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Section A :
Physique théorique.

Canonical dynamics of relativistic charged particles

by

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1. INTRODUCTION

The object of this paper is to give a formulation of relativistic dynamics of charged particles in terms of the geometry of symplectic manifolds.

Symplectic formulation of Hamiltonian dynamics is well described in references [*1*], [*2*] and [*3*]. Hamiltonian dynamics requires the existence of an absolute Newtonian time and therefore does not apply to relativistic particles. For relativistic particles it is necessary to consider the extended phase space including time and energy along with position and momentum. Dynamics is then described by a submanifold of the extended phase space.

A symplectic manifold together with a submanifold is called a canonical system. This concept is introduced in Section 2. In Section 3 Hamiltonian dynamics is formulated in terms of canonical systems. The application of canonical systems to relativistic dynamics of charged particles is given in Section 4. A different version of this dynamics based on Kaluza theory is discussed in Section 5. A summary of basic definitions and notation is given in the appendix.

2. CANONICAL SYSTEMS

2.1. Definition. — A canonical system is a triple (P, M, ω), where (P, ω) is a symplectic manifold and M is a submanifold of P.

(P, ω) is called the extended phase space and M is called the constraint submanifold of (P, M, ω).

2.2. DEFINITION. — A diffeomorphism $\gamma : \mathrm{P} \to \mathrm{P}'$ such that $\mathrm{M}' = \gamma(\mathrm{M})$ and $\omega = \omega' \cdot \overset{2}{\wedge} \mathrm{T}\gamma$ is called an isomorphism from $(\mathrm{P}, \mathrm{M}, \omega)$ to $(\mathrm{P}', \mathrm{M}', \omega')$.

2.3. The restriction $\mu = \omega \cdot \overset{2}{\wedge} \mathrm{T}\imath_{\mathrm{M}}$ of ω to M is a closed 2-form. If rank μ is constant, then the characteristic set of μ is an integrable distribution on M.

3. HAMILTONIAN CANONICAL SYSTEMS

3.1. DEFINITION (1). — A time dependent Hamiltonian H on a symplectic manifold (Y, η) is a function $\mathrm{H} : \underset{\sim}{\mathrm{R}} \times \mathrm{Y} \to \underset{\sim}{\mathrm{R}}$.

For each $t \in \underset{\sim}{\mathrm{R}}$, H_t denotes the function $\mathrm{H}_t : \mathrm{Y} \to \underset{\sim}{\mathrm{R}} : y \mapsto \mathrm{H}(t, y)$.

3.2. DEFINITION (2). — A time dependent vector field corresponding to a time dependent Hamiltonian H on (Y, η) is a mapping $h : \underset{\sim}{\mathrm{R}} \times \mathrm{Y} \to \mathrm{TY}$ such that, for each $t \in \underset{\sim}{\mathrm{R}}$, $h_t : \mathrm{Y} \to \mathrm{TY} : y \mapsto h(t, y)$ is a vector field on Y and satisfies $h_t \lrcorner\, \eta = -d\mathrm{H}_t$.

3.3. Given a time dependent Hamiltonian H on a symplectic manifold (Y, η) we construct a canonical system $(\mathrm{P}, \mathrm{M}, \omega)$ where $\mathrm{P} = \underset{\sim}{\mathrm{R}} \times \underset{\sim}{\mathrm{R}} \times \mathrm{Y}$, $\omega = \eta \circ \overset{2}{\wedge} \mathrm{T}pr_3 - dpr_1 \wedge dpr_2$, and $\mathrm{M} = \{ (\mathrm{E}, t, y) \mid \mathrm{E} = \mathrm{H}(t, y) \}$.

Let $\mu = \omega \cdot \overset{2}{\wedge} \mathrm{T}\imath_{\mathrm{M}}$ be the restriction of ω to M.

PROPOSITION. — Rank $\mu = \dim \mathrm{M} - 1$.

Proof. — For any canonical system, codim $\mathrm{M} \geqslant \dim \mathrm{M} - \text{rank}\ \mu \geqslant 0$. Since codim $\mathrm{M} = 1$, $\dim \mathrm{M} - \text{rank}\ \mu$ is 0 or 1. Further,

$$\dim \mathrm{M} = \dim \mathrm{P} - 1$$

is odd, and the rank of a 2-form is even. Hence rank $\mu = \dim \mathrm{M} - 1$.

Q. E. D.

COROLLARY. — The characteristic set N of μ is an integrable distribution on M. Integral manifolds of N are 1-dimensional.

3.4. The mapping $\chi : \underset{\sim}{\mathrm{R}} \times \mathrm{Y} \to \mathrm{M} : (t, y) \to (\mathrm{H}(t, y), t, y)$ is a diffeomorphism.

(1) See Def. 20.14 of [*1*].
(2) See Def. 20.16 of [*1*].

PROPOSITION. — A vector field v on M such that $dpr_2 \cdot T\iota_M \cdot v = 1$ is in N if and only if $h = Tpr_3 \cdot T\iota_M \cdot v \cdot \chi$ is the time dependent vector field on (Y, η) corresponding to H.

This proposition is equivalent to one given by Cartan which can be found in ref. [*1*], Prop. 20.20.

4. DYNAMICS OF CHARGED PARTICLES

4.1. The underlying structure of electrodynamics is a system (X, g_X, F), where X is a 4-dimensional manifold interpreted as space-time, g_X is an indefinite Riemannian metric on X, with signature $(+, -, -, -)$, interpreted as the gravitational field, and F is a closed 2-form on X interpreted as the electromagnetic field.

4.2. Dynamics of a charged particle with mass m and charge e can be formulated in terms of a canonical system (T^*X, M, ω), where

$$M = \{ p \in T^*X \mid g_X^*(p, p) = m^2 \} \quad \text{and} \quad \omega = d\theta_X - eF \cdot \overset{2}{\wedge} T\tau_X^*.$$

Elements of T^*X represent energy and momentum of the particle. Thus the constraint $g_X^*(p, p) = m^2$ reflects the relativistic relation between energy, momentum and mass.

Let $\mu = \omega \cdot \overset{2}{\wedge} T\iota_M$ be the restriction of ω to M.

PROPOSITION. — Rank $\mu = 6$.

Proof. — By the argument of Prop. 3.2 rank $\mu = \dim M - 1$, but $\dim M = \dim T^*X - 1 = 7$. Hence rank $\mu = 6$. Q. E. D.

COROLLARY. — The characteristic set N of μ is an integrable distribution on M. Integral manifolds of N are 1-dimensional.

4.3. Let w be the unique vector field on T^*X such that $w \lrcorner \omega = df$, where $f : T^*X \to \underset{\sim}{R} : p \mapsto (1/2m) g_X^*(p, p)$.

PROPOSITION. — There exists a unique vector field w' on M such that $T\iota_M \cdot w' = w \cdot \iota_M$. The vector field w' spans N.

Proof. — Since $M = f^{-1}(m/2)$ and $df \cdot w = 0$ there exists a unique vector field w' on M such that $T\iota_M \cdot w' = w \cdot \iota_M$. Moreover

$$w' \lrcorner \mu = (w \lrcorner \omega) \cdot T\iota_M = df \cdot T\iota_M = 0,$$

and so w' is in N. For every $p \in M$, $df_p \neq 0$ and $w'(p) \neq 0$. Hence w' spans N. Q. E. D.

COROLLARY. — Trajectories of w contained in M are integral manifolds of N.

4.4. Let $\pi : \mathrm{J} \to \mathrm{T^*X}$ be a curve and let $\lambda = \tau^*_{\mathrm{X}} \cdot \pi$ be its projection to X. Let $v(s)$ denote the tangent vector to π at $\pi(s)$ and $u(s)$, the tangent vector to λ at $\lambda(s)$. The horizontal [3] part of $v(s)$ is related to $u(s)$ by $\mathrm{T}\tau^*_{\mathrm{X}} \cdot hor \cdot v(s) = u(s)$. The vertical part of $v : \mathrm{J} \to \mathrm{T(T^*X)} : s \to v(s)$ determines uniquely a curve $\mathrm{D}\pi : \mathrm{J} \to \mathrm{T^*X}$, called the absolute derivative of π, such that, for each $s \in \mathrm{J}$, $ver(v(s)) \lrcorner\, d\theta_{\mathrm{X}} = \mathrm{D}\pi(s) \cdot \mathrm{T}\tau^*_{\mathrm{X}}$.

If π is a trajectory of w, then $v = w \cdot \pi$, and consequently π is a solution of $v(s) \lrcorner\, w = df_{\pi(s)}$, for each $s \in \mathrm{J}$. Composing this equation with *hor* and *ver*, respectively, results in an equivalent system of equations

$$(*) \qquad \mathrm{D}\pi(s) = eu(s) \lrcorner\, \mathrm{F},$$
$$(**) \qquad \pi(s) = mu(s) \lrcorner\, g_{\mathrm{X}}.$$

The curve π is in M if and only if

$$(***) \qquad g_{\mathrm{X}}(u(s), u(s)) = 1.$$

Equations (*), (**) and (***) are the familiar equations of motion of a charged particle in electromagnetic and gravitational fields.

4.5. Let F be exact and let A be any 1-form on X such that $d\mathrm{A} = \mathrm{F}$. The 1-form A is interpreted as the electromagnetic potential. The diffeomorphism $\gamma_{\mathrm{A}} : \mathrm{T^*X} \to \mathrm{T^*X} : p \mapsto p - e\mathrm{A}_x$, where $x = \tau^*_{\mathrm{X}}(p)$, is an isomorphism from $(\mathrm{T^*X}, \omega)$ to $(\mathrm{T^*X}, d\theta_{\mathrm{X}})$. The canonical system $(\mathrm{T^*X}, \gamma_{\mathrm{A}}(\mathrm{M}), d\theta_{\mathrm{X}})$, isomorphic to $(\mathrm{T^*X}, \mathrm{M}, \omega)$, gives an alternative canonical formulation of dynamics of charged particles. In this formulation the natural symplectic structure of T*X is used, however the elements of T*X no longer represent energy and momentum of the particle.

5. DYNAMICS OF PARTICLES IN KALUZA THEORY

5.1. The underlying structure of Kaluza theory [4] is a system $(\mathrm{Z}, g_{\mathrm{Z}}, \mathrm{G}, \mathrm{X}, \xi)$, where Z is a 5-dimensional manifold, g_{Z} is an indefinite Riemannian metric on Z with signature $(+, +, -, -, -)$ and $(\mathrm{Z}, \mathrm{G}, \mathrm{X}, \xi)$

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(3) The terms horizontal and vertical used here refer to the Riemannian connection in T*X.

(4) See Chapter XVII of [*4*].

is a principal fibre bundle. The structural group G is the additive group of real numbers. The metric g_Z is invariant under the action of the group G. Consequently, for every fundamental vector field u, $\mathscr{L}_u g_Z = 0$. In addition, the fundamental vector field k, corresponding to the real number 1 in the Lie algebra of G, satisfies $g_Z(k(z), k(z)) = 1$ for all $z \in Z$.

5.2. PROPOSITION. — There exists a unique connection in (Z, G, X, ξ) such that $\alpha = k \lrcorner g_Z$ is the connection form.

Proof. — Let u be the fundamental vector field on Z corresponding to a number a in the Lie algebra of G. Then $u = ak$, and, for each $z \in Z$, $\alpha(u(z)) = \alpha(ak(z)) = a\alpha(k(z)) = ag_Z(k(z), k(z)) = a$. Moreover,

$$\mathscr{L}_u \alpha = \mathscr{L}_u (k \lrcorner g_Z) = \mathscr{L}_u k \lrcorner g_Z + k \lrcorner \mathscr{L}_u g_Z = 0.$$

Hence there exists a unique connection in (Z, G, X, ξ) such that α is its connection form ([5]). Q. E. D.

We note that there exists a unique 2-form F on X such that $d\alpha = F \cdot \overset{2}{\wedge} T\xi$.

5.3. PROPOSITION. — There exists an indefinite Riemannian metric g_X in X, with signature $(+, -, -, -)$, such that, for each $z \in Z$ and every pair of horizontal vectors $\underset{\sim}{u}, \underset{\sim}{v} \in T_z Z$, $g_Z(\underset{\sim}{u}, \underset{\sim}{v}) = g_X(T\xi(\underset{\sim}{u}), T\xi(\underset{\sim}{v}))$.

Proof. — Let hor g_Z be defined by $\text{hor}\, g_Z(\underset{\sim}{u}, \underset{\sim}{v}) = g_Z(\underset{\sim}{u}, \underset{\sim}{v}) - \alpha(\underset{\sim}{u}) \cdot \alpha(\underset{\sim}{v})$ for each pair of vectors $\underset{\sim}{u}, \underset{\sim}{v}$ in the same fibre of TZ. For every vertical vector field w on Z, we have $w \lrcorner \text{hor}\, g_Z = 0$ and $\mathscr{L}_w \text{hor}\, g_Z = 0$. Since the fibres of ξ are connected, there exists a unique metric g_X in X such that $\text{hor}\, g_Z(\underset{\sim}{u}, \underset{\sim}{v}) = g_X(T\xi(\underset{\sim}{u}), T\xi(\underset{\sim}{v}))$. If $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are horizontal, then

$$g_Z(\underset{\sim}{u}, \underset{\sim}{v}) = g_X(T\xi(\underset{\sim}{u}), T\xi(\underset{\sim}{v})).$$

The metric g_X has signature $(+, -, -, -)$ since hor g_Z has signature $(0, +, -, -, -)$. Q. E. D.

5.4. The physical meaning of Kaluza theory follows from interpreting X as the space-time, the Riemannian metric g_X as the gravitational field, and the 2-form F (introduced in 5.2) as the electromagnetic field.

5.5. Dynamics of a charged particle in Kaluza theory can be formulated in terms of a canonical system $(T^*Z, \tilde{M}, d\theta_Z)$, where

$$\tilde{M} = \{ q \in T^*Z \mid g_Z^*(\text{hor}\, q, \text{hor}\, q) = m^2,\ q(k \cdot \tau_Z^*(q)) = e \},$$

and $\text{hor}\, q = q \cdot \text{hor}$.

([5]) See Chapter II, Prop. 1.1 of [5].

The horizontal part of an element $q \in T^*Z$ represents energy and momentum of the particle. Thus the constraint $g^*_Z(\text{hor}\, q, \text{hor}\, q) = m^2$ reflects the relativistic relation between energy and momentum, and mass. The remaining constraint $q(k \cdot \tau^*_Z(q)) = e$ relates the vertical part of q to the charge of the particle.

The action of G in Z can be extended to T*Z in a natural manner. For each $a \in G$, $\phi_a : Z \to Z : z \mapsto za$ is a diffeomorphism. The induced diffeomorphism κ_a: $T^*Z \to T^*Z$ is given by $\kappa_a(q) = q \cdot T\phi_a$, for each $q \in T^*Z$. It satisfies the identity $\tau^*_Z \cdot \kappa_a = \phi_a^{-1} \cdot \tau^*_Z$.

The form θ_Z is invariant under the action of G in T*Z. Since g^*_Z, α and and k are G invariant the submanifold M is invariant under the action of G in T*Z.

5.6. Let $\tilde{\mu} = d\theta_Z \cdot \overset{2}{\wedge} T\iota_{\tilde{M}}$ be the restriction of $d\theta_Z$ to $\tilde{M}$.

PROPOSITION. — Rank $\tilde{\mu} = 6$.

Proof. — Codim $\tilde{M} \geqslant \dim \tilde{M} - \text{rank}\, \tilde{\mu} \geqslant 0$. Since codim $\tilde{M} = 2$, $\dim \tilde{M} = 8$, and the rank of a 2-form is even, rank $\tilde{\mu}$ is 6 or 8.

Let l be a vector field on T*Z such that, for each $q \in T^*Z$, $l(q)$ is the tangent vector to the orbit of G through q. Then $\mathscr{L}_l \theta_Z = 0$, $T\tau^*_Z \cdot l = k \cdot \tau^*_Z$, and there exists a vector field n on M such that $T\iota_{\tilde{M}} \cdot n = l \cdot \iota_{\tilde{M}}$. Hence

$$\begin{aligned} n \lrcorner\, \tilde{\mu} &= n \lrcorner\, (d\theta_Z \cdot \overset{2}{\wedge} T\iota_{\tilde{M}}) \\ &= (l \lrcorner\, d\theta_Z) \cdot T\iota_{\tilde{M}} \\ &= -\, d(\theta_Z \cdot l) \cdot T\iota_{\tilde{M}} \\ &= -\, d(\theta_Z \cdot l \cdot \iota_{\tilde{M}}) \end{aligned}$$

But $\theta_Z \cdot l(q) = q \cdot T\tau^*_Z(l(q)) = q(k \cdot \tau^*_Z(q))$, and so $\theta_Z \cdot l \cdot \iota_{\tilde{M}} = e$. Therefore $n \lrcorner\, \tilde{\mu} = 0$.

We have thus shown existence of a non-singular vector field n in the characteristic set N of $\tilde{\mu}$, and so rank $\tilde{\mu} < \dim \tilde{M} = 8$. Hence rank $\tilde{\mu} = 6$. Q. E. D.

COROLLARY. — The characteristic set $\tilde{N}$ of $\tilde{\mu}$ is an integrable distribution on $\tilde{M}$. Integral manifolds of $\tilde{N}$ are 2-dimensional.

5.7. There exists a close relation between the canonical systems (T^*X, M, ω) and $(T^*Z, \tilde{M}, d\theta_Z)$ introduced in 4.2 and 5.5, respectively.

Let ζ: $T^*Z \to T^*X$ be a mapping such that, for each $q \in T^*Z$, $\zeta(q)$ is the unique covector in T*X satisfying the equality $\text{hor}\, q = \zeta(q) \cdot T\xi$.

N is the characteristic distribution of $\mu = \omega \cdot \overset{2}{\wedge} \mathrm{T}\iota_{\mathrm{M}}$, and $\tilde{\mathrm{N}}$ is the characteristic distribution of $\tilde{\mu} = d\theta_{\mathrm{Z}} \cdot \overset{2}{\wedge} \mathrm{T}\iota_{\tilde{\mathrm{M}}}$.

PROPOSITION.

(*i*) There exists a submersion ψ from $\tilde{\mathrm{M}}$ onto M such that

$$\iota_{\mathrm{M}} \cdot \psi = \zeta \cdot \iota_{\tilde{\mathrm{M}}} \quad \text{and} \quad \tilde{\mu} = \mu \cdot \overset{2}{\wedge} \mathrm{T}\psi.$$

(*ii*) A submanifold W of M is an integral manifold of N if and only if $\tilde{\mathrm{W}} = \psi^{-1}(\mathrm{W})$ is an integral manifold of $\tilde{\mathrm{N}}$.

Proof.

(*i*) For each $q \in \tilde{\mathrm{M}}$, $g^*_{\mathrm{Z}}(\mathrm{hor}\, q, \mathrm{hor}\, q) = m^2$ and so $g^*_{\mathrm{X}}(\zeta(q), \zeta(q)) = m^2$. Hence the restriction of ζ to $\tilde{\mathrm{M}}$ has its range in M. Since M is a regular submanifold of T*X there exists a differentiable mapping $\psi : \tilde{\mathrm{M}} \to \mathrm{M}$ such that $\iota_{\mathrm{M}} \cdot \psi = \zeta \cdot \iota_{\tilde{\mathrm{M}}}$.

Let $\mathrm{Q} : \mathrm{T}^*\mathrm{Z} \to \underset{\sim}{\mathrm{R}}$ be given by $\mathrm{Q}(q) = q(k \cdot \tau^*_{\mathrm{Z}}(q))$, then $\mathrm{ver}\, q = \mathrm{Q}(q)\alpha_z$, where $z = \tau^*_{\mathrm{Z}}(q)$. Let p be an arbitrary element of M and let q be an arbitrary element of $\zeta^{-1}(p)$. Then $q' = q - (\mathrm{Q}(q) - e)\alpha_z$, where $z = \tau^*_{\mathrm{Z}}(q)$, is also in $\zeta^{-1}(p)$, since $\mathrm{hor}\,(q') = \mathrm{hor}\,(q)$, and so

$$g^*_{\mathrm{Z}}(\mathrm{hor}\,(q'), \mathrm{hor}\,(q')) = g^*_{\mathrm{X}}(p, p) = m^2.$$

Moreover, $\mathrm{Q}(q') = \mathrm{Q}(q) - (\mathrm{Q}(q) - e) = e$. Therefore $q' \in \tilde{\mathrm{M}}$, and ψ is onto M.

For each $q \in \tilde{\mathrm{M}}$, $\mathrm{T}_q\psi(\underset{\sim}{u}) = 0$ if and only if $\mathrm{T}\iota_{\tilde{\mathrm{M}}}(\underset{\sim}{u})$ is proportional to $l(q)$, where l is the vector field on T*Z introduced in the proof of Prop. 5.5. Hence rank $\mathrm{T}_q\psi = \dim \tilde{\mathrm{M}} - 1 = 7 = \dim \mathrm{M}$, and so ψ is a submersion.

From the definition of ζ it follows that $\tau^*_{\mathrm{X}} \cdot \zeta = \xi \cdot \tau^*_{\mathrm{Z}}$. For each $\underset{\sim}{u} \in \mathrm{T}_q\mathrm{T}^*\mathrm{Z}$,

$$\begin{aligned}
\theta_{\mathrm{Z}}(\underset{\sim}{u}) &= q(\mathrm{T}\tau^*_{\mathrm{Z}}(\underset{\sim}{u})) \\
&= (\mathrm{hor}\,(q) + \mathrm{ver}\,(q))(\mathrm{T}\tau^*_{\mathrm{Z}}(\underset{\sim}{u})) \\
&= \zeta(q)(\mathrm{T}\xi \cdot \mathrm{T}\tau^*_{\mathrm{Z}}(\underset{\sim}{u})) + \mathrm{Q}(q)\alpha(\mathrm{T}\tau^*_{\mathrm{Z}}(\underset{\sim}{u})) \\
&= \zeta(q)(\mathrm{T}\tau^*_{\mathrm{X}} \cdot \mathrm{T}\zeta(\underset{\sim}{u})) + \mathrm{Q}(q)\alpha(\mathrm{T}\tau^*_{\mathrm{Z}}(\underset{\sim}{u})) \\
&= \theta_{\mathrm{X}}(\mathrm{T}\zeta(\underset{\sim}{u})) + \mathrm{Q}(q)\alpha(\mathrm{T}\tau^*_{\mathrm{Z}}(\underset{\sim}{u})).
\end{aligned}$$

Hence

$$\begin{aligned}
d\theta_{\mathrm{Z}} &= d\theta_{\mathrm{X}} \cdot \overset{2}{\wedge} \mathrm{T}\zeta + \mathrm{Q}d\alpha \cdot \overset{2}{\wedge} \mathrm{T}\tau^*_{\mathrm{Z}} + d\mathrm{Q} \wedge (\alpha \cdot \mathrm{T}\tau^*_{\mathrm{Z}}) \\
&= d\theta_{\mathrm{X}} \cdot \overset{2}{\wedge} \mathrm{T}\zeta + \mathrm{QF} \cdot \overset{2}{\wedge} \mathrm{T}\xi \cdot \overset{2}{\wedge} \mathrm{T}\tau^*_{\mathrm{Z}} + d\mathrm{Q} \wedge (\alpha \cdot \mathrm{T}\tau^*_{\mathrm{Z}}) \\
&= d\theta_{\mathrm{X}} \cdot \overset{2}{\wedge} \mathrm{T}\zeta + \mathrm{QF} \cdot \overset{2}{\wedge} \mathrm{T}\tau^*_{\mathrm{X}} \cdot \overset{2}{\wedge} \mathrm{T}\zeta + d\mathrm{Q} \wedge (\alpha \cdot \mathrm{T}\tau^*_{\mathrm{Z}}) \\
&= (d\theta_{\mathrm{X}} + \mathrm{QF} \cdot \overset{2}{\wedge} \mathrm{T}^*_{\mathrm{X}}) \cdot \overset{2}{\wedge} \mathrm{T}\zeta + d\mathrm{Q} \wedge (\alpha \cdot \mathrm{T}\tau^*_{\mathrm{Z}}).
\end{aligned}$$

But $Q \cdot \iota_{\tilde{M}}$ is a constant function, with value e, therefore

$$\begin{aligned}\tilde{\mu} &= d\theta_Z \cdot \overset{2}{\wedge} T\iota_M \\ &= (d\theta_X + eF \cdot \overset{2}{\wedge} T\tau^*_X) \cdot \overset{2}{\wedge} T\zeta \cdot \overset{2}{\wedge} T\iota_{\tilde{M}} \\ &= (d\theta_X + eF \cdot \overset{2}{\wedge} T\tau^*_X) \cdot \overset{2}{\wedge} T\iota_M \cdot \overset{2}{\wedge} T\psi \\ &= \omega \cdot \overset{2}{\wedge} T\iota_M \cdot \overset{2}{\wedge} T\psi \\ &= \mu \cdot \overset{2}{\wedge} T\psi.\end{aligned}$$

(*ii*) Let W be an integral manifold of N. Since ψ is a submersion $\tilde{W} = \psi^{-1}(W)$ is a submanifold of $\tilde{M}$. For each $\underset{\sim}{u} \in T\tilde{W}$,

$$\underset{\sim}{u} \lrcorner \tilde{\mu} = (T\psi(\underset{\sim}{u}) \lrcorner \mu) \cdot T\psi = 0,$$

and so $T\tilde{W} \subset \tilde{N}$. Moreover $\dim \tilde{W} - 2 = \dim \tilde{N}$, and $\tilde{W}$ is connected since each $\psi^{-1}(p)$ is connected. Hence $\tilde{W}$ is an integral manifold of $\tilde{N}$.

Conversely, let $\tilde{W}$ be an integral manifold of $\tilde{N}$. Let $\underset{\sim}{v} \in TW$ be arbitrary, and let $\underset{\sim}{u} \in T\tilde{W}$ be such that $\underset{\sim}{v} = T\psi(\underset{\sim}{u})$. Then $(\underset{\sim}{v} \lrcorner \mu) \cdot T\psi = \underset{\sim}{u} \lrcorner \tilde{\mu} = 0$, and since $T\psi$ is surjective, $\underset{\sim}{v} \lrcorner \mu = 0$. Hence $TW \subset N$. Moreover, W is connected and $\dim W = \dim N$. Therefore W is an integral manifold of N. Q. E. D.

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APPENDIX

A.1. SUMMARY OF DEFINITIONS

Manifolds considered in this paper are assumed to be c^{∞}, finite dimensional, Hausdorff and paracompact, unless otherwise stated.

A vector field on a manifold M is a section of the tangent bundle TM, and a differential k-form on M is a mapping from the bundle $\overset{k}{\wedge}$ TM to $\underset{\sim}{\mathrm{R}}$, linear on fibres.

Left interior product $v \lrcorner\, \omega$ of a k-form ω by a vector field v is the $(k-1)$-form such that for each $m \in \mathrm{M}$ and each $\underset{\sim}{w} \in \overset{k-1}{\wedge} \mathrm{T}_m\mathrm{M}$ $(v \lrcorner\, \omega)(\underset{\sim}{w}) = \omega(v(m) \wedge \underset{\sim}{w})$.

Let g be a metric on M and v a vector field on M. We denote by $v \lrcorner\, g$ the 1-form on M such that, for each $m \in \mathrm{M}$ and each $\underset{\sim}{u} \in \mathrm{T}_m\mathrm{M}$, $(v \lrcorner\, g)(\underset{\sim}{u}) = g(v(m), \underset{\sim}{u})$.

The characteristic space of a k-form ω at $m \in \mathrm{M}$ is the subspace $\mathrm{N}_m = \{ \underset{\sim}{u} \in \mathrm{T}_m\mathrm{M} \mid \underset{\sim}{u} \lrcorner\, \omega = 0 \}$ of $\mathrm{T}_m\mathrm{M}$. Rank ω at m is the codimension of N_m. The set $\mathrm{N} = \bigcup_{m \in \mathrm{M}} \mathrm{N}_m$ is called the characteristic set of ω. If rank of ω is constant on M then the characteristic set of ω is a differentiable distribution (6) on M, and it is called the characteristic distribution of ω. The characteristic distribution N of ω is integrable if ω is closed.

A manifold M′ is said to be a submanifold of a manifold M if it is a subset of M and if the natural injection $\mathrm{M}' \to \mathrm{M}$ is an immersion. The topology of a submanifold M′ of a manifold M, induced by its differential structure, need not be the same as the topology of M′ as a subset of M.

A principal fibre bundle (7) is a quadruple $(\mathrm{Z}, \mathrm{G}, \mathrm{X}, \xi)$, where Z and X are manifolds, G is a Lie group acting on Z on the right, and ξ is a differentiable mapping from Z to X. The following conditions are satisfied, for each $x \in \mathrm{X}$ there exists a neighbourhood U of x and a diffeomorphism $\sigma \colon \xi^{-1}(\mathrm{U}) \to \mathrm{U} \times \mathrm{G}$ such that $pr_1 \cdot \sigma(z) = \xi(z)$ and

$$pr_2 \cdot \sigma(za) = (pr_2 \cdot \sigma(z))a,$$

for each $z \in \xi^{-1}(\mathrm{U})$ and each $a \in \mathrm{G}$.

The action of G on Z induces a homomorphism from the Lie algebra $\mathfrak{G}$ of the group G to the Lie algebra of vector fields on Z. The image $\mathfrak{A}^*$ of an element $\mathfrak{A} \in \mathfrak{G}$ is called the fundamental vector field corresponding to $\mathfrak{A}$.

The vertical distribution VTZ on Z is defined by $\mathrm{VTZ} = \{ \underset{\sim}{v} \in \mathrm{TZ} \mid \mathrm{T}\xi(\underset{\sim}{v}) = 0 \}$. A vector field v in VTZ is called vertical. The fundamental vector fields on Z corresponding to a basis in $\mathfrak{G}$ span VTZ.

Assume G is a connected Lie group. If u is a vector field on Z such that, for every vertical vector field v, $[u, v]$ is vertical, then there exists a unique vector field u' on X satisfying $\mathrm{T}\xi \cdot u = u' \cdot \xi$. If ω is a differential k-form on Z such that, for every vertical vector field v, $v \lrcorner\, \omega = 0$ and $\mathscr{L}_v \omega = 0$, then there exists a unique k-form ω' on X satisfying $\omega = \omega' \cdot \overset{k}{\wedge} \mathrm{T}\xi$.

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(6) See Prop. 20.7 of [*1*].

(7) See Sec. 6.2.1 of [*6*].

If g is a degenerate metric on Z such that, for every vertical vector field v, $v \lrcorner g = 0$ and $\mathscr{L}_v g = 0$, then there exists a unique metric g' in X satisfying $g(\underset{\sim}{u}, \underset{\sim}{v}) = g'(\mathrm{T}\xi(\underset{\sim}{u}), \mathrm{T}\xi(\underset{\sim}{v}))$.

A connection in a principal fibre bundle (Z, G, X, ξ) is a differentiable distribution Q on Z, invariant under the action of G in Z, and such that, for each $z \in \mathrm{Z}$, Q_z is a complement of $\mathrm{VT}_z\mathrm{Z}$. Q is called the horizontal distribution.

Given a connection Q in (Z, G, X, ξ) there are two mappings hor: TZ $\to$ TZ and ver : TZ $\to$ TZ such that, for each $\underset{\sim}{v} \in \mathrm{TZ}$, $\mathrm{hor}(\underset{\sim}{v}) \in \mathrm{Q}$, $\mathrm{ver}(\underset{\sim}{v}) \in \mathrm{VTZ}$, and $\underset{\sim}{v} = \mathrm{hor}(\underset{\sim}{v}) + \mathrm{ver}(\underset{\sim}{v})$. The unique 1-form α on Z, with values in $\mathfrak{G}$, such that $\alpha \cdot \mathrm{hor} = 0$ and, for each $\mathfrak{A} \in \mathfrak{G}$ and each $z \in \mathrm{Z}$, $\alpha(\mathfrak{A}^*(z)) = \mathfrak{A}$, is called the connection form (9).

A symplectic manifold is a pair (P, ω) where P is a manifold and ω is a closed non-singular 2-form on P.

Let τ_{X}^*: T*X $\to$ X be the cotangent bundle of a manifold X. The canonical 1-form θ_{X} on T*X is defined by $\theta_{\mathrm{X}}(\underset{\sim}{u}) = (\tau_{\mathrm{T^*X}}(\underset{\sim}{u})(\mathrm{T}\tau_{\mathrm{X}}^*(\underset{\sim}{u}))$ for every $\underset{\sim}{u} \in \mathrm{T(T^*X)}$. The pair (T*X, $d\theta_{\mathrm{X}}$) is a symplectic manifold (10).

A.2. SUMMARY OF SYMBOLS

TM	tangent bundle space of M.
$\mathrm{T}_m\mathrm{M}$	tangent space of M at m.
τ_{M}	tangent bundle projection.
T*M	cotangent bundle space of M.
$\mathrm{T}_m^*\mathrm{M}$	cotangent space of M at m.
τ_{M}^*	cotangent bundle projection.
$\mathrm{T}\gamma$	derived mapping of a mapping γ.
ι_{M}	imbedding of a submanifold M.
g_{M}	Riemannian metric on M (indefinite).
g_{M}^*	scalar product for covectors induced by g_{M}.
θ_{M}	canonical 1-form on T*M.
d	exterior derivative.
$\mathscr{L}$	Lie derivative.
$\lrcorner$	left interior product.
$\wedge$	exterior product.
$\cdot$	composition of mappings,

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(8) This and subsequent results on projectability of geometric objets follow from Prop. 7.5.5 of [7].

(9) See Chapter II, Sec. 1, of [5].

(10) See Theorem 14.14 of [1].

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