ANNALES DE L'I. H. P., SECTION A

L. K. PATEL
P. C. VAIDYA

A generalized plane wave metric

Annales de l'I. H. P., section A, tome 15, nº 2 (1971), p. 169-175
<http://www.numdam.org/item?id=AIHPA_1971__15_2_169_0>

Ann. Inst. Henri Poincaré,
Vol. XV, n° 2, 1971, p. 169-175.

Section A :
Physique théorique.

A generalized plane wave metric

by

L. K. PATEL and P. C. VAIDYA
Department of Mathematics,
Gujarat University, Ahmedabad-9 (India).

1. INTRODUCTION

The theory of plane gravitational waves in general relativity has been discussed by many investigators. H. Takeno [*1*] has discussed the mathematical theory of plane gravitational waves in detail. A Peres [*2*] has studied the plane wave like line-element

$$(1.1) \quad ds^2 = -dx^2 - dy^2 - dz^2 + dt^2 - 2f(x, y, u)(dt - dz)^2$$

where $u = t - z$ and f is a function of x, y and u. The line-element (1.1) can also be expressed as

$$(1.2) \qquad ds^2 = -dx^2 - dy^2 + 2dudz + (1 - 2f)du^2$$

Vaidya and Pandya [*3*] have studied the metric (1.2) in connection with gravitational and electromagnetic radiation. In fact the solution of Peres is a particular case of a more general solution obtained by Pandya and Vaidya [*4*].

In Peres' solution, all components of the metric tensor are not functions of u. The object of the present investigation is to generalize Peres' metric in such a way that all the components of the metric tensor g_{ik} become functions of u. Of course, some of these components do depend upon x and y also.

2. GRAVITATIONAL WAVES

In Minkowskian space, consider an arbitrary smooth world line L that is every where time-like. Let u be the parameter along the world line. Let λ^i be the unit tangent vector at any point of L. Let A^i, B^i, C^i

be the three mutually orthogonal space-like unit vectors lying in the 3-space orthogonal to λ^i at the point under consideration. Thus we have the following relations:

$$(2.1) \qquad \lambda^i\lambda_i = 1, \quad A^iA_i = B^iB_i = C^iC_i = -1$$

and

$$(2.2) \qquad \lambda^iA_i = \lambda^iB_i = \lambda^iC_i = 0,$$

Here it should be noted that the raising and lowering of vector indices of λ^i, A^i, B^i and C^i is carried out with respect to the Minkowskian metric

$$n_{ik} = \text{diag}\,(-1, -1, -1, 1).$$

Let us define the new co-ordinates x, y, z and t in terms of the co-ordinates x^i by the following relations.

$$(2.3) \qquad x = x^iA_i, \quad y = x^iB_i, \quad z = x^iC_i, \quad t = x^i\lambda_i$$

Then clearly

$$(2.4) \qquad x_{,k} = A_k, \quad y_{,k} = B_k, \quad z_{,k} = C_k, \quad t_{,k} = \lambda_k$$

Here and in what follows a comma followed by a lower index will imply partial differentiation with respect to that index. Let

$$(2.5) \qquad Z_i = \lambda_i - C_i \quad \text{and} \quad p_i = A_i - B_i$$

It follows from (2.5) that

$$(2.6) \qquad Z_iZ^i = 0, \qquad p_ip^i = -2.$$

Thus Z^i is a null vector with respect to the Minkowskian metric and p^i is a space-like vector. In this paper we shall confine ourselves to the case in which λ^i, A^i, B^i and C^i are all constant vectors.

Consider a Riemannian 4-space whose metric is given by

$$(2.7) \qquad ds^2 = g_{ik}dx^idx^k$$

where the metric tensor g_{ik} is expressed by the following equation.

$$(2.8) \qquad g_{ik} = \eta_{ik} + Hp_ip_k + SZ_iZ_k$$

Here H is a function of $u = t - z$ and S is a function of x, y and u. The determinant g of the metric tensor g_{ik} can be easily computed. It is given by

$$(2.9) \qquad g = |g_{ik}| = -(1 - 2H).$$

As g is negative $1 - 2H$ should be positive. The vectors p_i and Z_i are orthogonal to each other.

$$Z_i p^i = 0 \tag{2.10}$$

The contravariant components of the metric tensor g_{ik} are given by

$$g^{ik} = \eta^{ik} - \frac{H}{1 - 2H} p^i p^k - SZ^i Z^k \tag{2.11}$$

It follows from (2.9), (2.10) and (2.11) that

$$g^{ik} Z_i = \eta^{ik} Z_i \tag{2.12}$$

and

$$g^{ik} Z_i Z_k = \eta^{ik} Z_i Z_k = Z^i Z_i = 0 \tag{2.13}$$

Thus raising and lowering of the vector indices of Z_i can be carried out with the Riemannian or Minkowskian metric. Also the null character of the congruence Z_i with respect to the Minkowskian metric implies its null character with respect to the Riemannian metric.

From (2.8), (2.10) and (2.11) we also have

$$g^{ik} p_i = \eta^{ik} p_i + \frac{2Hp^k}{1 - 2H} = \frac{p^k}{1 - 2H} \tag{2.14}$$

We shall continue to use the Minkowskian metric η_{ik} for raising and lowering of indices and any dependence on g_{ik} will the explicitly written out as in (2.14). The result (2.6) will be frequently used without mention.

The 3-index symbols for the metric (2.8) are given by

$$\Gamma_{i\,k}^{\,n} = \frac{1}{2} \left[\frac{2p^n H_{,(i} p_{k)}}{1 - 2H} + 2Z^n S_{,(i} Z_{k)} - n^{nl} H_{,l} p_i p_k - \eta^{nl} S_{,l} Z_i Z_k + \frac{H(S_y - S_x)}{1 - 2H} p^n Z_i Z_k \right] \tag{2.15}$$

Throughout this paper the following conventions are used:

Indices range and sum over 1, 2, 3, 4; a semicolon indicates covariant differentiation; round index brackets indicate symmetrization over the indices enclosed; square brackets indicate antisymmetrization; and the lower suffixes attached to functional symbols denote the derivatives of the function with respect to the corresponding variable, e. g.

$$S_y = \frac{\partial S}{\partial y}, \quad S_{xy} = \frac{\partial^2 S}{\partial y \partial x}, \quad H_{uu} = \frac{\partial^2 H}{\partial u^2}, \text{ etc.}$$

It is clear from (2.15) that

$$\Gamma^n_{ik} Z_n = 0 \tag{2.16}$$

The result (2.16) imply that the null congruence Z_i is geodetic.

In our case the expression for the Ricci tensor reduces to

$$R_{ik} = \frac{1}{1-2H}\left[-H_{uu} - \frac{H_u^2}{1-2H} - \frac{1-H}{2}(S_{xx}+S_{yy}) + HS_{xy}\right]Z_i Z_k \tag{2.17}$$

The Riemann curvature tensor R_{hijk} for the metric (2.8) is given by

$$\begin{aligned} R_{hijk} = {} & 2\left[H_{uu} + \frac{H_u^2}{1-2H}\right] p_{[i}Z_{j]}p_{[h}Z_{k]} \\ & + 2S_{xx}A_{[j}Z_{i]}A_{[k}Z_{h]} + 2S_{yy}B_{[j}Z_{i]}B_{[k}Z_{h]} \\ & + 2S_{xy}\{A_{[i}Z_{j]}B_{[k}Z_{h]} + B_{[i}Z_{j]}A_{[k}Z_{h]}\} \end{aligned} \tag{2.18}$$

For gravitational waves we have

$$R_{ik} = 0. \tag{2.19}$$

The results (2.17) and (2.19) imply that

$$S_{xx} + S_{yy} - \frac{2H}{1-H}S_{xy} = -\frac{2}{1-H}\left(H_{uu} + \frac{H_u^2}{1-2H}\right) \tag{2.20}$$

For gravitational waves, S and H have to satisfy the equation (2.20). The choice of any one of S and H is at ourdisposal.

If $S = 0$, then from (2.18), (2.19) and (2.20) we obtain $R_{hijk} = 0$ and the space-time becomes flat.

If $H = 0$, then $R_{ik} = 0$ implies $S_{xx} + S_{yy} = 0$ and we get the space-time of peres.

Thus it is clear that if we choose H in such a way that $H \neq 0$, $1-2H>0$ and $H_{uu} + (H_u^2/1 - 2H) \neq 0$, then we get the gravitational field which is different from that discussed by Peres.

From (2.18) we have:

A necessary and sufficient condition that a space-time given by (2.8) be Minkowskian is

$$S_{ab} = 0 \quad \text{and} \quad H_{uu} + \frac{H_u^2}{1-2H} = 0, \quad a, b = x, y \tag{2.21}$$

Thus, when S is a linear function of x and y whose coefficients are functions of u and H satisfies $H_{uu} + \dfrac{H_u^2}{1-2H} = 0$, then the space-time discussed here becomes flat.

3. CO-EXISTANCE OF ELECTROMAGNETIC WAVES

In this section we shall show that the solution obtained in the previous section can be generalized to the case in which the electromagnetic waves co-exist with the gravitational waves. The field equations of electromagnetic field in general relativity are

$$R_{ik} = -8\pi E_{ik} \tag{3.1}$$

and the maxwell equations are

$$\begin{aligned} F_{ik,n} + F_{kn,i} + F_{ni,k} &= 0 \\ F^{ik}{}_{;k} &= 0 \end{aligned} \tag{3.2}$$

Here F_{ik} is the antisymmetric tensor describing electromagnetic field and E_{ik} is the electromagnetic energy tensor defined by

$$E_{ik} = \frac{1}{4} g_{ik} F_{lm} F_{ab} g^{la} g^{mb} - F_{il} F_{km} g^{lm} \tag{3.3}$$

If ϕ_i is the 4-potential of the electromagnetic field then

$$F_{ik} = \phi_{i,k} - \phi_{k,i} \tag{3.4}$$

Let us choose the 4-potential ϕ_i of the electromagnetic field as

$$\phi_i = D(x, y, u) Z_i \tag{3.5}$$

Looking to the nature of our problem this choice of ϕ_i seems appropriate. Now,

$$F_{ik} = D_{,k} Z_i - D_{,i} Z_k \tag{3.6}$$

Clearly

$$g^{im} h^{kn} F_{mn} F_{ik} = 0 \tag{3.7}$$

Thus the electromagnetic field is null with respect to Riemannian metric. (2.8). The electromagnetic energy tensor E_{ik} is given by

$$E_{ik} = \left[D_x^2 + D_y^2 + \frac{H(D_y - D_x)^2}{1 - 2H} \right] Z_i Z_k \tag{3.8}$$

The results (2.17), (3.1) and (3.8) imply that

$$(3.9)\quad 8\pi\left[D_x^2 + D_y^2 + \frac{H(D_y - D_x)^2}{1 - 2H}\right] = \frac{1}{1-2H}\left[H_{uu} + \frac{H_u^2}{1-2H} + \frac{1-H}{2}(S_{xx} + S_{yy}) - HS_{xy}\right]$$

The Maxwell equations (3.2) are equivalent to

$$(3.10)\quad D_{xx} + D_{yy} - \frac{2H}{1-H}D_{xy} = 0$$

Hence, for electromagnetic waves D and S have to satisfy equations (3.9) and (3.10) and H remains arbitrary.

However if we consider S as a function of D, equations (3.9) and (3.10) imply

$$(3.11)\quad \frac{16\pi - \dfrac{d^2S}{dD^2}}{2}[(1 - H)(D_x^2 + D_y^2) - 2HD_xD_y] = H_{uu} + \frac{H_u^2}{1-2H}.$$

Let us consider a particular case in which

$$(3.12)\quad \frac{d^2S}{dD^2} = 16\pi \quad \text{i.e.}\quad S = 8\pi D^2 + \alpha D + \beta$$

where α and β are constants.

Equation (3.11) reduces to

$$(3.13)\quad H_{uu} + \frac{H_u^2}{1-2H} = 0 \quad \text{i.e.}\quad H = \frac{1-(au+b)^2}{2}$$

where a and b are constants.

For electromagnetic waves in this case, H and D have to satisfy (3.10) and (3.13). When $H = 0$ (i.e. $a = 0$, $b = 1$), the electromagnetic field discussed above reduces to the electromagnetic field studied by Takeno [5]. It may however be noted that Takeno [5] has not taken S as a function of D.

REFERENCES

[1] H. Takeno, *The Mathematical Theory of plane gravitational waves in general relativity.* Scientific reports of research institute for theoretical Physics, Hiroshima University, No. 1, 1961.

[2] A. PERES, Some gravitational waves. *Phys. Rev. let.*, t. **3**, 1959, p. 571.
[3] P. C. VAIDYA and I. M. PANDYA, Radiation: gravitational and electromagnetic. *Curr. Sci.*, t. **29**, 1960, p. 268.
[4] I. M. PANDYA and P. C. VAIDYA, Wave solutions in general relativity. I. *Proc. nat. Sci. India*, t. **27**, 1961, p. 620.
[5] H. TAKENO, On the space-time of Peres. *Tensor*, N. S., t. **11**, 1961, p. 99.

(Reçu le 3 février 1971).

Directeur de la publication : GUY DE DAMPIERRE.

IMPRIMÉ EN FRANCE

DÉPOT LÉGAL ÉD. N° 1795*a*. IMPRIMERIE BARNÉOUD S. A. LAVAL, N° 6244. 9-1971.