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Section A :
Physique théorique.

On a differential equation approach to quantum field theory: causality property for the solutions of Thirring's model

by

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ABSTRACT. — A causality property is established for the solutions of Thirring's model, where the fields are operator valued functions of the space time variables. This property is just that one expects for the convolution of a Wightman field with a compactely supported C^∞ function.

RÉSUMÉ. — On établit une propriété de causalité pour les solutions d'un modèle de Thirring où les champs sont des fonctions des variables d'espace-temps à valeurs opérateurs. Cette propriété est bien celle qui convient pour un champ de Wightman convolué avec une fonction de classe C^∞ à support compact.

1. — INTRODUCTION

The opportunity of connecting the theory of non linear differential equations to quantum field theory has been repeatedly stressed [*1*], and recently some results in this direction have been obtained, by applying non linear semigroup theory to the analysis of some field theoretical models [2].

According to this treatment, the physical model was defined by a Cauchy problem of the first order in the time variable t for a quantity $\Phi(t, x)$ defined for any positive time t and any space point x: more specifically, $\Phi(t, x)$ was supposed to be, for any $t \geqslant 0$ and $x \in \mathbb{R}^s$, a bounded operator acting in the Hilbert space $\mathcal{H}$ of the physical states. No requirement was made about the commutators or anticommutators of Φ taken at different space-time points. Within this framework, the existence and uniqueness of $\Phi(t, x)$ was proved for the Thirring and the Federbush models.

To suppose the $\Phi(t, x)$'s to be operator valued *functions* is of course a very severe restriction, as it forbids Φ to be a field in Wightman's sense [3] [4]. However, if we take a spin 1/2 Wightman field A, and a function $f \in \mathfrak{D}(\mathbb{R} \times \mathbb{R}^s)$ the quantity

$$\mathrm{B}(t, x) = \int dt' dx' f(t' - t, x' - x) \mathrm{A}(t', x')$$

defines an operator valued function of t, x, which is of course not covariant nor local, still shares many properties with the above introduced Φ. Thus we may hope to approximate, in a sense to be furtherly specified, a « true » Wightman field by a sequence of functions like Φ. As a first step in this direction, one should try to establish on a rigorous ground the identification between B and Φ. For doing that, further conditions are needed for Φ: namely, if d is the size of the support of f in $\mathbb{R} \times \mathbb{R}^s$, the local (anti-)commutativity of A entails:

$$[\mathrm{B}(t, x), \mathrm{B}^{\#}(t', x')]_+ = 0 \qquad (\mathrm{B}^{\#} = \mathrm{B}, \mathrm{B}^*)$$

if

$$(t - t')^2 - (x - x')^2 < - d^2 :$$

therefore the same must be valid for Φ.

The aim of the present paper is to show that, in the case of the Thirring model, the condition

$$[\Phi(t, x), \Phi^{\#}(t', x')]_+ = 0$$

if

$$(t - t')^2 - (x - x')^2 < - d^2$$

can actually be imposed, and it is compatible with the equation $\Phi(t, x)$ obeys. More specifically, we shall show that if the initial data $\chi(x)$ satisfy

$$[\chi(x), \chi^*(x')]_+ = 0 \qquad \text{if} \qquad |x - x'| > d,$$

then

$$[\Phi(t, x), \Phi^{\#}(t', x')]_+ = 0$$

if

$$| x - x' + t - t' | > d \qquad \text{and if} \qquad | x - x' - (t - t') > d,$$

and, in particular, if

$$(t - t')^2 - (x - x')^2 < - d^2.$$

2. — FORMULATION OF THE PROBLEM

If the fields are regarded as functions of the time t ($t \geqslant 0$) and of the space variable $\vec{x} \equiv (x_1 \dots x_s)$, whose values are bounded operators in the Hilbert space of physical states, they are realized as mappings from the space $M_+ = \mathbb{R}_+ \times \mathbb{R}^s$ to a C^* algebra X. We furtherly make a definite assumption about the dependence on x: we namely suppose that for any t, the field is a continuous function of x, vanishing at infinity (our framework should allow, however, even a more general dependence on x). Thus a j-component field may be viewed as a function from $\mathbb{R}$ to the space $Y = \bigoplus_{i=1}^{j} Y_i$, where each Y_i is the space $C^0(\mathbb{R}^s; X)$:

$$\Phi : t \mapsto \Phi(t) = \begin{pmatrix} \varphi_1(t) \\ \vdots \\ \varphi_j(t) \end{pmatrix}; \quad \varphi_i(t) \in Y_i = C^0(\mathbb{R}^s; X), \quad i = 1 \dots j$$

$$(\varphi_i(t))(x) \equiv \varphi_i(t, x) \in X.$$

By introducing in Y_i and Y the norms

$$| \varphi_i(t) |_{Y_i} = \sup_{\vec{x} \in \mathbb{R}^s} | \varphi_i(t, x) |_X$$

($| \cdot |_X$ being the norm of X),

$$| \Phi(t) |_Y = \sum_{i=1}^{j} | \varphi_i(t) |_{Y_i}$$

each Y_i is turned into a C^*-algebra for the product

$$\varphi_i(t)\varphi_i'(t) : (\varphi_i(t)\varphi_i'(t))(x) = \varphi_i(t, x)\varphi_i'(t, x),$$

and Y into a Banach space.

The equations we shall consider are of the form

$$\begin{cases} \dfrac{d}{dt}\Phi(t) + L\Phi(t) + T\Phi(t) = 0 \\ \Phi(0) = \chi \end{cases} \tag{2.1}$$

where $\Phi(t)$ $(t \geqslant 0)$, χ, are elements of Y, L is a linear closed operator in Y generating a semigroup $S_0(t)$; and T is a non linear operator defined on the whole of Y. A more general equation is

$$\Phi(t) = S_0(t)\chi + \int_0^t S_0(t-s)T\Phi(s)ds \tag{2.2}$$

which is equivalent to (2.1) if the mapping $t \mapsto \Phi(t)$ is differentiable. Solutions of (2.2) which do not need to be differentiable are called *mild solutions* of (2.1) [5].

The equations of the Thirring model [6]

$$i\partial^\mu \gamma_\mu \Phi = g\Phi^\dagger \gamma^\mu \Phi \gamma_\mu \Phi$$

may be written (without any assumption about C. A. R.) as

$$\left\{\begin{aligned} &\partial_t\varphi_1(t, x) + \partial_x\varphi_1(t, x) + 2ig\varphi_2^*(t, x)\varphi_2(t, x)\varphi_1(t, x) = 0 \\ &\partial_t\varphi_2(t, x) - \partial_x\varphi_2(t, x) + 2ig\varphi_1^*(t, x)\varphi_1(t, x)\varphi_2(t, x) = 0 \\ &\varphi_1(0, x) = \chi_1(x) \\ &\varphi_2(0, x) = \chi_2(x) \end{aligned}\right. \tag{2.3}$$

and therefore are of the form (2.1) with $j = 2$, $s = 1$:

$$\left\{\begin{aligned} &Y = C^0(\mathbb{R}; X) \oplus C^0(\mathbb{R}; X); \qquad \Phi(t) = \begin{pmatrix}\varphi_1(t)\\ \varphi_2(t)\end{pmatrix} \\ &(\varphi_i(t))(x) \equiv \varphi_i(t, x) \in X,\ i = 1, 2; \qquad L = \begin{pmatrix}\partial_x & 0 \\ 0 & -\partial_x\end{pmatrix} \\ &\text{and} \\ &\qquad T\Phi(t) = 2ig\begin{pmatrix}\varphi_2^*(t)\varphi_2(t)\varphi_1(t)\\ \varphi_1^*(t)\varphi_1(t)\varphi_2(t)\end{pmatrix}; \qquad g = \bar{g}. \end{aligned}\right. \tag{2.4}$$

In Ref. [2] it was shown that: *i)* L generates a semigroup of (linear) contractions, *ii)* T is locally Lipshitz continuous, *i. e.*,

$$|T\Psi - T\Psi'|_Y \leqslant K(\tau)\,|\Psi - \Psi'|_Y$$

for $\Psi, \Psi' \in Y$, $|\Psi|_Y, |\Psi'|_Y \leqslant \tau$ (*) and, *iii)* T satisfies the « condition M »: $\forall \tau > 0$, $\exists \lambda > 0$ such that $|T\Psi - \zeta\Psi|_Y > |\Psi|_Y$, $\forall \Psi \in Y$, $|\Psi|_Y \leqslant \tau$ and $\zeta \geqslant \lambda$. As a consequence [7], a unique mild solution of (2.3) (*i. e.* (2.1) with (2.4)), exists and it is generated by a non linear semigroup $S(t)$:

$$\Phi(t) = S(t)\chi \tag{2.5}$$

(*) The constant $K(r)$ is called the r-Lipschitz norm of T, and in our case

$$K(r) \leqslant 3r^2 \cdot 2\,|g| = 6\,|g|\,r^2.$$

where $S(t)$ satisfies

$$| S(t)\Psi |_Y \leqslant | \Psi |_Y, \qquad \forall \Psi \in Y \tag{2.6}$$

$$| S(t)\Psi - S(t)\Psi' |_Y \leqslant \exp(tK(\tau)) | \Psi - \Psi' |_Y \tag{2.7}$$

for any Ψ, $\Psi' \in Y$, $| \Psi |_Y \, | \Psi' |_Y \leqslant \tau$.

We want now to impose a further condition on our equation (2.3), namely we require that the initial data anticommute beyond a given distance d:

$$\left.\begin{aligned} &[\chi_i(x), \chi_j(y)]_+ = 0 \quad \forall i, j; \quad \forall x, y \\ &[\chi_i(x), \chi_j^*(y)]_+ = 0 \begin{cases} \forall x, y & \text{if } i \neq j \\ \text{and for } |x - y| > d & \text{if } i = j. \end{cases} \end{aligned}\right\} \tag{2.8}$$

From the explicit action of $S_0(t)$:

$$S_0(t)\begin{pmatrix} \chi_1(s, x) \\ \chi_2(s, x) \end{pmatrix} = \begin{pmatrix} \chi_1(s, x + t) \\ \chi_2(s, x - t) \end{pmatrix}$$

we deduce that the free solution $\Phi_0(t) = S_0(t)\chi$ satisfies

$$\begin{aligned} &[\varphi_{(0),1}(t, x), \varphi^*_{(0),1}(t', x')]_+ = 0 && \text{if } | x - x' + t - t' | > d \\ &[\varphi_{(0),2}(t, x), \varphi^*_{(0),2}(t', x')]_+ = 0 && \text{if } | x - x' - (t - t') | > d \\ &\left.\begin{aligned} &[\varphi_{(0),1}(t, x), \varphi^*_{(0),2}(t', x')]_+ = 0 \\ &[\varphi_{(0),2}(t, x), \varphi^*_{(0),1}(t', x')]_+ = 0 \\ &[\varphi_{(0),i}(t, x), \varphi_{(0),j}(t', x')]_+ = 0 \quad \forall i, j \end{aligned}\right\} && \text{for any } x, x', t, t'. \end{aligned}$$

In particular,

$$\left.\begin{aligned} &[\varphi_{(0),i}(t, x), \varphi^{\#}_{(0),j}(t', x')]_+ = 0 \quad \forall i, j \\ &\quad \text{if} \quad | x - x' + t - t' | > d \quad \text{and if} \quad | x - x' - (t - t') | > d \end{aligned}\right\} \tag{2.9}$$

The aim of the present paper is to show that the solution $\Phi(t)$ of (2.3) has anticommutation relations like (2.9), which expresses the causality for a « delocalized » field like B.

3. — ITERATING PROCEDURE

It is convenient to introduce a sequence of recursive solutions of the integral equation (2.2):

$$\left.\begin{aligned} \Phi_{(k)}(t) &= S_0(t)\chi + 2ig\int_0^t S_0(t - s)T\Phi_{(k-1)}(s)\,ds \\ \Phi_{(0)}(t) &= S_0(t)\chi \end{aligned}\right\} \tag{3.1}$$

and we ask about the convergence of $\Phi_{(k)}(t)$ towards $\Phi(t)$ as k goes to infinity. We then have:

LEMMA 1. — Let $\Phi(t)$ be the solution of (2.1)-(2.4) for the initial data χ. Then there is a positive number τ depending on $|\chi|_Y$ such that

$$\lim_{k\to\infty} \Phi_{(k)}(t) = \Phi(t) \quad \text{in Y for any } t \leqslant \tau.$$

Proof. — Let us compute $|\Phi_{(k)}(t)|_Y$. Putting $\rho = |\chi|_Y$, we have, as the r-Lipschitz norm of T is $\leqslant 6|g|r^2$,

$$|\Phi_{(1)}(t)|_Y \leqslant \rho + \int_0^t 6|g|\rho^2\rho ds = \rho(1 + 6|g|t\rho^2) \leqslant \rho(1 + a)$$

where $\alpha = 2^2 \cdot 6|g|t\rho^2$. If we choose $\alpha < 1/2$, we have $1 + \alpha \leqslant 1 + \sum_{j=1}^{\infty} \alpha^j = 2$;

therefore

$$|\Phi_{(2)}(t)|_Y \leqslant \rho + \int_0^t 6|g|(2\rho)^2 \cdot \rho(1 + 2^2 \cdot 6|g|t\rho^2)ds$$
$$= \rho + \int_0^t 6|g|(2\rho)^2\rho(1 + \alpha)ds = \rho(1 + \alpha + \alpha^2),$$

and repeating the same argument it is easily seen that

$$|\Phi_{(k)}(t)|_Y \leqslant \rho \sum_0^k \alpha^j \leqslant 2\rho, \quad \forall k.$$

Furthermore, it is easy to check that

$$|\Phi_{(k)}(t) - \Phi_{(k-1)}(t)|_Y \leqslant c(t6|g|(2\rho)^2)^k$$

which converges to zero as $k \to \infty$, always provided $\alpha < 1/2$. Thus the limit of $\Phi_{(k)}(t)$ exists, and as it must be a solution of (2.1) with the initial data χ, the uniqueness of the solution shows that $\lim_{k\to\infty} \Phi_{(k)}(t) = \Phi(t)$.

The condition $\alpha < 1/2$ reads, in terms of t,

$$t < (\ 6|g|2^3 t)^{-1} \qquad \text{i. e.}$$
$$t < (48|g||\Phi|_Y^2)^{-1}$$

Thus the Lemma is proved with $\tau < (48|g||\Phi|_Y^2)^{-1}$.

Because of the semigroup character of $S(t)$, we may write, for $t \geqslant t' \geqslant 0$:

$$\Phi(t) = S_0(t - t')\Phi(t') + 2ig\int_{t'}^t S_0(t - s)T\Phi(s)ds \qquad (3.2)$$

This defines another iterating procedure:

$$\left.\begin{aligned} \Phi^{t'}_{(k)}(t) &= S_0(t-t')\Phi(t') + 2ig\int_{t'}^{t} S_0(t-s)T\Phi^{t'}_{(k-1)}(s)ds \\ \Phi^{t'}_{(0)}(t) &= S_0(t-t')\Phi(t') \end{aligned}\right\} \tag{3.3}$$

Then it is easy to prove:

LEMMA 2. — Let τ be the positive quantity determined in Lemma 1. Then

$$\lim_{k\to\infty} \Phi^{t'}_{(k)}(t) = \Phi(t) \quad \text{in } Y$$

for any t: such that $t - t' \leqslant \tau$.

Proof. — Repeating the argument of Lemma 1 one finds that

$$\lim_{k\to\infty} \Phi^{t'}_{(k)}(t) = \Phi(t)$$

for any t such that

$$t - t' \leqslant \tau', \quad \text{where} \quad \tau' < \left(48\,|\,g\,|\,|\,\Phi(t')\,|_Y^2\right)^{-1} \tag{3.4}$$

Thus, if $t - t' \leqslant \tau = \left(48\,|\,g\,|\,\chi\,|_y^2\right)^{-1}$, applying (2.6), it is evident that $t - t' \leqslant \tau'$, which proves our result.

4. — ANTICOMMUTATION RELATIONS

The results of section 3 are completely independent of the hypothesis (2.8), and hold in a more general context. We now introduce (2.8), thus even (2.9) is valid. We shall prove:

LEMMA 3. — $\Phi_{(k)}(t)$ satisfies the following anticommutation relations for any k:

Let $\mathcal{O}_d =$
$\left\{(x, x', t, t'); x, x' \in \mathbb{R}; t, t' \in \mathbb{R}_+ \,\middle|\, |x-x'+t-t'| > d; |x-x'-(t-t')| > d\right\}$

Then

$$[\varphi_{(k),i}(t, x), \varphi^{\#}_{(k),j}(t', x')]_+ = 0 \quad \forall i, j$$

if
$$(x, x', t, t') \in \theta_d \tag{4.1}$$

where $\psi^{\#}$, denotes both ψ and ψ^*.

Remark. — The region $\mathcal{O}_d$ contains the region $\mathcal{O}_d^m$ whose points (x, x', t, t') are such that

$$(t - t')^2 - (x - x')^2 < -d^2$$

Proof. — If $k = 0$, (4.1) is a particular case of (2.9),. Let us next suppose (4.1) to be valid for $k \leqslant h$: we will show that the same holds for $k = h + 1$.

Take for instance

$$\varphi_{(h+1),1}(t, x) = \varphi_{(0),1}(t, x) + 2ig\int_0^t ds\varphi^*_{(h),2}(s, x - s + t)\varphi_{(h),2}(s, x - s + t)\varphi_{(h),1}(s, x - s + t)$$

$$\varphi^*_{(h+1),2}(t', x') = \varphi^*_{(0),2}(t', x') + 2ig\int_0^{t'} ds'\varphi_{(h),2}(s', x' + s' - t')\varphi^*_{(h),1}(s', x', + s' - t')\varphi^*_{(h),1}(s', x' + s' - t')$$

One can easily show that the anticommutator between any one of the terms

$$\varphi_{(0),1}(t, x),\ \varphi^*_{(h),2}(s, x - s + t),\ \varphi_{(h),2}(s, x - s + t),\ \varphi_{(h),1}(s, x - s + t) \quad \forall s : 0 \leqslant s \leqslant t$$

and any one of the terms

$$\varphi^*_{(0),2}(t', x'),\ \varphi^*_{(h),2}(s', x' + s' - t')\varphi_{(h),1},\ \varphi^*_{(h),1}(s', x, + s' - t'),\ \varphi_{(h),1}(s', x' + s' - t') \quad (\forall s', 0 \leqslant s' \leqslant t')$$

vanishes for $(x, x', t, t') \in \mathcal{O}_d$. As an example, take

$$[\varphi_{(h),1}(s, x - s + t), \varphi^*_{(h),2}(s', x' + s' - t')]_+ \quad \forall s, s', 0 \leqslant s \leqslant t,\ 0 \leqslant s' \leqslant t'$$

which vanishes for (x, x', t, t') belonging to

$$\bigcap_{\substack{0 \leqslant s \leqslant t \\ 0 \leqslant s' \leqslant t'}} \left\{(x, x', t, t') \,\middle|\, |x - s + t - x' - s' + t' + s - s'| > d\right\}$$
$$\cap \bigcap_{\substack{0 \leqslant s \leqslant t \\ 0 \leqslant s' \leqslant t'}} \left\{(x, x', t, t') \,\middle|\, |x - s + t - x' - s' + t' - (s - s')| > d\right\}$$
$$= \bigcap_{0 \leqslant s' \leqslant t'} \left\{(x, x', t, t') \,\middle|\, |x + t - x' + t' - 2s'| > d\right\}$$
$$\cap \bigcap_{0 \leqslant s \leqslant t} \left\{(x, x', t, t') | \, x + t - x' + t' - 2s \,| > d\right\}$$
$$= \left\{(x, x', t, t') \,\middle|\, |x - x' + t + t' - 2s'| > d\right\}\Big|_{s'=t'}$$
$$\cap \left\{(x, x', t, t') \,\middle|\, |x - x' + t + t' - 2s| > d\right\}\Big|_{s=t}$$
$$= \left\{(x, x', t, t') \,\middle|\, |x - x' + t - t'| > d\right\}$$
$$\cap \left\{(x, x', t, t') \,\middle|\, |x - x' - (t - t')| > d\right\}$$

i. e. for $(x, x', t, t') \in \mathcal{O}_d$.

Thus, anticommutators between $\varphi_{(i+1),h}(t, x)$ and $\varphi^{\#}_{(h+1),j}(t', x')$ all vanish for $(x, x', t, t') \in \mathcal{O}_d$, and the result for $\Phi_{(k)}$ follows by induction, q. e. d.

Then it is easy to establish our main theorem:

THEOREM 4.

$$[\varphi_i(t, x), \varphi_j^{\#}(t', x')]_+ = 0 \quad \forall i, j$$

if

$$(x, x', t, t') \in \mathcal{O}_d.$$

Proof. — For $t, t' \leqslant \tau$, where τ is given by Lemma 1, the assertion of the theorem is true because of Lemmas 1 and 3. Then suppose the same to hold for $t, t' \leqslant n\tau$. Then we may consider the sequence $\{ \Phi^{n\tau}_{(k)} \}$: repeating the argument of Lemma 3 each term $\Phi^{n\tau}_{(k)}(t)$ can be shown to satisfy

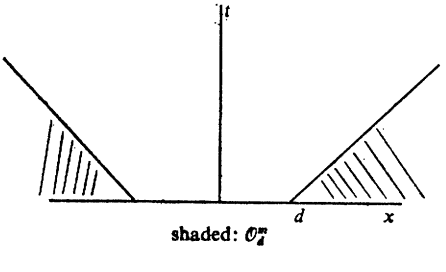
$$[\varphi^{n\tau}_{(k),i}(t, x), \varphi^{n\tau}_{(k),j}(t', x')]_+ = 0 \quad \forall i, j$$

if

$$(x, x', t, t') \in \mathcal{O}_d,$$

and because of Lemma 2, the assertion of the theorem is established for any $t, t' \leqslant (n + 1)\tau$. The result then follows by induction on n, q. e. d.

Remarks. — Theorem 1 shows that the anticommutators of the « field » vanish in a region $\mathcal{O}_d$ which is larger then $\mathcal{O}_d^m$ (see fig. 1). This depends essentially on the « massless » character of the operator L; and may be related to the fact that $\Delta_m^+(t, x)$ has support in the whole forward cone for $m \neq 0$, whereas $\Delta_0^+(t, x)$ has support only on the surface of the forward cone [8].



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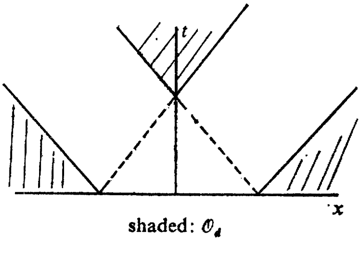

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