U}_{ij} = \mathrm{A}_{ij} + \mathrm{B}_{ij} + \mathrm{C}_{ij} \qquad i, j = 1, 2, 3,$$

where of course e. g. $\mathrm{A}_{ij} = a_i^* a_j$. We now inquire as before about the operators made out of the $a\ b\ c$'s etc. which act within the one dimensional space S. Apart from S itself, there must be the SU (3) invariant scalar products of the spinors $a\ b\ c$, and the functions thereof. These are e. g. $a^*b = \sum_{\alpha=1}^{3} a_\alpha^* b_\alpha$. They generate another U (3) algebra which is isomorphic in its structure to the $\mathrm{U} = \mathrm{A} + \mathrm{B} + \mathrm{C}$ algebra. The associated outer automorphism is just the exchange of $(a\, b\, c)$ with $(1\, 2\, 3)$. The scalar S' isomorphic to S is then:

$$\frac{1}{J\,!\,\sqrt{(J+1)\,!}}\sqrt{\frac{2}{(J+1)(J+2)}}\left\{\det\begin{vmatrix}\tau_1 & \xi_1 & \eta_1\\ \tau_2 & \xi_2 & \eta_2\\ \tau_3 & \xi_3 & \eta_3\end{vmatrix}\right\}^J$$

which is clearly equal to S.

Thus $S' = S$ is the SU (3) interpretation of Regge's transposition symmetry. Expressed in SU (3) 3j symbols, it reads (a class II symmetry)

$$\begin{pmatrix}(k_1\ k_2\ k_3) & (K_1\ K_2\ K_3) & (\lambda_1\ \lambda_2\ \lambda_3)\\ J & J & J\end{pmatrix} = \begin{pmatrix}(k_1\ K_1\ \lambda_1) & (k_2\ K_2\ \lambda_2) & (k_3\ K_3\ \lambda_3)\\ J & J & J\end{pmatrix}.$$

It is clear that this property can be trivially carried over to the generalized SU (n) Wigner coefficient which couples n completely symmetrical representations of SU (n) to the identity. Let us also note that Schwinger's generating function ref. [2] fits in nicely in the present framework.

3.4. The recursion formulae.

They can be classified into three classes:

(i) the recursion relations which relate 3j coefficients with the same values of $j_1\, j_2\, j_3$ but different values of the m's. These have a direct meaning with respect to SU (2) $J = J^{(1)} + J^{(2)} + J^{(3)}$. They are obtained by writing:

$$[H_{[k]}, J_\pm] = 0$$

(ii) the recursion formulae which relate 3j symbols with the same value of $J = j_1 + j_2 + j_3$ but different values of the individual values of $j_1\, j_2\, j_3$. These have a direct meaning with respect to SU (3). They can be obtained by commuting the T's with both sides of

$$\begin{pmatrix}2\times J\\ [k]\end{pmatrix} = \sqrt{\frac{(J+1)(J+2)}{2}}\begin{pmatrix}2\times J & [K] & [\lambda]\\ [k] & J & J\end{pmatrix}\begin{pmatrix}J\\ [K]\end{pmatrix}\begin{pmatrix}J\\ [\lambda]\end{pmatrix}$$

or more in the spirit of invariant theory by writting (in the notations of the preceding subsection)

$$[U, S] = 0$$

(iii) the recursion relations which involve $3j$ symbols with different values of J. They are accounted for easily along the lines of the preceding subsection or more simply by applying D^*_{ij} to the above formula which expresses $\begin{pmatrix} 2 \times J \\ [k] \end{pmatrix}$ in terms of $\begin{pmatrix} J \\ [K] \end{pmatrix}\begin{pmatrix} J \\ [\lambda] \end{pmatrix}$.

4. — CONCLUSION

We have fitted the problem of the Regge symmetries into a broader group theoretic framework. However we did not attempt at all to generalize our point of view to an arbitrary Lie group. In our opinion the important property which must be understood at a deeper level is the splitting of the SU (n) algebra we discussed in section 2.

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APPENDIX

We carry out the identification of our Lie algebra L_n with a skew Hermitean realization of $\underline{SO}^*(2n)$ [8]. As a $2n \times 2n$ matrix Lie algebra, $\underline{SO}^*(2n)$ is the set of the matrices

$$\begin{vmatrix} \alpha & \gamma \\ -\gamma & \alpha \end{vmatrix} + i \begin{vmatrix} \beta & \delta \\ \delta & -\beta \end{vmatrix} \equiv S_U + i S_P \tag{1}$$

where α, β, δ are real antisymmetric $n \times n$ matrices and γ is a real symmetric $n \times n$ matrix. The algebra $\underline{U}(n)$ can be embedded in $\underline{SO^*}(2n)$ through the following isomorphism:

$$S_U = \begin{vmatrix} \alpha & \gamma \\ -\gamma & \alpha \end{vmatrix} \leftrightarrow U = \alpha + i\gamma \qquad U \in \underline{U}(n). \tag{2}$$

Let us remark that the corresponding representation obtained for the group U (n) by exponentiation ($M \to \exp tM$, t a real parameter) is unitary since $S_U^* = - S_U$ whereas it is not so for the group SO* $(2n)$ because of $(iS_P)^* = iS_P$. This is not surprising since SO* $(2n)$ is non compact. On the contrary the infinite dimensional representation obtained from the T's, the D's and the D*'s is unitary. To perform the identification we define first skew Hermitean operators:

$$\begin{aligned} S_{jk} &= i(T_{jk} + T_{kj}) \qquad & A_{jk} &= T_{jk} - T_{kj} \\ s_{jk} &= i\left(D_{jk} + D_{jk}^*\right) \qquad & a_{jk} &= D_{jk} - D_{jk}^* \end{aligned} \tag{3}$$

The new commutation relations then read:

$$\begin{aligned} [A_{ij}, A_{kl}] &= \delta_{jk}A_{il} - \delta_{jl}A_{ik} + \delta_{il}A_{jk} - \delta_{ik}A_{jl} \\ [A_{ij}, S_{kl}] &= \delta_{jk}S_{il} + \delta_{ji}S_{ik} - \delta_{il}S_{jk} - \delta_{ik}S_{jl} \\ [S_{ij}, S_{kl}] &= -\delta_{jk}A_{il} - \delta_{jl}A_{ik} - \delta_{il}A_{jk} - \delta_{ik}A_{jl} \end{aligned} \tag{4}$$

We defer writing those relations involving the s ans a's and we identify subalgebra (4).

According to (1) and (2) we get:

$$\begin{vmatrix} \alpha_{ik} & 0 \\ 0 & \alpha_{ik} \end{vmatrix} \leftrightarrow A_{ik}, \quad \begin{vmatrix} 0 & \sigma_{ik} \\ -\sigma_{ik} & 0 \end{vmatrix} \leftrightarrow S_{ik} \tag{5}$$

where the elements of the matrices α_{ik} and σ_{ik} are

$$\begin{aligned} (\alpha_{ik})_{\lambda\mu} &= \delta_{i\lambda}\delta_{k\mu} - \delta_{k\lambda}\delta_{i\mu} \\ (\sigma_{ik})_{\lambda\mu} &= \delta_{i\lambda}\delta_{k\mu} + \delta_{k\lambda}\delta_{i\mu} \end{aligned} \tag{6}$$

Let us now write down the remaining commutators:

$$\begin{aligned} [s_{ij}, S_{kl}] &= -[a_{ij}, A_{kl}] = -\delta_{jk}a_{il} + \delta_{ik}a_{jl} - \delta_{il}a_{jk} + \delta_{jl}a_{ik} \\ [a_{ij}, S_{kl}] &= [s_{ij}, A_{kl}] = \delta_{jk}s_{il} - \delta_{ik}s_{jl} + \delta_{il}s_{jk} - \delta_{jl}s_{ik} \\ [a_{ij}, a_{kl}] &= [s_{ij}, s_{kl}] = -[A_{ij}, A_{kl}] \\ [a_{ij}, s_{kl}] &= -\delta_{jk}S_{il} + \delta_{jl}S_{ik} - \delta_{il}S_{jk} + \delta_{ik}S_{jl}. \end{aligned} \tag{7}$$