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On a class of infinite products occurring in quantum statistical mechanics

by

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ABSTRACT. — We study the class of infinite products

$$f_c^L(\lambda) = \sum_{n=0}^{\infty} a_n^L \lambda^n = \prod_{j=1}^{\infty} \left\{ 1 + \lambda c(k_j)^2 \right\}, \quad k_j = j \frac{2\pi}{L} \quad \text{and} \quad L \in \mathbb{R}^+,$$

where $c(k) = \frac{A}{k^m}$, $k \geq 0$, $A > 0$, $m > \frac{1}{2}$, which occurs naturally in quantum statistical mechanics. In particular, we compute the limits

$$\lim_{\substack{n \rightarrow \infty \\ L \rightarrow \infty}} \left\{ \frac{2n}{L} = d \right\} \frac{a_{n+1}^L}{a_n^L} \quad \text{and} \quad \lim_{\substack{n \rightarrow \infty \\ L \rightarrow \infty}} \left\{ \frac{2n}{L} = d \right\} \frac{1}{L} \text{Log } a_n^L$$

which are relevant in the problem of the thermodynamic limit of the BCS superconducting state. By the same way, we get new results concerning the infinite products of the form

$$g_\rho(\mu) = \sum_{n=0}^{\infty} b_n \mu^n = \prod_{j=1}^{\infty} \left\{ 1 + \mu \frac{A}{j^{1/\rho}} \right\}, \quad A > 0, \quad 0 < \rho < 1.$$

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In particular we are able to compute the limits

$$\lim_{n \rightarrow \infty} n^{1/\rho} \frac{b_{n+1}}{b_n} \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{1}{2n} \text{Log } n^{1/2} b_n^{1/2}$$

RÉSUMÉ. — Nous étudions la classe de produits infinis

$$f_c^L(\lambda) = \sum_{n=0}^{\infty} a_n^L \lambda^n = \prod_{j=1}^{\infty} \left\{ 1 + \lambda c(k_j)^2 \right\}, \quad k_j = j \frac{2\pi}{L} \quad \text{et} \quad L \in \mathbb{R}^+,$$

où $c(k) = \frac{A}{k^m}$, $k \geq 0$, $A > 0$, $m > \frac{1}{2}$, qui se présentent naturellement en mécanique statistique quantique. En particulier nous calculons les limites

$$\lim_{\substack{n \rightarrow \infty \\ L \rightarrow \infty} \left\{ \frac{2n}{L} = d \right\}} \frac{a_{n+1}^L}{a_n^L} \quad \text{et} \quad \lim_{\substack{n \rightarrow \infty \\ L \rightarrow \infty} \left\{ \frac{2n}{L} = d \right\}} \frac{1}{L} \text{Log } a_n^L$$

qui ont leur importance dans la limite thermodynamique de l'état de la supraconductivité de Bardeen-Cooper-Schrieffer. Simultanément nous obtenons des résultats nouveaux relatifs aux produits infinis de la forme

$$g_\rho(\mu) = \sum_{n=0}^{\infty} b_n \mu^n = \prod_{j=1}^{\infty} \left\{ 1 + \mu \frac{A}{j^{1/\rho}} \right\}, \quad A > 0, \quad 0 < \rho < 1$$

En particulier nous sommes en mesure de calculer les limites

$$\lim_{n \rightarrow \infty} n^{1/\rho} \frac{b_{n+1}}{b_n} \quad \text{et} \quad \lim_{n \rightarrow \infty} \frac{1}{2n} \text{Log } n^{1/2} b_n^{1/2}$$

I. INTRODUCTION

It is now well known that algebras of observables are useful in the kinematical description, from a quantum mechanical point of view, of systems of interacting particles. For that purpose, we constructed, in a preceding paper [1], a family of states (positive linear functionals of norm 1) over a Clifford-C*-algebra, each of which characterized by a real function c of real argument and describing a « condensate » of pairs of fermions with density d .

In that work, we studied a class of entire functions of the complex variables λ , depending on the function c and on a length L , defined by the infinite products

$$(1) \quad f_c^L(\lambda) = \prod_{j=1}^{\infty} \{ 1 + \lambda c(k_j)^2 \}, \quad L \in \mathbb{R}^+, \quad k_j = j \frac{2\pi}{L}$$

More precisely, if we write

$$(2) \quad f_c^L(\lambda) = \sum_{n=0}^{\infty} a_n^L \lambda^n$$

where

$$(3) \quad a_n^L = \sum_{0 < j_1 < j_2 < \dots < j_n} c(k_{j_1})^2 c(k_{j_2})^2 \dots c(k_{j_n})^2$$

the main point was the existence of the limit

$$(4) \quad \gamma(d) = \lim_{\substack{n \rightarrow \infty \\ L \rightarrow \infty}} \frac{a_{n+1}^L}{a_n^L}, \quad d \in \mathbb{R}^+$$

and we were able to prove the following theorem:

THEOREM. — Let c be a real, bounded, decreasing, square integrable function of a real positive variable k , tending to zero, when k tends to infinity, faster than $k^{-1/2}$. Let d and L be positive reals.

Then the limit (4) exists. Moreover, defining the function $g(d)$ as

$$(5) \quad g(d) = \lim_{\substack{n \rightarrow \infty \\ L \rightarrow \infty}} \frac{1}{L} \text{Log } a_n^L,$$

then $g(d)$ exists, is convex and differentiable, and

$$(6) \quad \gamma(d) = e^{2g'(d)}$$

Finally one has the following integral formula

$$(7) \quad d = \frac{1}{\pi} \int_0^{\infty} \frac{c^2(k)}{\gamma(d) + c^2(k)} dk$$

The methods employed are rather involved and do not fully exploit the analyticity properties of f_c^L . On the other hand, the theorem quoted

above is only an existence theorem, so that except for very special choices for c ([1], (109)), it does not allow to compute explicitly the quantities $g(d)$ and $\gamma(d)$ as functions of d .

Our purpose is to restate that theorem, using now entire functions technics. To that end, we have to make a different choice for our class of functions c , giving up definiteness at the origin (and so boundedness and square integrability) but requiring an homogeneity condition. The quantities a_n^L , $g(d)$ and $\gamma(d)$ are now explicitly given.

By the same way, we study the class of infinite products

$$(8) \quad g_\rho(\mu) = \prod_{j=1}^{\infty} \left\{ 1 + \mu \frac{A^2}{j^{1/\rho}} \right\}, \quad A > 0, \quad 0 < \rho < 1$$

obtaining some new results for $\rho \neq \frac{1}{2}$.

II. CHOICE OF THE CLASS OF FUNCTIONS c

Let us consider, for the moment, positive functions $c(k)$, defined for $k > 0$, and vanishing at infinity faster than $k^{-1/2}$. The infinite products (1) are then convergent and define entire functions f_c^L whose zeros lie on the real negative axis and are given by

$$(9) \quad \lambda_j = -\frac{1}{c(k_j)^2}$$

By definition ([4], I, 4), ([5], 2.5.2), the convergence exponent of the sequence (9) is the greatest lower bound ρ of the reals α such that

$$(10) \quad \sum_{j=1}^{\infty} \left\{ \frac{1}{c(k_j)^2} \right\}^{-\alpha} = \sum_{j=1}^{\infty} c(k_j)^{2\alpha} < +\infty$$

If, by now, we restrict ourselves to functions such that

$$(11) \quad c(k) \sim \frac{A}{k^m}, \quad k \rightarrow \infty, \quad m > \frac{1}{2}, \quad A > 0$$

a necessary and sufficient condition for the convergence of (10) is

$$(12) \quad 2m\alpha = 1 + \varepsilon, \quad \varepsilon > 0$$

so that

$$(13) \quad \rho = \operatorname{Inf}_{\varepsilon > 0} \left\{ \alpha: \alpha = \frac{1}{2m} + \frac{\varepsilon}{2m} \right\} = \frac{1}{2m} < 1$$

On the other hand, the f_c^L are infinite canonical products of order ρ and genus p ([4], I, Th. 7), ([5], 2.6.5), the genus p being related to the order ρ according to

$$(14) \quad p \leq \rho \leq p + 1, \quad p \text{ integer.}$$

In the case of (11), we can then conclude that the entire functions f_c^L are characterised by

$$(15) \quad p = 0, \quad 0 < \rho = \frac{1}{2m} < 1, \quad \frac{1}{2} < m + \infty$$

Further more, let $n(r)$, $r \in \mathbb{R}^+$, the function giving the number of zeros of f_c^L with modulus less than or equal to r . One can show ([4], I, lemma 1), ([5], 2.5.8) that

$$(16) \quad \rho = \overline{\lim}_{r \rightarrow \infty} \frac{\operatorname{Log} n(r)}{\operatorname{Log} r}$$

and one defines the density Δ of the sequence (9) as

$$(17) \quad \Delta = \overline{\lim}_{r \rightarrow \infty} \frac{n(r)}{r^\rho}$$

We are going to estimate these quantities, restricting now to functions c such that (11) holds, which are monotonic and differentiable at least for k large and such that

$$(18) \quad c'(k) \sim \frac{-B}{k^{m+1}}, \quad k \rightarrow \infty, \quad B > 0$$

Turning back to the definition of $n(r)$, we can write that

$$(19) \quad n(r) = \operatorname{Max} \left\{ j: \frac{1}{c(k_j)^2} \leq r \right\} = \operatorname{Max} \left\{ j: k_j \leq v\left(\frac{1}{\sqrt{r}}\right) \right\} = \left[\frac{L}{2\pi} v\left(\frac{1}{\sqrt{r}}\right) \right]$$

where v is the inverse function of c (defined at least for r large enough) and where the squared brackets mean « the largest integer contained in ». Thanks to our hypothesis (18), it is an easy task to show that

$$(20) \quad n(r) \sim \frac{L}{2\pi} A^{2\rho} r^\rho, \quad r \rightarrow \infty$$

(a result which agrees with (16)) and that

$$(21) \quad \Delta = \frac{L}{2\pi} A^{2\rho}$$

From these estimations, it is actually possible to deduce that ([4], I, Th. 25), ([5], 4.1.1)

$$(22) \quad \text{Log } f_c^L(re^{i\theta}) \sim e^{i\rho\theta} \frac{L}{2} A^{2\rho} (\text{cosec } \pi\rho)r^\rho, \quad -\pi < \theta < +\pi, \quad r \rightarrow \infty$$

or, taking the real part, that

$$(23) \quad \text{Log } |f_c^L(re^{i\theta})| \sim \frac{L}{2} A^{2\rho} (\text{cosec } \pi\rho) \cos \rho\theta \cdot r^\rho, \quad -\pi < \theta < +\pi, \quad r \rightarrow \infty$$

uniformly with respect to θ if $-\pi + \varepsilon \leq \theta \leq \pi - \varepsilon$.

Then, the indicator function $h_c^L(\theta)$ of f_c^L is given by ([4], I, 15), ([5], 2.1.8)

$$(24) \quad h_c^L(\theta) = \overline{\lim}_{r \rightarrow \infty} \frac{\text{Log } |f_c^L(re^{i\theta})|}{r^\rho} = \frac{L}{2} A^{2\rho} (\text{cosec } \pi\rho) \cos \rho\theta, \quad -\pi < \theta < +\pi$$

and also for $-\pi \leq \theta \leq +\pi$ as h_c^L is a continuous function, defined by periodicity for other values of θ .

From now on, we are able to compute the type of f_c^L according to the formula ([4], I, 1 and Th. 29), ([5], 2.1.4)

$$(25) \quad \tau(L) = \overline{\lim}_{r \rightarrow \infty} \frac{\text{Log } \text{Max}_{|\lambda|=r} |f_c^L(\lambda)|}{r^\rho} = \text{Max}_\theta |h(\theta)| = \frac{L}{2} A^{2\rho} \text{cosec } \pi\rho$$

as well as, by ([4], I, Th. 2), ([5], 2.2.10)

$$(26) \quad \tau(L) = \frac{1}{e\rho} \overline{\lim}_{n \rightarrow \infty} n(a_n^L)^{\rho/n}$$

Comparing formulas (20) and (16), we see that the $\overline{\lim}$ occurring in (16) is in fact a limit, as well as the ones in (17), (24), (25) and consequently in (26).

Unfortunately, the comparison of formulas (25) and (26) does not allow to estimate the limit (4) because of a lack of uniformity with respect to L . A simple case where uniformity can be recovered is given by

$$(27) \quad a_n^L = \left(\frac{L}{2\pi}\right)^{n/\rho} b_n \quad b_n \text{ independent of } L$$

The following proposition, the proof of which is trivial, shows that (27) holds if and only if $c(k)$ is a homogeneous function of degree $-\frac{1}{2\rho}$:

PROPOSITION. — If

$$f_c^L(\lambda) = \prod_{j=1}^{\infty} \{ 1 + \lambda c(k_j)^2 \} = \sum_{n=0}^{\infty} a_n^L \lambda^n$$

where $k_j = j \frac{2\pi}{L}$, then $a_n^L = \left(\frac{L}{2\pi}\right)^{n/\rho} b_n$, b_n independent of L , if and only if

$$c(k) = \frac{A}{k^{1/2\rho}} = \frac{A}{k^m},$$

$\rho = \frac{1}{2m}$ being the order of f_c^L .

So our conclusion is that entire functions technics can be easily applied to our problem if we restrict ourselves to the class of functions

$$(28) \quad c(k) = \frac{A}{k^m}, \quad \frac{1}{2} < m < \infty, \quad A > 0, \quad 0 < k < \infty$$

for which conditions (11) and (18) are evidently fulfilled.

III. THE MAIN THEOREM

From now on, we can write, thanks to (25),

$$(29) \quad f_c^L(\lambda) = \sum_{n=0}^{\infty} \left(\frac{L}{2\pi}\right)^{n/\rho} b_n \lambda^n = \sum_{n=0}^{\infty} \frac{(\tau(L)^{1/\rho} \lambda)^n}{\left(\frac{A^{2\rho} \pi}{\sin \pi \rho}\right)^{n/\rho} \frac{1}{b_n}}$$

and

$$(30) \quad g_\rho(\mu) = \prod_{j=1}^{\infty} \left\{ 1 + \mu \frac{A^2}{j^{1/\rho}} \right\} = \sum_{n=0}^{\infty} b_n \mu^n$$

Moreover, the expressions

$$(31) \quad \frac{a_n^L}{\left(\frac{\tau(L)e\rho}{n}\right)^{n/\rho}} = \frac{b_n}{\left(\frac{A^{2\rho} \pi}{\sin \pi \rho}\right)^{n/\rho} \left(\frac{e\rho}{n}\right)^{n/\rho}}$$

and (where Γ means the usual gamma function)

$$(32) \quad \frac{a_{n+1}^L}{a_n^L} \left/ \frac{\left(\frac{\tau(L)e\rho}{n+1} \right)^{\frac{n+1}{\rho}}}{\left(\frac{\tau(L)e\rho}{n} \right)^{n/\rho}} \right. = \frac{b_{n+1}}{\left(\frac{A^{2\rho}\pi}{\sin \pi\rho} \right)^{\frac{n+1}{\rho}} \left(\frac{e\rho}{n+1} \right)^{\frac{n+1}{\rho}}} \frac{\left(\frac{A^{2\rho}\pi}{\sin \pi\rho} \right)^{\frac{n}{\rho}} \left(\frac{e\rho}{n} \right)^{\frac{n}{\rho}}}{b_n}$$

$$\xrightarrow{n \rightarrow \infty} \frac{b_{n+1}}{\left(\frac{A^{2\rho}\pi}{\sin \pi\rho} \right)^{\frac{n+1}{\rho}} \frac{1}{\Gamma\left(\frac{n+1}{\rho} + 1\right)}} \frac{1}{b_n \frac{\left(\frac{A^{2\rho}\pi}{\sin \pi\rho} \right)^{\frac{n}{\rho}} \Gamma\left(\frac{n}{\rho} + 1\right)}{1}} = \frac{\chi(n+1; \rho)}{\chi(n; \rho)}$$

are now independent of L . Therefore the result we are aiming at is equivalent to

$$(33) \quad \lim_{n \rightarrow \infty} \frac{\chi(n+1; \rho)}{\chi(n; \rho)} = 1$$

or

$$(34) \quad b_n = \left(\frac{A^{2\rho}\pi}{\sin \pi\rho} \right)^{n/\rho} \frac{\chi(n; \rho)}{\Gamma\left(\frac{n}{\rho} + 1\right)} \quad \text{where} \quad \lim_{n \rightarrow \infty} \frac{\chi(n+1; \rho)}{\chi(n; \rho)} = 1$$

Incidentally it is interesting to remark that

$$(35) \quad g_\rho(\mu) = \sum_{n=0}^{\infty} \frac{\left(\frac{A^{2\rho}\pi}{\sin \pi\rho} \right)^{n/\rho} \chi(n; \rho)}{\Gamma\left(\frac{n}{\rho} + 1\right)} \mu^n$$

is of order ρ and type

$$(36) \quad \tau = \frac{A^{2\rho}\pi}{\sin \pi\rho} = \tau(2\pi)$$

as is also the Mittag-Leffler function [6]

$$(37) \quad E_\rho(\mu) = \sum_{n=0}^{\infty} \frac{\left(\frac{A^{2\rho}\pi}{\sin \pi\rho} \right)^{n/\rho}}{\Gamma\left(\frac{n}{\rho} + 1\right)} \mu^n$$

which shows the close relation between E_ρ and g_ρ and asserts the well

known fact that E_ρ is, in some sense [7], the simplest entire function of a given order and type.

Another way of writing (33) is the following (provided the limits exists):

$$\begin{aligned}
 (38) \quad & \lim_{\substack{n \rightarrow \infty \\ L \rightarrow \infty}} \left\{ \frac{a_{n+1}^L}{a_n^L} \right\}_{L=d} = \lim_{\substack{n \rightarrow \infty \\ L \rightarrow \infty}} \left\{ \frac{2n}{L} \right\}_{L=d} \left(\frac{\tau(L)e\rho}{n+1} \right)^{\frac{n+1}{\rho}} \bigg/ \left(\frac{\tau(L)e\rho}{n} \right)^{n/\rho} \\
 & = \lim_{\substack{n \rightarrow \infty \\ L \rightarrow \infty}} \left\{ \frac{2n}{L} \right\}_{L=d} \left(\frac{LA^{2\rho}e\rho \operatorname{cosec} \pi\rho}{2(n+1)} \right)^{\frac{n+1}{\rho}} \bigg/ \left(\frac{LA^{2\rho}e\rho \operatorname{cosec} \pi\rho}{2n} \right)^{n/\rho} \\
 & = \lim_{n \rightarrow \infty} \left(\frac{A^{2\rho}e\rho \operatorname{cosec} \pi\rho}{d} \right)^{1/\rho} \frac{1}{\left(1 + \frac{1}{n}\right)^{n/\rho}} \frac{1}{\left(1 + \frac{1}{n}\right)^{1/\rho}} = \left(\frac{A^{2\rho}e\rho \operatorname{cosec} \pi\rho}{d} \right)^{\frac{1}{\rho}}
 \end{aligned}$$

So, once the existence of the limit is proved, then necessarily

$$(39) \quad \gamma(d) = \left(\frac{A^{2\rho}e\rho \operatorname{cosec} \pi\rho}{d} \right)^{1/\rho}$$

and hence

$$(40) \quad \lim_{n \rightarrow \infty} n^{1/\rho} \frac{b_{n+1}}{b_n} = (\pi A^{2\rho}e\rho \operatorname{cosec} \pi\rho)^{1/\rho}$$

Before proving our main theorem, let us give a lemma:

LEMMA. — We have the inequalities:

$$(41) \quad \frac{a_n^L}{a_{n-1}^L} > \frac{a_{n+1}^L}{a_n^L}$$

and

$$(42) \quad \frac{b_n}{b_{n-1}} > \frac{b_{n+1}}{b_n}$$

These inequalities are well known in the theory of entire functions ([5], 2.8.2) but we restate the proof by sake of completeness.

One has:

$$\frac{f'(\lambda)}{f(\lambda)} = \sum_{j=1}^{\infty} \frac{c(k_j)^2}{1 + \lambda c(k_j)^2}$$

and

$$\left\{ \frac{f'(\lambda)}{f(\lambda)} \right\}' = \sum_{j=1}^{\infty} \frac{-c(k_j)^4}{[1 + \lambda c(k_j)^2]^2}$$

So $f(\lambda)f''(\lambda) < f'(\lambda)^2$ if $\lambda \in \mathbb{R}$ or, else, $f^{(n-1)}(\lambda)f^{(n+1)}(\lambda) < f^{(n)}(\lambda)^2$ by appli-

cation of the same inequality to $f^{(n-1)}(\lambda)$, which is also an entire function of the same order, type and genus as f . It follows that

$$\frac{a_{n+1}^L}{a_n^L} < \frac{n}{n+1} \frac{a_n^L}{a_{n-1}^L} < \frac{a_n^L}{a_{n-1}^L}.$$

It is interesting to remark that, by adapting to our case the proof of ([1 bis], lemma 3, (80)), one has conversely

$$(43) \quad \begin{aligned} n \frac{b_n}{b_{n-1}} &\leq (n+1) \frac{b_{n+1}}{b_n} + A^2 \\ \frac{a_n^L}{a_{n-1}^L} &\leq \frac{n+1}{n} \frac{a_{n+1}^L}{a_n^L} + \frac{1}{\pi d} A^2 \end{aligned}$$

We are now able to prove our theorem.

THEOREM. — Let $c(k) = \frac{A}{k^m}$ with $k > 0$, $A > 0$, $m > \frac{1}{2}$, L and d be positive, and

$$(44) \quad \begin{cases} f_c^L(\lambda) = \sum_{n=0}^{\infty} a_n^L \lambda^n = \prod_{j=1}^{\infty} \{1 + \lambda c(k_j)^2\}, & k_j = j \frac{2\pi}{L} \\ g_\rho(\mu) = \sum_{n=0}^{\infty} b_n \mu^n = \prod_{j=1}^{\infty} \left\{1 + \mu \frac{A^2}{j^{1/\rho}}\right\}, & \rho = \frac{1}{2m}, \quad 0 < \rho < 1 \end{cases}$$

Then

$$(45) \quad \lim_{\substack{n \rightarrow \infty \\ L \rightarrow \infty}} \left\{ \frac{2n}{L} = d \right\} \frac{1}{L} \operatorname{Log} a_n^L = g(d) = \frac{d}{2\rho} \operatorname{Log} \left(\frac{A^{2\rho} e\rho \operatorname{cosec} \pi\rho}{d} \right)$$

and

$$(46) \quad \lim_{n \rightarrow \infty} \frac{1}{2\pi} \operatorname{Log} n^{1/\rho} b_n^{1/n} = \frac{1}{2\pi\rho} \operatorname{Log} (\pi A^{2\rho} e\rho \operatorname{cosec} \pi\rho) = g\left(\frac{1}{\pi}\right)$$

Moreover

$$(47) \quad \lim_{\substack{n \rightarrow \infty \\ L \rightarrow \infty}} \left\{ \frac{2n}{L} = d \right\} \frac{a_{n+1}^L}{a_n^L} = \gamma(d) = \left(\frac{A^{2\rho} \rho \operatorname{cosec} \pi\rho}{d} \right)^{1/\rho}$$

$$(48) \quad \lim_{n \rightarrow \infty} n^{1/\rho} \frac{b_{n+1}}{b_n} = \gamma = (\pi A^{2\rho} \rho \operatorname{cosec} \pi\rho)^{1/\rho} = \gamma\left(\frac{1}{\pi}\right)$$

and

$$(49) \quad \begin{cases} \gamma(d) = e^{2g'(d)} \\ \gamma = e^{2g'\left(\frac{1}{\pi}\right)} \end{cases}$$

Finally one has the following integral formulas

$$(50) \quad \begin{cases} d = \frac{1}{\pi} \int_0^\infty \frac{c(k)^2}{c(k)^2 + \gamma(d)} dk = \frac{1}{\pi} \int_0^\infty \frac{dk}{1 + \frac{\gamma(d)}{A^2} k^{2m}} \\ 1 = \int_0^\infty \frac{c(k)^2}{c(k)^2 + \gamma} dk = \int_0^\infty \frac{dk}{1 + \frac{\gamma}{A^2} k^{2m}} \end{cases}$$

Proof (our proof is similar to that used recently by Dobrushin and Minlos [9]). — Formulas (26) and (36) tell us that

$$\forall \varepsilon > 0, \exists N(\varepsilon): n > N(\varepsilon) \Rightarrow \left(1 - \frac{\varepsilon}{\tau\rho}\right)^{n/\rho} < \frac{b_n}{\left(\frac{\tau\rho}{n}\right)^{n/\rho}} < \left(1 + \frac{\varepsilon}{\tau\rho}\right)^{n/\rho}$$

which gives rise to the equivalent formulas

$$\left| \frac{1}{n} \text{Log } b_n - \frac{1}{\rho} \text{Log } \frac{\tau\rho}{n} \right| < \frac{\varepsilon}{\tau\rho^2}$$

or

$$\left| \frac{\text{Log } b_n}{\frac{n}{\rho} \text{Log } \frac{\tau\rho}{n}} - 1 \right| < \frac{\varepsilon}{\tau\rho} \left| \frac{1}{\text{Log } \frac{\tau\rho}{n}} \right| < \varepsilon' \quad \text{for } n \text{ large enough}$$

Consequently,

$$\begin{aligned} \lim_{\substack{n \rightarrow \infty \\ L \rightarrow \infty \left\{ \begin{smallmatrix} 2n \\ L \end{smallmatrix} \right\} = d}} \frac{1}{L} \text{Log } a_n^L &= \lim_{\substack{n \rightarrow \infty \\ L \rightarrow \infty \left\{ \begin{smallmatrix} 2n \\ L \end{smallmatrix} \right\} = d}} \frac{d}{2\rho} \left\{ \text{Log } \frac{L}{2\pi} + \frac{\rho}{n} \text{Log } b_n \right\} \\ &= \lim_{\substack{n \rightarrow \infty \\ L \rightarrow \infty \left\{ \begin{smallmatrix} 2n \\ L \end{smallmatrix} \right\} = d}} \frac{d}{2\rho} \left\{ \text{Log } \frac{L}{2\pi} + \text{Log } \frac{\tau\rho}{n} \right\} = \frac{d}{2\rho} \text{Log } \frac{\tau\rho}{\pi d} = g(d) \end{aligned}$$

and we get (45) or, in the same way, (46).

On the other hand, we have, from (27),

$$\text{Log } \frac{a_{n+1}^L}{a_n^L} = \frac{1}{\rho} \text{Log } \frac{L}{2\pi} + \text{Log } \frac{b_{n+1}}{b_n}$$

It is sufficient to prove (47) to get also (48) and (49). We shall proceed in two steps, proving successively that

$$a) \quad \lim_{\substack{n \rightarrow \infty \\ L \rightarrow \infty \left\{ \begin{smallmatrix} 2n \\ L \end{smallmatrix} \right\} = d}} \frac{a_{n+1}^L}{a_n^L} \geq e^{2g'(d)}$$

$$b) \quad \overline{\lim}_{\substack{n \rightarrow \infty \\ L \rightarrow \infty}} \left. \begin{matrix} 2n \\ L = d \end{matrix} \right\} \frac{a_{n+1}^L}{a_n^L} \leq e^{2g'(d)}$$

a) Inequality (41) allows to write, for m positive integer,

$$\frac{a_{n+1+m}^L}{a_n^L} = \frac{a_{n+1+m}^L}{a_{n+m}^L} \cdot \frac{a_{n+m}^L}{a_{n+m-1}^L} \cdot \frac{a_{n+m-1}^L}{a_{n+m-2}^L} \cdots \frac{a_{n+1}^L}{a_n^L} \leq \left(\frac{a_{n+1}^L}{a_n^L} \right)^{m+1}$$

or else

$$\frac{a_{n+1}^L}{a_n^L} \geq \left(\frac{a_{n+1+m}^L}{a_n^L} \right)^{\frac{1}{m+1}} = e^{\frac{L}{m+1} \frac{1}{L} \text{Log}(a_{n+m+1}^L - a_n^L)}$$

Taking the limit of both sides for $n \rightarrow \infty, L \rightarrow \infty, \frac{2n}{L} = d, m \rightarrow \infty, \frac{2(m+1)}{L} = \varepsilon$, we get

$$\lim_{\substack{n \rightarrow \infty \\ L \rightarrow \infty}} \left. \begin{matrix} 2n \\ L = d \end{matrix} \right\} \frac{a_{n+1}^L}{a_n^L} \geq e^{\frac{2}{\varepsilon} [g(d+\varepsilon) - g(d)]}$$

which proves the desired result thanks to the arbitrariness of ε .

b) Inequality (41) allows to write, for m positive integer,

$$\frac{a_n^L}{a_{n-m}^L} = \frac{a_n^L}{a_{n-1}^L} \frac{a_{n-1}^L}{a_{n-2}^L} \cdots \frac{a_{n-m+1}^L}{a_{n-m}^L} \geq \left(\frac{a_{n+1}^L}{a_n^L} \right)^m$$

or else

$$\frac{a_{n+1}^L}{a_n^L} \leq \left(\frac{a_n^L}{a_{n-m}^L} \right)^{\frac{1}{m}} = e^{\frac{L}{m} \frac{1}{L} \text{Log}(a_n^L - a_{n-m}^L)}$$

Taking the limit of both sides for $n \rightarrow \infty, L \rightarrow \infty, \frac{2n}{L} = d, m \rightarrow \infty, \frac{2m}{L} = \varepsilon$, we get

$$\overline{\lim}_{\substack{n \rightarrow \infty \\ L \rightarrow \infty}} \left. \begin{matrix} 2n \\ L = d \end{matrix} \right\} \frac{a_{n+1}^L}{a_n^L} \leq e^{\frac{2}{\varepsilon} [g(d) - g(d-\varepsilon)]}$$

which again proves the desired result (b) thanks to the arbitrariness of ε .

Finally, formulas (50) can be easily deduced from the identity ([10], 3.241, 2):

$$\int_0^\infty \frac{dk}{1+k^{2m}} = \frac{\pi}{2m} \operatorname{cosec} \frac{\pi}{2m}, \quad m > \frac{1}{2}$$

To end this part, we want to mention that it is possible to compute the

quantities b_n , and consequently a_n^L , with the help of the ζ function of Riemann, using methods patterned from the Fredholm theory of integral equations [11]. This result cannot be considered new. It is contained, for instance, into the formulas ([10], 8.334, 1, 8.321, 2)

$$(51) \quad g_\rho(\mu) = \prod_{j=1}^{\infty} \left\{ 1 + \mu \frac{A^2}{j^{2m}} \right\} = \frac{1}{\mu A^2} \prod_{k=1}^{2m} \frac{1}{\Gamma \left[-(-\mu A^2)^{\frac{1}{2m}} \exp \frac{2\pi k i}{2m} \right]}$$

and

$$(52) \quad \frac{1}{\Gamma(z+1)} = \sum_{k=0}^{\infty} d_k z^k$$

where

$$d_0 = 1, \quad d_{n+1} = \frac{\sum_{k=0}^n (-1)^k s_{k+1} d_{n-k}}{n+1}, \quad s_1 = C, \quad s_n = \zeta(n) \quad \text{for } n \geq 2$$

and $C = \text{Euler's constant} = 0,57721 \dots$

However, for sake of completeness, we give here a direct proof of it. From (30) we can deduce that

$$(53) \quad \frac{g'_\rho(\mu)}{g_\rho(\mu)} = \sum_{j=1}^{\infty} \frac{c(j)^2}{1 + \mu c(j)^2} = \sum_{j=1}^{\infty} \sum_{n=0}^{\infty} (-1)^n c(j)^{2n+2} \mu^n = \sum_{n=0}^{\infty} (-1)^n \sigma_{n+1} \mu^n$$

provided that

$$(54) \quad |\mu c(j)^2| < 1, \quad j = 1, 2, \dots \quad \text{i. e.} \quad |\mu| < \frac{1}{c(1)^2} = \frac{1}{A^2}$$

if we define

$$(55) \quad \sigma_p = \sum_{j=1}^{\infty} c(j)^{2p} = \sum_{j=1}^{\infty} A^{2p} \frac{1}{j^{p/\rho}} = A^{2p} \zeta(p/\rho)$$

But, on the other hand,

$$(56) \quad g_\rho(\mu) = \sum_{n=0}^{\infty} b_n \mu^n, \quad g'_\rho(\mu) = \sum_{n=0}^{\infty} (n+1) b_{n+1} \mu^n$$

so that formula (53) gives rise to the relation

$$(57) \quad \sum_{n=0}^{\infty} (n+1) b_{n+1} \mu^n = \left\{ \sum_{m=0}^{\infty} b_m \mu^m \right\} \left\{ \sum_{q=0}^{\infty} (-1)^q \sigma_{q+1} \mu^q \right\}$$

from which we deduce that

$$(58) \quad b_{n+1} = \sum_{m+q=n}^{\infty} (-1)^q b_m \sigma_{q+1}; \quad b_0 = 1$$

Solving this system of equations, we get the following formula, which is quite familiar in the theory of integral equations (see for instance [11])

$$(59) \quad b_n = \frac{1}{n!} \begin{vmatrix} \sigma_1 & 1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ \sigma_2 & \sigma_1 & 2 & 0 & \cdot & \cdot & \cdot & 0 \\ \sigma_3 & \sigma_2 & \sigma_1 & 3 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & n-1 \\ \sigma_n & \sigma_{n-1} & \sigma_{n-2} & \cdot & \cdot & \cdot & \cdot & \sigma_1 \end{vmatrix}$$

For instance

$$(60) \quad \begin{cases} b_1 = A^2 \zeta\left(\frac{1}{\rho}\right) \\ b_2 = \frac{A^2}{2} \left[\zeta\left(\frac{1}{\rho}\right)^2 - \zeta\left(\frac{2}{\rho}\right) \right] \\ \text{etc.} \end{cases}$$

IV. AN EXAMPLE

As an illustration of the precedings results, let us now study the well known case where

$$(61) \quad c(k) = \frac{1}{k}, \quad A = 1, \quad m = 1, \quad \tau(L) = \frac{L}{2}, \quad \tau = \pi, \quad \rho = \frac{1}{2}$$

We have then :

$$(62) \quad \begin{cases} f_{1/k}^L(\lambda) = \prod_{j=1}^{\infty} \left\{ 1 + \lambda \left(\frac{L}{2\pi} \right)^2 \frac{1}{j^2} \right\} = \frac{\sin \frac{iL}{2} \sqrt{\lambda}}{\frac{iL}{2} \sqrt{\lambda}} \\ g_{1/2}(\mu) = \prod_{j=1}^{\infty} \left\{ 1 + \frac{\mu}{j^2} \right\} = \frac{\sin i\pi \sqrt{\lambda}}{i\pi \sqrt{\lambda}} \end{cases}$$

and consequently,

$$(63) \quad a_n^L = \left(\frac{L}{2\pi}\right)^{2n} \frac{\pi^{2n}}{(2n+1)!}; \quad b_n = \frac{\pi^{2n}}{(2n+1)!}$$

We can then immediately see that the ratio (31) is independent of L and that

$$(64) \quad \chi\left(n; \frac{1}{2}\right) = \frac{1}{2n+1}; \quad \lim_{n \rightarrow \infty} \frac{\chi\left(n+1; \frac{1}{2}\right)}{\chi\left(n; \frac{1}{2}\right)} = 1$$

Moreover one get directly that

$$(65) \quad \lim_{\substack{n \rightarrow \infty \\ L \rightarrow \infty \atop \frac{2n}{L} = d}} \frac{a_{n+1}^L}{a_n^L} = \left(\frac{1}{2d}\right)^2 \quad \text{and} \quad \lim_{n \rightarrow \infty} n^2 \frac{b_{n+1}}{b_n} = \frac{\pi^2}{4}$$

in agreement with formulas (47) and (48).

In the same way, we can compute the limits

$$(66) \quad \lim_{\substack{n \rightarrow \infty \\ L \rightarrow \infty \atop \frac{2n}{L} = d}} \frac{1}{L} \text{Log } a_n^L = d(1 - \text{Log } 2d)$$

$$\lim_{n \rightarrow \infty} \frac{1}{2\pi} \text{Log } n^2 b_n^{\frac{1}{n}} = \frac{1}{\pi} \left(1 - \text{Log } \frac{2}{\pi}\right)$$

These results are consistent with formulas (45) and (46).

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