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Predictive relativistic mechanics

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Section A :
Physique théorique.

Predictive relativistic mechanics (*)

by

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SUMMARY. — Several papers on Predictive Relativistic Mechanics are reviewed. The meaning of the so-called « No Interaction Theorems » is examined, and solutions to some open problems are proposed.

INTRODUCTION

Predictive Relativistic Mechanics of Isolated Systems (P. R. M. of I. S.) was first considered by P. A. Dirac in 1949 [*1*]. His formulation of the problem was incomplete, and when it was conveniently completed it appeared to be too restrictive. This is the main conclusion of a famous theorem which was first proved in a particular case by D. G. Currie, T. F. Jordan, E. C. G. Sudarshan in 1963 [*2*]. This theorem and its known generalisations are sometimes referred to, improperly, as « No Interaction Theorems ».

P. R. M. was reformulated in 1966 by D. G. Currie [*3*], in 1967 by R. N. Hill [*4*], in 1969 by Ph. Droz-Vincent [5] and only very recently by myself [*6*]. I shall review the main results of these papers which have led us to the conclusion that Predictive Mechanics is definitely consistent with Special Relativity.

Then I shall present the « No Interaction Theorems » and examine carefully what they really mean. Some general ideas on how an Hamiltonian formalism can be developped in the framework of P. R. M. will finish this lecture.

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1. POINCARÉ INVARIANT SYSTEMS : GENERAL RESULTS

a. — Let us consider an isolated system of N point-like structure less particles. I shall say that the mechanics of the system is Predictive if the equations of motion are given by a system of ordinary second order differential equations

$$\frac{dx_a^i}{dt} = v_a^i \quad , \quad \frac{dv_a^i}{dt} = \mu_a^i(x_b^j, v_c^k) \tag{1}$$

$(i, j, \ldots = 1, 2, 3 ; a, b, c, \ldots = 1, 2, \ldots, N)$ as in Newtonian Mechanics.

This formulation of P. R. M. in Special Relativity raises immediately several difficulties. From the mathematical point of view, the first difficulty, and consequently our first task, will be to give an unquestionable definition of Poincaré Invariant Systems (P. I. S.) of type (1). We shall see in a moment that this is possible and easy.

From the physical point of view the obvious difficulties are the following: Causality, as we understand this word now, is violated in P. R. M. and Radiation is not induded in the theory. I do not believe that these are real difficulties if we keep in mind that we are dealing with Isolated Systems only, and that the classical theory to be presented will eventually be quantized. In other words, if the domain of validity of the theory is suitably understood.

b. — Let:

$$x_a^i = \varphi_a^i(\underset{0}{x}{}_b^j, \underset{0}{v}{}_c^k ; t) \tag{2}$$

be the general solution of system (1), $(\underset{0}{x}{}_b^j, \underset{0}{v}{}_c^k)$ beeing the initial conditions at $t = 0$. Let us consider Minkowski Space-time M_4 and an arbitrary galilean coordinate system. To each set of physically admissible initial conditions $(v_a^2 < 1)$ we can associate a set of N time-like trajectories with parametric equations

$$x_a^0 = t \quad , \quad x_a^i = \varphi_a^i(\underset{0}{x}{}_b^j, \underset{0}{v}{}_c^k ; t) \tag{3}$$

To the general solution (2) will thus correspond a 6N parameters family Γ of sets of N trajectories. We say that (1) is a P. I. S. if the Poincaré group maps Γ into itself.

Let $(\underset{0}{x}', \underset{0}{v}')$ be the initial conditions of the images of a set of trajectories (3) by a Poincaré transformation corresponding to some parameters Λ^K (K, L, ... = 0, 1, ..., 9) $(\underset{0}{x}', \underset{0}{v}')$ will be functions of $(\underset{0}{x}, \underset{0}{v})$ and Λ^K:

$$\underset{0}{x}{}'^i_a = f^i_a(\underset{0}{x}{}^j_b, \underset{0}{v}{}^k_c; \Lambda^K) \quad , \quad \underset{0}{v}{}'^i_a = g^i_a(\underset{0}{x}{}^j_b, \underset{0}{v}{}^k_c; \Lambda^K) \tag{4}$$

Since the image of any point $(x^0_a = t \quad , \quad x^i_a = \varphi^i_a)$ is given by

$$\begin{aligned} x^{0'}_a &= L^{0'}_a(t - A^0) + L^{0'}_i[\varphi^i_a(\underset{0}{x}, \underset{0}{v}; t) - A^i] \\ x^{j'}_a &= L^{j'}_a(t - A^0) + L^{j'}_i[\varphi^i_a(\underset{0}{x}, \underset{0}{v}; t) - A^i] \end{aligned} \tag{5}$$

the geometric definition given above can be translated as follows:

System (1) is a P. I. S. iff there exist functions (4) such that

$$x^{i'}_a = \varphi^i_a(\underset{0}{x}{}'^j_b, \underset{0}{v}{}'^k_c; x^{0'}_a) \tag{6}$$

is true for each set of initial conditions $(\underset{0}{x}, \underset{0}{v})$, each element (Λ^K) of the Poincaré group, and each value of the parameter t (*).

c. — Let V_{6N} be the tangent bundle to configuration space. The $x^{i's}_a$ will be taken as the first 3N coordinates of a point of V_{6N} and the $v^{i's}_a$ as the last 3N ones. Similarly if $\vec{\Xi}$ is a vector field on V_{6N} it will be represented by (ξ^i_a, θ^i_a). The $\xi^{i's}_a$ being the first 3N components and the $\theta^{i's}_a$ the last 3N ones.

Let us now introduce the ten vector fields (**):

$$\begin{aligned} &\vec{H} : (- v^i_a, \; - \mu^i_a) \\ &\vec{P}_j : (- \varepsilon_a \delta^i_j, \; 0) \\ &\vec{J}_j : (- \eta^i_{\cdot jk} x^k_a, \; - \eta^i_{\cdot jk} v^k_a) \\ &\vec{K}_j : (- v^i_a x_{aj}, \; \varepsilon_a \delta^i_j - v^i_a v_{aj} - \mu^i_a x_{aj}) \end{aligned} \qquad (\varepsilon_a = 1) \tag{7}$$

whose components contain as unknowns the functions μ^i_a only. The first important result of P. R. M. is the following:

1. — System (1) is a P. I. S. if and only if the Lie-Brackets of (7) satisfy

(*) A more precise definition of locally invariant systems by the connected component of the identity will certainly be needed in the future.

(**) η_{ijk} is the Levi-Civitta symbol. The latin indices will always be raised or lowered without change of sign.

the commutation relations caracteristic of the Poincaré Lie algebra:

$$
(8) \qquad
\begin{array}{lll}
[\vec{P}_i, \vec{H}] = 0 & [\vec{J}_i, \vec{H}] = 0 & [\vec{K}_i, \vec{H}] = \vec{P}_i \\
[\vec{P}_i, \vec{P}_j] = 0 & [\vec{J}_i, \vec{P}_j] = \eta_{ijk}\vec{P}^k & [\vec{K}_i, \vec{P}_j] = \delta_{ij}\vec{H} \\
[\vec{J}_i, \vec{J}_j] = \eta_{ijk}\vec{J}_k & [\vec{K}_i, \vec{J}_j] = \eta_{ijk}\vec{K}^k & [\vec{K}_i, \vec{K}_j] = -\eta_{ijk}\vec{J}_k
\end{array}
$$

2. — If (1) is a P. I. S., appropriate parametrisations (Λ^K) of the Poincaré group do exist such that the finite transformations generated by (7) coincide with those defined by (4) (Dropping the subindex 0).

These finite transformations provide a realisation of the Poincaré group acting on V_{6N} and play a central role in the theory.

When we explicit equations (8) we obtain (*):

$$
(9) \qquad
\begin{array}{c}
\varepsilon_a \dfrac{\partial \mu_b^i}{\partial x_a^j} = 0 \quad ; \quad \eta^s_{\cdot kr}\left(x_a^k \dfrac{\partial \mu_b^i}{\partial x_a^s} + v_a^k \dfrac{\partial \mu_b^i}{\partial v_a^s} \right) = \eta^i_{\cdot kr}\mu_b^k \\
v_a^s(x_{ar} - x_{br}) \dfrac{\partial \mu_b^i}{\partial x_a^s} + \left[v_a^s v_{ar} + \mu_a^s(x_{ar} - x_{br}) - \varepsilon_a \delta_r^s \right] \dfrac{\partial \mu_b^i}{\partial v_a^s} = 2\mu_b^i v_{br} + v_b^i \mu_{br}.
\end{array}
$$

These equations were first proved to be necessary conditions by D. G. Currie [*3*] and R. N. Hill [*4*]. I proved that they are also sufficient in [*6*].

d. — If we are dealing with a system of particles of the same type, then it is necessary to complement equations (9) with the supplementary conditions:

$$
(10) \qquad \mu_a^i(x_{Sb}^j, v_{Sc}^k ; \lambda_{Sd}) = \mu_{Sa}^i(x_b^j, v_c^k ; \lambda_d)
$$

where S is any permutation of the integers (1, 2, ..., N) and where λ_d refers to any constants caracteristic of the particles involved e. g., the masses.

On the other hand it is clear that some other supplementary conditions are still needed for a physically meaningful theory.

Obviously we want the functions μ_a^i to tend to zero when the distances between any pair of particles go to infinity (**):

$$
(11) \qquad \lim_{|x_a - x_b| \to \infty} \mu_a^i = 0 \qquad (a \neq b)
$$

And we want also the speed of the particles to remain less than one. It is not difficult to see that this will be the case if

$$
(12) \qquad v_a^r - 1 = 0 \Rightarrow v_a^i \mu_{ai} = 0
$$

(*) The sommation convention is used also for the indices $a, b, \ldots$

(**) A more precise « Separability Condition » can be stated.

One solution of equations (9) and (10), for N = 2, has been obtained by D. G. Currie and T. F. Jordan [7]. But no solution of equations (9) satisfying the supplementary conditions (10), (12) is yet known.

e. — For obvious reasons I shall call the formalism presented above, the manifestly time-symmetric description of P. R. M. of I. S. We shall see in the Appendix that an equivalent manifestly covariant description of the theory does exist.

2. ENERGY, MOMENTUM, ANGULAR MOMENTUM AND CENTER OF MASS

Among the P. I. S. we always have the free particles systems. Let us consider, for these systems, the total energy $H \equiv P^0$, the total momentum P^i, the total angular momentum $J_k = \frac{1}{2}\eta_{kij}H^{ij}$, and the Center of Mass coordinates $R^i \equiv H^{-1}K^i \equiv H^{-1}H^{i0}$:

$$(13) \qquad \begin{array}{ll} P^0 = \varepsilon^a m_a[1-(v^a)^2]^{-\frac{1}{2}} \equiv \varepsilon^a h_a \ , & P^i = \varepsilon^a h_a v^i_a \equiv \varepsilon^a p^i_a \\ H^{ij} = \varepsilon^a(x^i_a p^j_a - x^j_a p^i_a) \ , & H^{i0} = \varepsilon^a h_a x^i_a \end{array}$$

A straightforward calculation proves that under any finite transformation (4) (For free particles systems these transformations are explicitly given in [6]). These quantities transform according to formulae:

$$(14) \qquad \begin{array}{ll} P^\alpha(f, g) = L^{\alpha'}_\beta P^\beta(x, v) & (\alpha, \beta, \ldots = 0, 1, 2, 3) \\ H^{\alpha\beta}(f, g) = L^{\alpha'}_\rho L^{\beta'}_\sigma[H^{\rho\sigma}(x, v) - A^\rho P^\sigma(x, v) + A^\sigma P^\rho(x, v)] & \end{array}$$

In the general case, I shall consider formulae (14) as part of the definition of the total energy, momentum, angular momentum and center of mass coordinates. But as part only because it can be proved that for every P. I. S. many sets of such quantities can be obtained which transform according to formulae (14).

In [6] I proved that H, P^i, J^i and K^i transform as in (14) if and only if the following relations are satisfied:

$$(15) \qquad \left\{ \begin{array}{lll} \mathscr{L}(\vec{H})H = 0 & \mathscr{L}(\vec{H})P_j = 0 & \mathscr{L}(\vec{H})J_j = 0 \\ \mathscr{L}(\vec{P}_i)H = 0 & \mathscr{L}(\vec{P}_i)P_j = 0 & \mathscr{L}(\vec{P}_i)J_j = \eta_{ijk}P^k \\ \mathscr{L}(\vec{J}_i)H = 0 & \mathscr{L}(\vec{J}_i)P_j = \eta_{ijk}P^k & \mathscr{L}(\vec{J}_i)J_j = \eta_{ijk}J^k \end{array} \right.$$

$$(16) \qquad \mathscr{L}(\vec{K}_i)H = P_i \qquad \mathscr{L}(\vec{K}_i)P_j = \delta_{ij}H$$

$$\mathscr{L}(\vec{K}_i)J_j = \eta_{ijk}K^k \tag{17}$$

$$\begin{cases} \mathscr{L}(\vec{H})K_j = -P_j & \mathscr{L}(\vec{P}_i)K_j = -\delta_{ij}H \\ \mathscr{L}(\vec{J}_i)K_j = \eta_{ijk}K^k & \mathscr{L}(\vec{K}_i)K_j = -\eta_{ijk}J^k \end{cases} \tag{18}$$

$\mathscr{L}$ beeing the Lie-Derivative symbol. I shall need latter these equations.

3. THE REDUCED GENERATOR SYSTEM

a. — Since it has already been suggested in the litterature, I shall call the subgroup of the Poincaré group generated by the Euclidean group and time translations: the Aristotle group. I shall now prove that the problem of obtaining P. I. S. can be reduced to a simpler problem which can be completely solved.

Let us consider an arbitrary galilean system of coordinates of M_4 and let:

$$x_a^0 = t \quad , \quad x_a^i = \hat{\varphi}_a^i(v^\zeta ; t) \qquad (\zeta, \eta, \ldots = 1, 2, \ldots, 6N - 3) \tag{19}$$

be the parametric equations of a $6N - 3$ parameters family Σ of sets of N trajectories. I shall say that this family Σ is Aristotle invariant if the Aristotle group maps Σ into itself.

If we consider a parametrisation γ^k of the Rotation group $(R_j^{i'})$ we can say more explicitly that Σ is Aristotle invariant if there exist functions:

$$v'^\zeta = v'^\zeta(v^\eta ; \gamma^k, A^\alpha) \tag{20}$$

such that:

$$R_j^{i'}[\hat{\varphi}_a^j(v^\zeta ; t) - A^j] = \hat{\varphi}_a^i(v'^\eta ; t - A^0) \tag{21}$$

for each set (v^ζ), each element of the Aristotle group, and each value of t.

From Σ we can construct a 6N parameters family Γ by taking the images of Σ for all pure Lorentz transformations, and it is easy to prove that Γ is Poincaré invariant. Consequently Γ can be considered as the general solution of a P. I. S. of type (1).

b. — Let us assume for simplicity that $N = 2$. To obtain Aristotle invariant families Σ we can proceed as follows:

1) Let

$$\frac{dx^i}{dt} = v^i \quad , \quad \frac{dv^i}{dt} = \mu^i(x^j, v^k) \tag{22}$$

be a second order differential system, μ^i beeing the components of a vector function, and let

$$x^i = x^i(\underset{0}{x}^j, \underset{0}{v}^k ; t) \tag{23}$$

be its general solution.

2) Let us consider a vector function

$$\mathrm{R}^i = \mathrm{R}^i(x_1^j, x_2^k ; x^r, v^s) \tag{24}$$

satisfying the following conditions:

2*a*) R^i behave as the coordinates of a point of euclidean space under the translation group:

$$\mathrm{R}^i(x_1^j - \mathrm{A}^j, x_2^k - \mathrm{A}^k ; x^r, v^s) = \mathrm{R}^i(x_1^j, x_2^k ; x^r, v^s) - \mathrm{A}^i \tag{25}$$

2*b*) The functions (24) and $x_1^i - x_2^i = x^i$ can be inverted to yield functions:

$$x_a^i = x_a^i(x^j, v^k, \mathrm{R}^s) \qquad (a = 1, 2) \tag{26}$$

Then it can be proved that the functions:

$$x_a^i = \hat{\varphi}_a^i(\underset{0}{x}_1^j, \underset{0}{x}_2^k, \underset{0}{v}^r ; t) \equiv x_a^i[x^j(\underset{0}{x}, \underset{0}{v} ; t), v^k(\underset{0}{x}, \underset{0}{v} ; t), \mathrm{R}^s(\underset{0}{x}_1, \underset{0}{x}_2 ; \underset{0}{x}, \underset{0}{v})] \tag{27}$$

where $\underset{0}{x} = \underset{0}{x}_1 - \underset{0}{x}_2$, define an Aristotle invariant family Σ of pairs of trajectories.

Σ generates a P. I. S. as seen in *a*. But since it is true that every P. I. S. can be obtained in this way we have here, in some implicit sense, the general solution of equations (9).

I shall call system (22): the Reduced Generator System; the functions (24): the Splitting Functions; and the frame of reference used to construct the family Σ together with every one else obtained from it by a transformation of the Aristotle group: the Aristotle Class of Frames of Reference.

4. « NO INTERACTION THEOREMS »

a. — It is now a quite natural question to ask wether or not it is possible to give a consistent definition of Hamiltonian P. I. S.

We shall see that the answer to this question is not obvious because the first « natural » definitions which were considered led to « No Interaction Theorems ».

Since the appropriate language to discuss this problem is the language of symplectic geometry, I shall review here some of the relevant definitions and results of this formalism (see for instance [*8*], [*9*]).

A 2-form σ is said to be a symplectic form of V_{6N} if it has maximum rank ($\det \sigma \neq 0$) and if it is closed:

$$d\sigma = 0 \tag{28}$$

d beeing the exterior differential operator.

Every symplectic form σ can be written, in many ways, as

$$\sigma = dq^i_a \wedge dp^a_i \tag{29}$$

where $\wedge$ is the exterior product symbol and q^i_a and p^a_i are functions of (x^i_b, v^k_c). Such functions are called canonical coordinates of σ.

If $\vec{\Xi}$ is a vector field on V_{6N}, we say that $\vec{\Xi}$ is the generator of a one parameter group of canonical transformations if

$$\mathscr{L}(\vec{\Xi})\sigma = 0 \tag{30}$$

Since for any p-form:

$$\mathscr{L}(\vec{\Xi})\sigma = i(\vec{\Xi})d\sigma + di(\vec{\Xi})\sigma \tag{31}$$

where $i(\vec{\Xi})$ is the interior product symbol, from (28) and (30) it follows that there exists a function Ξ such that:

$$i(\vec{\Xi})\sigma = -\, d\Xi \tag{32}$$

Ξ is defined only up to an additive constant:

$$\Xi \to \Xi + \text{Const.} \tag{33}$$

If $\underset{1}{\Phi}$ and $\underset{2}{\Phi}$ are two functions the Poisson Bracket of these functions is by definition the function:

$$[\underset{1}{\Phi}, \underset{2}{\Phi}] = -\, \sigma^{-1}(d\underset{1}{\Phi}, d\underset{2}{\Phi}) \tag{34}$$

that is to say, minus the value of σ^{-1} for $(d\underset{1}{\Phi}, d\underset{2}{\Phi})$. For every set of canonical coordinates (q, p) this definition reduces to the usual one.

If Φ is a function and $\vec{\Xi}$ satisfies (30) then:

$$\mathscr{L}(\vec{\Xi})\Phi = -\, [\Phi, \Xi] \tag{35}$$

If $\vec{\underset{1}{\Xi}}$ and $\vec{\underset{2}{\Xi}}$ satisfy (30) then $[\vec{\underset{1}{\Xi}}, \vec{\underset{2}{\Xi}}]$ does satisfy (30) also and the function associated to this vector by formula (32) is, up to an additive constant:

$$\underset{[1,2]}{\Xi} = [\underset{1}{\Xi}, \underset{2}{\Xi}] \tag{36}$$

b. — It looks of course quite natural to say that a P. I. S. is Hamiltonian if it is possible to define functions $p_i^a(x_b^j, v_c^k)$ and construct a function $H(x_a^i, p_j^b)$ such that system (1) be equivalent to:

$$v_a^i = \frac{\partial H}{\partial p_i^a} \quad , \quad \frac{dp_i^a}{dt} = -\frac{\partial H}{\partial x_a^i} \tag{37}$$

This definition would be equivalent to the following one: A P. I. S. is Hamiltonian if it is possible to construct on V_{6N} a symplectic form σ satisfying the following conditions:

1*a.* — σ is invariant under the one parameter group generated by $\vec{H}$:

$$\mathscr{L}(\vec{H})\sigma = 0 \tag{38}$$

2. — σ can be written as

$$\sigma = dx_a^i \wedge dp_i^a \tag{39}$$

Or, in other words: the x'^s are part of a canonical set of coordinates of σ. The Hamiltonian H would then be any of the functions associated to $\vec{H}$ by formula (32).

Let us assume that conditions (38) and (39) are satisfied, and let us consider the image σ^* of σ by any transformation of the Euclidean subgroup generated by $\vec{P}_i$ and $\vec{J}_i$. Since $\vec{H}$ commutes with these generators and since for this subgroup the transformation (4) are just those of euclidean geometry, it turns out that σ^* would satisfy again conditions (38) and (39) with some new functions p_i^{*a}. Thus unless $\sigma^* = \sigma$ we would be faced with a problem of unicity of the Hamiltonian.

Consequently it is quite natural to assume, instead of condition 1*a.* —, the more restrictive conditions

$$1b. \text{ — } \mathscr{L}(\vec{H})\sigma = 0 \quad , \quad \mathscr{L}(\vec{P}_i)\sigma = 0 \quad , \quad \mathscr{L}(\vec{J}_i)\sigma = 0 \tag{40}$$

which will insure that $\sigma^* = \sigma$.

We do not have any motivation to assume:

$$1c. \text{ — } \mathscr{L}(\vec{H})\sigma = 0 \quad , \quad \mathscr{L}(\vec{P}_i)\sigma = 0 \quad , \quad \mathscr{L}(\vec{J}_i)\sigma = 0 \quad , \quad \mathscr{L}(\vec{K}_i)\sigma = 0 \tag{41}$$

because $\vec{K}_i$ do not commute with $\vec{H}$, but as a first try we may consider this last condition and work out its implications.

Let us assume that a P. I. S. exists for which wa can construct a symplectic form σ satisfying (39) and (41). To each vector field (7), formula (32) associates one function H, P_i, J_i, K_i defined up to a constant. From (36) it follows that the Poisson Brackets of these functions will satisfy the same commutation relations (8) except for some neutral elements which might appear in the right hand sides. But if this were the case for a given set of functions H, P_i, J_i, K_i it is well known (see for instance [*2*]) that a new set could be selected, which would correspond to an appropriate choice of the constants in (33), for which we would have exactly:

$$(42)\quad \begin{gathered} [P_i, H] = 0 \qquad [J_i, H] = 0 \qquad [P_i, P_j] = 0 \\ [J_i, P_j] = \eta_{ijk}P^k \qquad [J_i, J_j] = \eta_{ijk}J^k \end{gathered}$$

and:

$$(43)\quad [K_i, H] = P_i \quad , \quad [K_i, P_j] = \delta_{ij}H \quad , \quad [K_i, J_j] = \eta_{ijk}K^k \quad , \quad [K_i, K_j] = -\eta_{ijk}J^k$$

From (35) applied to x^i_a and to $\vec{P}_i$, $\vec{J}_i$, $\vec{K}_i$ it follows:

$$(44)\quad [x^i_a, P_j] = \delta^i_j \quad , \quad [x^i_a, J_j] = \eta^i{}_{\cdot jk}x^k_a \quad , \quad [x^i_a, K_j] = x_{aj}[x^i_a, H]$$

D. G. Currie, T. F. Jordan and E. C. G. Sudarshan [*2*], for N = 2, J. T. Cannon and T. F. Jordan [*10*], for N = 3, and H. Leutwyler [*11*] and R. N. Hill [*12*] for any N, have proved that if functions H, P_i, J_i, K_i exist satisfying (42), (43) and (44) then:

$$(45)\quad [[x^i_a, H], H] = 0$$

which means $\mu^i_a = 0$. In other words the only P. I. S. for which a symplectic form σ can be constructed satisfying (39) and (41) are the free particles systems. This is the first « No Interaction Theorem ».

b. — Since we did not have any deep motivation to include condition (41) in the definition of Hamiltonian P. I. S., the next logical step is to drop (41) and keep (39) and (40) only. If we do that, the situation is the following. Formula (32) still defines H, P_i, J_i, and the constants in (33) can be chosen such that these functions satisfy (42). From these and (35) it follows that relations (15) will be satisfied. But relations (16) will not be necessarily satisfied and relations (17) and (18) will not have any meaning until we define somehow the functions K_i. Consequently the interpretation of H, P_i and J_i as total energy, momentum and angular momentum will be unjustified unless we make some additional assumptions.

For this purpose the more economical assumptions we can make are the following:

3*a*. — Functions H and P_i satisfy relations (16).

3*b*. — $\mathscr{L}(\vec{K}_i)J_j = -\mathscr{L}(\vec{K}_j)J_i$, one at least of these quantities beeing non zero.

In fact, under assumption 3*b* formula (17) can be considered as the definition of K_i and it can be proved that these functions satisfy equations (18).

T. F. Jordan [*13*] and myself have proved, for N = 2, that the only P. I. S. for which we have (39), (40) and for which conditions 3 above are satisfied are the free particle systems. This is the second known « No Interaction Theorem ».

Since I shall need some elements of my proof of this theorem I shall sketch it. From (39) and (40) it follows that functions p_i^1 and p_i^2 can be introduced such that:

$$P_i = p_i^1 + p_i^2 \quad , \quad J_i = \eta_{ijk}(x_1^j p_1^k + x_2^j p_2^k) \tag{46}$$

Consequently if we define Φ as:

$$\Phi \equiv J_i(x_1^i - x_2^i) - \eta_{ijk} x_1^i x_2^j P^k \tag{47}$$

we have identically:

$$\Phi \equiv 0 \tag{48}$$

and $\vec{\Xi}_K$ (K, L, S = 0, 1, ..., 9) beeing any of the vector fields (7) we obtain from (48) as necessary conditions:

$$\mathscr{L}(\vec{\Xi}_K)\Phi = 0 \quad , \quad \mathscr{L}(\vec{\Xi}_K)\mathscr{L}(\vec{\Xi}_L)\Phi = 0 \quad , \quad \mathscr{L}(\vec{\Xi}_K)\mathscr{L}(\vec{\Xi}_L)\mathscr{L}(\vec{\Xi}_S)\Phi = 0 \tag{49}$$

when these necessary conditions are worked out using (15), (18) we get $\mu_1^i = \mu_2^i = 0$.

Clearly these « No Interaction Theorems » do not prove that Predictive Mechanics is inconsistent with Special Relativity. They prove only that to give a definition of Hamiltonian P. I. S. is more difficult than we thought it would be.

5. HAMILTONIAN POINCARÉ INVARIANT SYSTEMS

a. — I shall examine here, in view of the above results, some possibilities that remain open to give consistent definitions of Hamiltonian P. I. S.

First of all let me emphasize that Jordan's theorem has been proved only for N = 2, and Jordan's proof as well as mine are based on particular relations which are valid only for this case. For instance no relation equivalent to (48) exists for N > 2. Consequently we have to keep in mind that this theorem might be false for N > 2. If this were the case then we should have to handdle the case N = 2 by a separate method. This problem desserves further work.

R. N. Hill and E. H. Kerner [*14*] propose to define an Hamiltonian P. I. S. as a system for which a symplectic form σ can be constructed satisfying conditions (41) and such that canonical coordinates q_a^i as in (29) exist satisfying the assymptotic condition:

$$\lim_{|x_b - x_c| \to \infty} (q_a^i - x_a^i) = 0 \qquad (b \neq c) \tag{50}$$

Wether or not this the right answer to the problem is still not clear, and new results are needed to prove the usefulness of this definition.

b. — Personally I am very much inclined towards another solution which I shall now present.

Let us consider the Reduced Generator System of a P. I. S. I shall say that the latter is Hamiltonian if the first is Hamiltonian in the usual sense.

The difficulty with this definition is that it leaves open the problem of defining the total energy, momentum, angular momentum, and the center of mass coordinates. Consequently I define these quantities independently as follows:

1. — For each frame of reference of the Aristotle Class, the total energy H is the Hamiltonian and the total angular momentum is the angular momentum of the Reduced Generator System. The total momentum is zero: $P_i = 0$. And the coordinates of the center of mass are the Splitting Functions R^i.

2. — For any other reference frame the quantities H, P_i, J_i and R^i are defined by formulae (41).

Of course the idea of these definitions is to identify the Reduced Generator System with the Internal Motion System; the Aristotle Class of Frames of Reference with the Center of Mass Frames of Reference, and for these, the Splitting Functions with the Center of Mass Coordinates.

There still remain a difficulty if we compare the situation now with the corresponding galilean one. Namely that the Internal Motion System and the Center of Mass Coordinates can be given quite independently. For N = 2 a connection can be established as follows. I shall assume that the quantities H, P_i, J_i and R_i as defined above have to be such that, Φ beeing the function (47)

$$\text{(51)} \qquad \{ \Phi \}_{P_i=0} = 0 \qquad \{ \mathscr{L}(\vec{\Xi}_K)\Phi \}_{P_i=0} = 0$$

These supplementary conditions are of course suggested by (48) and (49), and when they are worked out they are shown to be equivalent to the following ones, valid only for the center of mass frames of reference:

$$\text{(52)} \qquad \begin{array}{l} J_i(x_1^i - x_2^i) = 0 \quad , \quad J_i v_1^i = 0 \quad , \quad J_i v_2^i = 0 \\ R^i = H^{-1}(h_1 x_1^i + h_2 x_2^i) \quad , \quad h_1 + h_2 = H \end{array}$$

with h_1 and h_2 scalar first integrals of the internal motion system.

The geometrical meaning of (52) is very clear: for the center of mass frames of reference, the trajectories of both particles lie on a plane orthogonal to the angular momentum, and the center of mass lies on the segment joining the positions of the two particles.

A final and natural assumption is to give to h_1 and h_2 the same expressions, as functions of H, that they have for free systems, namely:

$$h_1 = (2H)^{-1}(H^2 + m_1^2 - m_2^2) \quad , \quad h_2 = (2H)^{-1}(H^2 + m_2^2 - m_1^2)$$

m_1 and m_2 beeing the masses of the particles involved.

Of course this approach to Hamiltonian P. I. S. and Hill and Kerner's definition are not *a priori* contradictory. It would be a nice result to prove that the requirements of both approaches can be met simultaneously.

APPENDIX

The main idea which led to a manifestly covariant description of P. R. M. is due to Ph. Droz-Vincent [5], but some more work was needed to establish a clear connection between Droz-Vincent's approach and the time-symmetric description of P. R. M. of I. S. which has been presented in this lecture.

Let us consider, in M_4, an ordinary second order system of differential equations:

$$\text{(A1)} \qquad \frac{dx_a^\alpha}{d\tau} = u_a^\alpha \quad , \quad \frac{du_a^\alpha}{d\tau} = \xi_a^\alpha(x_b^\beta, u_c^\gamma)$$

To each set of initial conditions $(\underset{0}{x_a^\alpha}, \underset{0}{u_b^\beta})$ will correspond a set of N space-time trajectories, and to the general solution it will thus correspond a family of such sets depending in general on 8N essential parameters, instead of the 6N degrees of freedom of P. I. S. as defined in § 1.

But let us assume that the functions ξ_a^α satisfy the following conditions:

$$\text{(A2)} \qquad \begin{cases} u_a^\alpha \xi_{a\alpha} = 0 \\ u_a^\alpha \dfrac{\partial \xi_b^\beta}{\partial x_a^\alpha} + \xi_a^\alpha \dfrac{\partial \xi_b^\beta}{\partial u_a^\alpha} = 0 \qquad \text{for} \qquad a \neq b \end{cases}$$

The first set of conditions (A2) tell us that we can consistently restrict the general solution to those particular solutions which correspond to initial conditions satisfying the constraints ([1]).

$$\text{(A3)} \qquad \underset{0}{u_a^\alpha} \underset{0}{u_{a\alpha}} = - m_a^2$$

where m_a are some constants which we can interpret as the masses of the particles whose motion is supposed to be described by (A1). The second set of conditions (A2) can be proved to imply the following property: If x_a^α are any set of points lying on a given set of trajectories obtained from some initial conditions $(\underset{0}{x_a^\alpha}, \underset{0}{u_a^\alpha})$, and if u_a^α are the corresponding tangent vectors, then the trajectories which correspond to the initial conditions (x_a^α, u_b^β) are the same as those which correspond to $(\underset{0}{x_a^\alpha}, \underset{0}{u_b^\beta})$. Consequently when conditions (A2) are satisfied the general solution of (A1) restricted by the constraints (A3) depends only on 6N essential parameters.

Up to here this just proves that predictivity and manifest covariance are consistent with each other. The further assumptions which are needed for system (A1) to be relevant in a theory of isolated systems are:

$$\text{(A4)} \qquad \begin{gathered} \varepsilon_a \frac{\partial \xi_b^\beta}{\partial x_a^\alpha} = 0 \\ (\delta_\sigma^\rho \eta_{\mu\gamma} - \delta_\mu^\rho \eta_{\sigma\gamma})\xi_a^\gamma = x_b^\beta\left(\frac{\partial \xi_a^\rho}{\partial x_b^\sigma}\eta_{\beta\mu} - \frac{\partial \xi_a^\rho}{\partial x_b^\mu}\eta_{\beta\sigma}\right) + u_b^\beta\left(\frac{\partial \xi_a^\rho}{\partial u_b^\sigma}\eta_{\beta\mu} - \frac{\partial \xi_a^\rho}{\partial u_b^\mu}\eta_{\beta\sigma}\right) \end{gathered}$$

where $\eta_{\mu\gamma}$ are the coefficients of the Minkowski metric. The first set of equations mean that the functions ξ_a^α are invariant under the space-time translation group. The second set that the vectors ξ_a^α do not contain any constants other than scalar constants.

([1]) I am using the Minkows ki metric with signature + 2.

It can be proven that to each set of functions μ_a^i of a P. I. S. it corresponds a set of functions ξ_a^α satisfying (A2) and (A4) and the other way around. In other words: the time symmetric description and the manifestly covariant description are equivalent description of P. I. S.

Obviously the manifestly covariant description is in many senses more fondamental and simpler than the time-symmetric one. But since it introduces spurious degrees of freedom in the problem I believe that both descriptions will be useful in the future development of the theory.

It is a pleasure to knowledge many stimulating discussions with Dr. L. MAS.

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