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Section A :
Physique théorique.

About O'Raifeartaigh's theorem

by

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INTRODUCTION

This paper is devoted to the study of a conjecture that has been made sometime ago. The statement is that any semi-simple Lie algebra $\underline{G}$ having Poincaré $\underline{\mathfrak{T}}$ as a subalgebra also contains a dilatation (*i. e.* an element d commuting with the Lorentz generators $M^{\mu\nu}$ and having the translation generators for eigenvectors: $[d.P^{\mu}] = \lambda P^{\mu}$, $\lambda \neq 0$).

The existence of such a d would have proved that the reduction to Poincaré of any unitary representation of any semi-simple Lie algebra only contained a direct integral on masses, from 0 to ∞, of representations of $\underline{\mathfrak{T}}$, with possibly other representations corresponding to the discrete eigenvalue $m = 0$ ($\underline{G}$ now having the subspace $\underline{\mathfrak{T}} \oplus d$ as a subalgebra, the irreducible representations of which reduced on $\underline{\mathfrak{T}}$ give rise to a mass integral from 0 to ∞ or to the point spectrum $m = 0$).

It is shown here that the conjecture, which would have given a result somewhat more precise than O'Raifeartaigh's theorem and of purely algebraic nature, is not true. A rule is given to construct counterexamples of arbitrarily high dimension for any type of algebra.

I. — PRELIMINARIES

A. In the following $\underline{G}$ is any real simple Lie algebra having $\underline{\mathfrak{T}}$ as a subalgebra (the case when $\underline{G}$ is only semi-simple trivially reduces to this one, a semi-simple algebra being the direct sum of simple components such that each component contains an homomorphic image of $\underline{\mathcal{T}}$ one at least being faithful).

The classification of real non compact simple Lie algebras is well known [1]; the different series—directly issued from Cartan's classification for compact algebras—are defined as Lie algebras of real or complex matrices of a certain type acting in some $\mathbb{R}^n$ or $\mathbb{C}^n$ (any finite dimensional representation of $\underline{G}$ defines the algebra).

We shall now prove two lemmas which will allow us to leave aside the question of the series to which $\underline{G}$ belongs.

B. Given $\underline{G}$ by some finite irreducible representation ρ_n in V^n (V^n is $\mathbb{R}^n$ or $\mathbb{C}^n$) we call B the bilinear invariant symetric and definite form:

$$B(X, Y) = -\operatorname{Tr} \rho_n(X)\,\rho_n(Y) \qquad X, Y \in \underline{G}$$

(if ρ_n is the adjoint representation of $\underline{G}$, B is the Killing form.)

We know that the $\rho_n(P^\mu)$ are nilpotent matrices (by $[[\rho_n(M^{\mu\nu}), \rho_n(P^\lambda)], \rho_n(P^\lambda)] = 0$ and Jacobson lemma). Moreover as they commute with one another we have:

$$B(P^\mu, P^\nu) = 0$$

The definiteness of B then tells us that there exist four elements X^μ in $\underline{G}$ (we can choose them such that they transform as the components of a four-vector) such that

$$B(P^\mu, X^\nu) = g^{\mu\nu}$$

Now $\underline{G}$ may contain other four-vectors. We call U_i^μ a basis for these vectors with the conditions:

$$B(P^\mu, U_i^\nu) = B(X^\mu, U_i^\nu) = 0$$

$$B(U_i^\mu, U_j^\nu) = g^{\mu\nu}\gamma_{ij}$$

where $\gamma_{ij} = 0$ if $i \neq j$, $\gamma_{ii} = \pm 1$ ($\underline{G}$ being not compact we cannot enforce $\gamma_{ii} = +1$).

Moreover $\underline{G}$ also contains a subalgebra D of scalar (*i. e.* commuting with the $M^{\mu\nu}$) elements, which is not trivial: being simple $\underline{G}$ admits a Cartan decomposition : $\underline{G} = \underline{K} \oplus \underline{P}$ ($\underline{K}$ maximal compact subalgebra, $[\underline{K}, \underline{P}] = \underline{P}$, $[\underline{P}, \underline{P}] = \underline{K}$) such that $M^{ij} \in \underline{K}$, $M^{0i} \in \underline{P}$ ($i = 1, 2, 3$); then we have in a canonical way: $P^\mu = K^{(\mu)} + P^{(\mu)}$ ($K^{(\mu)} \in \underline{K}$, $P^{(\mu)} \in \underline{P}$) and the quantities

$$Y_\mu = K^{(\mu)} - P^{(\mu)}$$

are the covariant components of a four-vector; now one can find a basis of $\underline{G}$ such that the four matrices AdP^μ have all their elements equal to zero on and under the diagonal and that AdX_μ is just the transposed matrix of AdP^μ, $\forall_\mu$; one gets then easily convinced that $Add \equiv [AdP^\mu, AdX_\mu] \neq 0$ (the repeated indices are to be summed over).

Now for any $d \in D$ we may write:

$$[d, P^\mu] = \lambda P^\mu + \sum_i \lambda_i U_i^\mu$$

where

$$4\lambda = B(d, [P^\mu, X_\mu])$$

$$4\lambda_i = B(d, [P^\mu, U_{i\mu}])$$

clearly a d such that $\lambda \neq 0$, $\lambda_i = 0 \ \forall\, i$ is a dilatation.

LEMMA 1. — There exists a dilatation d in $\underline{G}$ if and only if the element $[P^\mu, X_\mu]$ is linearly independant from the $[P^\mu, U_{i\mu}]$.

PROOF. — The condition is obviously necessary. Let us show it is also sufficient. From Lorentz invariance we deduce that the form B is not degenerate on D. Let E be the vector space of the $[P^\mu, U_{i\mu}]$ and call F the direct sum $E \oplus [P^\mu, X_\mu]$. If $[P^\mu, X_\mu] \notin E$ then we have

$$F \supset E \qquad \text{and so} \qquad E^\perp \supset F^\perp$$

($E^\perp$ and $F^\perp$ being defined by $B(E, E^\perp) = B(F, F^\perp) = 0$).

Clearly a d such that $d \in E^\perp$, $d \notin F^\perp$ obeys

$$[d, P^\mu] = \lambda P^\mu \qquad \text{q. e. d.}$$

We shall call an « outer-dilatation » an element d' of $\mathcal{L}(V_n, V_n)$ such that

$$[d', \rho_n(M^{\mu\nu})] = 0, \ [d', \rho_n(P^\mu)] = \lambda' \rho_n(P^\mu), \qquad \lambda' \neq 0$$

LEMMA 2. — If there exists for $\rho_n(\underline{G})$ an outer-dilatation, then there exists a dilatation d in $\underline{G}$.

PROOF. — We can by linearity extend B to all the $n \times n$ matrices of $\mathcal{L}(V_n, V_n)$. We then have:

$$B(d', [P^\mu, X'_\mu]) = \lambda' B(P^\mu, X'_\mu) = 4\lambda' a$$

with $X'^\mu = aX^\mu + \sum_i b_i U_i^\mu$. So we can assert that $[P^\mu, X'_\mu] \neq 0$ when $a \neq 0$ *i. e.* $F \supset E$. Application of lemma 1 then gives lemma 2.

II. — DESCRIPTION OF THE FINITE DIMENSIONAL REPRESENTATIONS OF $\underline{\mathcal{T}}$

A. It is well known [*2*] that all finite dimensional representations ρ of a semi-direct product of an abelian Lie algebra $\underline{\mathcal{A}}$ by a semi-simple one $\underline{S}$, can be given, by a suitable choice of basis, a canonical form; a representation ρ_N, acting in a space V_N, gets the following triangular form by « blocks »:

$$\rho_N(\underline{\mathcal{A}} + \underline{S}) = \begin{bmatrix} \rho_1(\underline{S}) & \alpha_{12}t_{12}(\underline{\mathcal{A}}) & \dots & \alpha_{1m}t_{1m}(\underline{\mathcal{A}}) \\ 0 & \rho_2(\underline{S}) & \dots & \alpha_{2m}t_{2m}(\underline{\mathcal{A}}) \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \dots & \rho_m(\underline{S}) \end{bmatrix}$$

where:

(i) the $\rho_i(\underline{S})$ are real or complex irreducible representations of $\underline{S}$,

(ii) the $t_{ij}(\underline{\mathcal{A}})$ are well determined matrices. If we call ρ_0 the representation (we suppose it irreducible) by which $\underline{S}$ acts on $\underline{\mathcal{A}}$ in the semi-direct product, the $t_{ij}(\underline{\mathcal{A}})$ are in fact unique (after normalization) if $\overset{*}{\rho}_0$ appears only once in the reduction of the tensor product $\overset{*}{\rho}_j \otimes \rho_i$ (the star means contragredient). The $t_{ij}(\underline{\mathcal{A}})$ are zero if $\overset{*}{\rho}_j \otimes \rho_i$ does not contain $\overset{*}{\rho}_0$,

(iii) the α_{ij} are some constants which satisfy algebraic relations coming from the condition

$$[\rho_N(A), \rho_N(B)] \equiv C(A, B) = 0 \qquad A, B \in \underline{\mathcal{A}}$$

all this structure proceeds from the nilpotency of the matrices $\rho(\underline{\mathcal{A}})$ (see I). We shall not analyse it in further details.

B. We shall rather give another description of ρ_N by mean of diagrams, which will prove more convenient for our study ($\underline{\mathcal{A}} = R^4$, $\underline{S} = \underline{\mathfrak{L}}$ the Lorentz algebra, $\rho_\iota = D_{j_1 j_2}$ with j_1, j_2 integers or half integers, $\rho_0 = D_{\frac{1}{2}\frac{1}{2}}$).

Let $\vec{O}j_1$, $\vec{O}j_2$ be two orthogonal axes. We first associate to $\rho_N(\underline{\mathcal{T}})$ the set of points (j_1, j_2) corresponding to the subrepresentations ρ_i which appear when ρ is given the canonical form (*i*); then to each point we associate a set of numbers: first the multiplicity of the point (number of equivalent ρ_i), then the indices *i* of the different equivalent ρ_i. There may be some arbitrariness in the ordering; it is easily seen that it is irrelevant in the following.

Now, starting from a ρ_i, only those t_{ij} are different from O for which the product $\rho_i \otimes \rho_j$ contains the representation $D_{\frac{1}{2}\frac{1}{2}}$. We draw a line in our diagram for such « allowed » transitions ($Dj_1 j_2 \to Dj_1 + \frac{1}{2}\varepsilon, j_2 + \frac{1}{2}\varepsilon'$; $\varepsilon, \varepsilon' = \pm 1$) (i, i). We shall be interested only in connected diagrams (for which we can go from any i to any j by « allowed » transitions): the existence of n disconnected subdiagrams only means that the initial representation ρ is a direct sum of n subrepresentations.

We then get such drawings as fig. 1:

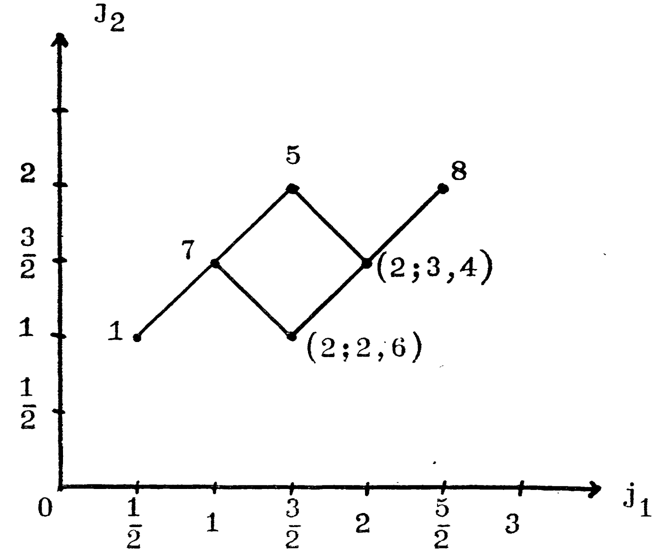

FIG. 1.

(we have not mentionned the multiplicity when equal to one).

The last condition (*iii*) is somewhat more involved. Let us write a block element of the commutator $C(P^\mu, P^\nu)$. It reads

$$(1) \qquad C_{ij}(P^\mu, P^\nu) - \sum_k \alpha_{ik}\alpha_{kj}[t_{ik}(P^\mu)t_{kj}(P^\nu) - t_{ik}(P^\nu)t_{kj}(P^\mu)]$$

This block matrix defines a linear application from the space of antisymetric tensors with two indices (in which $\underline{\mathcal{L}}$ acts by a $(1, 0) \oplus (0, 1)$ representation) in the space $\mathcal{L}(V_i, V_j)$ (linear transformations from V_i, subspace of ρ_i, to V_j, subspace of ρ_j, in which $\underline{\mathcal{L}}$ acts by $\rho_i \otimes \rho_j$).

From the interpretation of t_{mn} we see that, starting from a number i,

(1) may give effective conditions on the α only if the points k and j are $\left(\text{up to rotations of } \frac{n\pi}{2}\right)$ in the positions given by fig. 2 in the j_1, j_2 plane:

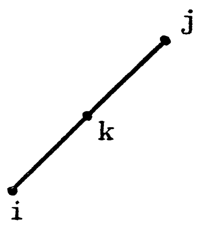

FIG. 2, 1.

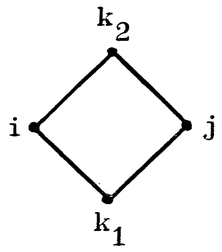

FIG. 2, 2.

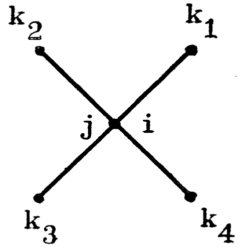

FIG. 2, 3.

Now from the interpretation of C_{ij} we see that for the case of fig. 2, 1, the bracketed quantity in (1) is zero (Wigner-Eckart theorem: if i is (j_1, j_2), j is $(j_1 + 1, j_2 + 1)$).

For the case of fig. 2, 2 we have:

$$[t_{ik}(P^\mu)t_{kj}(P^\nu) - t_{ik}(P^\mu)t_k(P_j{}^\nu)] = a_{ij}^k M_{ij}(P^\mu, P^\nu)$$

where the M_{ij} are standard matrices, k-independant, and the a_{ij}^k non zero constants (Wigner-Eckart theorem: if i is $(j_1 j_2)$, j is $(j_1, j_2 + 1)$ only the transition by mean of (0, 1) occurs). The condition (1) this times leads to

$$\text{(2)} \qquad \alpha_{ik_1}\alpha_{k_1j}C_{ij}^{k_1} + \alpha_{ik_2}\alpha_{k_2j}C_{ij}^{k_2} = 0$$

(k_1 and k_2 indicate the two possible ways for going from i to j; clearly in the case of multiplicity this notation covers a set of indices).

We shall suppose in the following that the restriction $\rho(\underline{\mathcal{L}})$ contains no equivalent ρ_i (no multiplicity) so we shall not be interested in the case of fig. 2, 3 where j is the same representation as i.

III. — POSSIBLE REPRESENTATIONS OF OUTER-DILATATIONS

A. Having seen the general form of $\rho_N(\underline{\mathcal{T}})$ in V_N we now look for elements d' of $\mathcal{L}(V_N, V_N)$ satisfying:

$$\text{(3)} \qquad [d', \rho_N(P^\mu)] = -\lambda'\rho_N(P^\mu) \qquad \lambda' \neq 0$$

$$\text{(4)} \qquad [d', \rho_N(M^{\mu\nu})] = 0$$

With the restriction of no multiplicity for the ρ_i condition (4) requires d to be diagonal by blocks *i. e.* multiple λ_i of the identity in each V_i.

Then equation (3) becomes:

$$-\lambda'\alpha_{ij}t_{ij}(P^\mu) = (\lambda_i - \lambda_j)\alpha_{ij}t_{ij}(P^\mu) \qquad \lambda \neq 0 \tag{5}$$

B. Equation (5) can be easily studied with the diagrams we have introduced. To each point i we associate the potential λ_i in the following way: we give the potential $\lambda_1 = 0$ to the point 1, and we go along all possible lines (*i. e.* « allowed » transitions t_{ij} with $\alpha_{ij} \neq 0$) with the convention that going form one end i to the other j of a possible elementary segment, we add a potential difference $\pm \lambda'$ according as $i \gtrless j$.

Now (5) has a solution or not according as, whatever way we follow, we come back to 1 with potential 0 or $\neq 0$. We show in fig. 3 and 4 examples of the two situations.

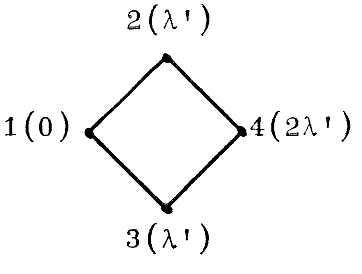

FIG. 3.
Solution: $\lambda_1 = 0$, $\lambda_2 = \lambda_3 = \lambda'$, $\lambda_4 = 2\lambda'$.

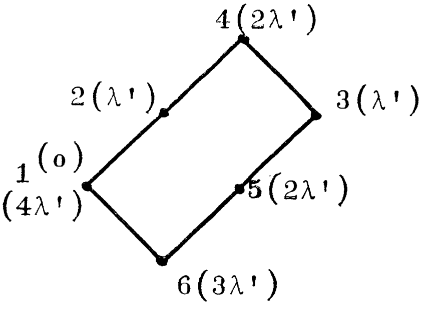

FIG. 4.
No solution.

One easily sees that the diagram of fig. 4 is (up to the ordering and a rotation of $\frac{n\pi}{2}$) the diagram with the smallest number of points such that equation (5) has no solution ($\lambda' \neq 0$) (such a diagram as fig. 3 with numbers 3 and 4 exchanged cannot define a ρ since clearly it does not satisfy condition (2)— which is (because of the ordering) $\alpha_{12}\alpha_{23} = 0$, $\alpha_{23}\alpha_{34} = 0$—so that one at least of the indicated segment must be cut to save the commutativity).

Diagrams of the type of fig. 4 (no other solution than $\lambda = 0$) are found for arbitrarily high number of points.

IV. — COUNTEREXAMPLES

A. With help of diagrams of III and lemmas of I, we are now in position to draw conclusions.

Let $\rho_N(\underline{G})$ be a finite dimensional representation of a real simple algebra $\underline{G}$ containing Poincaré $\underline{\mathcal{P}}$, $\rho_N(\underline{\mathcal{P}})$ its restriction to $\underline{\mathcal{P}}$ such that $\rho_N(\underline{\mathcal{L}})$ contains no equivalent irreducible representations, D the diagram associated with $\rho_N(\underline{\mathcal{P}})$. Then:

a) if D is a diagram of fig. 4 type there is a dilatation in $\underline{G}$;

b) if D is a diagram of fig. 3 type there is no dilatation in $\underline{G}$ which is a counterexample to the conjecture we quoted.

B. We shall limit ourselves to two classes of counterexamples: when $\underline{G}$ is $\underline{SL}(n, \mathbb{C})$ or $\underline{SL}(n, \mathbb{R})$ (Lie algebras of $n \times n$ complex or real matrices with 0 trace).

1° For $\underline{SL}(n, \mathbb{C})$ the counterexamples are very easily obtained: given any diagram of the type of fig. 4, and a choice for the axes j_1 and j_2 (they are defined up to horizontal and vertical translations of lengths $p \times \frac{1}{2}$, $q \times \frac{1}{2}$) we specify a representation $\rho_m(\underline{\mathcal{P}})$, of dimension m, which is clearly a subalgebra of the algebra $\underline{SL}(m, \mathbb{C})$ isomorphic to $\underline{\mathcal{P}}$. We can conclude that for this value of m, and this way of defining the subalgebra $\underline{\mathcal{P}}$, the algebra $\underline{SL}(m, \mathbb{C})$ has no dilatation.

One then sees that the diagram which gives the lowest m is the one of fig. 5 (with any allowed ordering). We find $m = 35$.

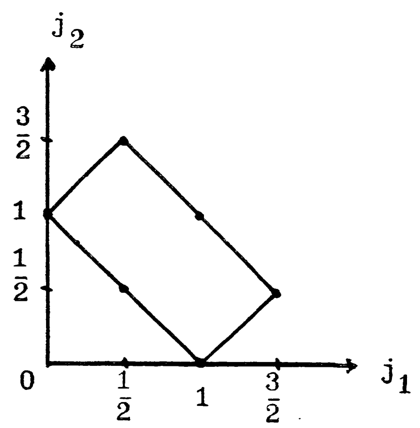

FIG. 5.

2° One must be slightly more careful for $\underline{SL}(m, \mathbb{R})$: the $\rho_i(\underline{\mathfrak{L}})$ must now be real irreducible representations *i. e.* either D_{jj} or the sum $D_{j_1j_2} \oplus D_{j_2j_1}$ with $j_1 \neq j_2$ both integers or half integers. The easiest way of obtaining real representations of $\underline{\mathfrak{T}}$ is to consider only diagrams which are in the plane j_1j_2 between the axis j, and the bissector of the two axis, with the convention that to the point j_1j_2 is associated the Lorentz representation $D_{j_1j_2} \oplus D_{j_2j_1}$. So again to any diagram of the typ of fig. 4, with a choice for the axes j_1 and j_2 compatible with what we just said, we associate a representation $\rho_D(\underline{\mathfrak{T}})$ of dimension m, which is a real subrepresentation of the algebra of SL(m, $\mathbb{R}$) isomorphic to $\underline{\mathfrak{T}}$ and for which there is no dilatation.

In this case the diagram which gives rise to the lowest m (always up to the ordering) is the one of fig. 6 for which $m = 81$.

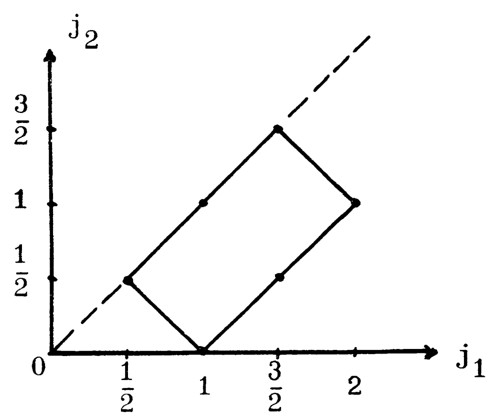

FIG. 6.

In conclusion we can notice that all the non compact algebras the physicists have already considered in elementary particle physics, up to U(6, 6), containis a dilatation (whichever way Poincaré defined) and so lead to an infinite continuous mass spectrum. Now, seeing the high dimension of algebras with no dilatation we exhibit (2 448 for SL(35$\mathbb{C}$), 6 560 for SL(81$\mathbb{R}$)), we can speculate that the situation will not change in the days to come; probably such possibilities allowed by O'Raifeartaigh's theorem as a point spectrum outside O or a continuous spectrum of finite range—if they really exist—will not be encountered.

On the other hand we can most likely assert by now that O'Raifeartaigh's theorem actually lays in the domain of mathematical analysis.

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