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Section A :
Physique théorique.

Canonical formulation of newtonian dynamics

by

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1. INTRODUCTION

The object of the present Note is formulation of Newtonian dynamics, as close to the canonical formulation as possible, within the framework of Galilean relativity. The traditional canonical formulation relies heavily on the Cartesian product structure of space-time, which is introduced by a distinguished inertial frame, and which is essential for the existence of momenta, and is used to define the Hamiltonian. This Cartesian product structure is incompatible with Galilean relativity, and is not used in this paper. Consequently, motions of bodies are not described in a phase-space in terms of a Hamiltonian. Instead, motions are obtained as integral manifolds of the characteristic distribution of a 2-form [1]. This characterization of motions closely resembles that obtained in the theory of canonical systems which includes the Hamiltonian dynamics as a special case [2].

2. STRUCTURE OF GALILEAN SPACE-TIME

Definition. — An affine space associated to a vector space E is a set X and a mapping $\alpha : X \times X \to E$ such that

(i) for each $x \in X$, and each $\mathbf{e} \in E$, there exists $x' \in X$ such that $\alpha(x, x') = \mathbf{e}$;

(ii) for each x, x', and x'' in X,

$$\alpha(x, x') + \alpha(x', x'') = \alpha(x, x'').$$

The element x', satisfying $\alpha(x, x') = \mathbf{e}$, is uniquely determined by x and e, and will be denoted by $x + \mathbf{e}$. We also write $x_1 - x_2$ for $\alpha(x_2, x_1)$.

We assume that Galilean space-time X is an affine space associated to a 4-dimensional real vector space E. In E there is a distinguished non-zéro form θ. The kermel of θ, $\mathrm{Ker}\ \theta = \{ \mathbf{e} \in \mathrm{E} \mid \theta(\mathbf{e}) = \mathbf{0} \}$, is a Euclidean space with a metric g.

3. KINEMATICS

Let $\gamma : \mathrm{J} \to \mathrm{X}$ be a curve in X; here J denotes an open interval in R. For each $t \in \mathrm{J}$, the tangent vector $v(t)$ to γ at the point $\gamma(t)$ is given by

$$v(t) = \lim_{s \to 0} \frac{1}{s} (\gamma(t + s) - \gamma(t)).$$

Let V be a subset of E defined by

$$\mathrm{V} = \{ \mathbf{e} \in \mathrm{E} \mid \theta(\mathbf{e}) = 1 \}.$$

It is an affine space associated to Ker θ. A curve $\gamma : \mathrm{J} \to \mathrm{X}$ such that, for each $t \in \mathrm{J}$, the tangent vector $v(t)$ to γ at $\gamma(t)$ belongs to V is called a motion.

Let $\gamma : \mathrm{J} \to \mathrm{X}$ be a motion. The mapping $v : \mathrm{J} \to \mathrm{V}$ such that, for each $t \in \mathrm{J}$, $v(t)$ is the tangent vector to γ at $\gamma(t)$, is called the velocity of the motion γ. The mapping $a : \mathrm{J} \to \mathrm{Ker}\ \theta$ such that, for each $t \in \mathrm{J}$,

$$a(t) = \lim_{s \to 0} \frac{1}{s} (v(t + s) - v(t))$$

is called the acceleration of the motion γ.

The Cartesian product $\mathrm{X} \times \mathrm{V}$ of affine spaces X and V, associated to the vector spaces E and Ker θ, respectively, is an affine space associated to $\mathrm{E} \times \mathrm{Ker}\ \theta$.

PROPOSITION. — *There exists a unique differential* 2-*form*

$$\omega : \quad \mathrm{X} \times \mathrm{V} \to [(\mathrm{E} \times \mathrm{Ker}\ \theta) \wedge (\mathrm{E} \times \mathrm{Ker}\ \theta)]^*$$

such that, for each $(x, \mathbf{v}) \in \mathrm{X} \times \mathrm{V}$, *and each simple* 2-*vector* $(\mathbf{e}, \mathbf{u}) \wedge (\mathbf{e}', \mathbf{u}')$ *in* $(\mathrm{E} \times \mathrm{Ker}\ \theta) \wedge (\mathrm{E} \times \mathrm{Ker}\ \theta)$,

$$\omega_{(x,\mathbf{v})}((\mathbf{e}, \mathbf{u}) \wedge (\mathbf{e}', \mathbf{u}')) = g(\mathbf{e} - \theta(\mathbf{e})\mathbf{v}, \mathbf{u}') - g(\mathbf{e}' - \theta(\mathbf{e}')\mathbf{v}, \mathbf{u}).$$

The form ω *is closed, and, for each* $(x, \mathbf{v}) \in \mathrm{X} \times \mathrm{V}$, $\omega_{(x,\mathbf{v})}$ *restricted to* $(\mathrm{Ker}\ \theta \times \mathrm{Ker}\ \theta)$ *is non-singular.*

Proof. — The right hand side of the equation above is well defined, since $\mathbf{e} - \theta(\mathbf{e})\mathbf{v}$, $\mathbf{e}' - \theta(\mathbf{e}')\mathbf{v}$, $\mathbf{u}$, and $\mathbf{u}'$ belong to Ker θ, and g is defined in Ker θ. Hence, the condition aboves defines a unique 2-form ω. Differentiability of ω is obvious.

For each simple 3-vector $(\mathbf{e}, \mathbf{u}) \wedge (\mathbf{e}', \mathbf{u}') \wedge (\mathbf{e}'', \mathbf{u}'') \in \overset{3}{\wedge} (\mathrm{E} \times \mathrm{Ker}\ \theta)$, and each $(x, \mathbf{v}) \in \mathrm{X} \times \mathrm{V}$,

$$
\begin{aligned}
&d\omega_{(x,\mathbf{v})}((\mathbf{e}, \mathbf{u}) \wedge (\mathbf{e}', \mathbf{u}') \wedge (\mathbf{e}'', \mathbf{u}'')) \\
&\quad = \theta(\mathbf{e})\, g(\mathbf{u}', \mathbf{u}'') - \theta(\mathbf{e}')\, g(\mathbf{u}, \mathbf{u}'') + \theta(\mathbf{e}'')\, g(\mathbf{u}, \mathbf{u}') - \theta(\mathbf{e})\, g(\mathbf{u}'', \mathbf{u}') \\
&\qquad + \theta(\mathbf{e}')\, g(\mathbf{u}'', \mathbf{u}) - \theta(\mathbf{e}'')\, g(\mathbf{u}', \mathbf{u}) = 0.
\end{aligned}
$$

Hence, $d\omega = 0$.

If $(\mathbf{e}, \mathbf{u})$ and $(\mathbf{e}', \mathbf{u}')$ belong to Ker $\theta \times$ Ker θ, then, for each $(x, \mathbf{v}) \in \mathrm{X} \times \mathrm{V}$, $\omega_{(x,\mathbf{v})}((\mathbf{e}, \mathbf{u}) \wedge (\mathbf{e}', \mathbf{u}')) = g(\mathbf{e}, \mathbf{u}') - g(\mathbf{e}', \mathbf{u})$, and it vanishes, for all $(\mathbf{e}', \mathbf{u}')$ in Ker $\theta \times$ Ker θ, if and only if $\mathbf{e} = \mathbf{u} = \mathbf{0}$. Hence, for each $(x, \mathbf{v}) \in \mathrm{X} \times \mathrm{V}$, the restriction of $\omega_{(x,\mathbf{v})}$ to Ker $\theta \times$ Ker θ is non-singular.

4. THE SECOND LAW OF DYNAMICS

The first law of dynamics is already contained in the assumption of an affine structure of space-time. We now proceed to formulate the second law.

A force f is a mapping from $\mathrm{X} \times \mathrm{V}$ to Ker θ.

The second Law of Dynamics. — A motion $\gamma : \mathrm{J} \to \mathrm{X}$ is dynamically admissible for a body of mass m, $m > 0$, under the action of a force f, if and only if, for each $t \in \mathrm{J}$.

$$ma(t) = f(\gamma(t), v(t)),$$

where $a : \mathrm{J} \to$ Ker θ is the acceleration, and $v : \mathrm{J} \to \mathrm{V}$ is the velocity of the motion γ.

5. CANONICAL DYNAMICS

Given a force f, and $m > 0$, we define ρ to be the unique 2-form in $\mathrm{X} \times \mathrm{V}$ such that, for each $(x, \mathbf{v}) \in \mathrm{X} \times \mathrm{V}$, and each simple 2-vector

$$(\mathbf{e}, \mathbf{u}) \wedge (\mathbf{e}', \mathbf{u}') \in (\mathrm{E} \times \mathrm{Ker}\ \theta) \wedge (\mathrm{E} \times \mathrm{Ker}\ \theta),$$

$$
\begin{aligned}
&\rho_{(x,\mathbf{v})}((\mathbf{e}, \mathbf{u}) \wedge (\mathbf{e}', \mathbf{u}')) \\
&\quad = m\, \omega_{(x,\mathbf{v})}((\mathbf{e}, \mathbf{u}) \wedge (\mathbf{e}', \mathbf{u}')) - g(f(x, \mathbf{v}), e - \theta(\mathbf{e})\mathbf{v})\, \theta(\mathbf{e}') \\
&\qquad + g(f(x, \mathbf{v}), \mathbf{e}' - \theta(\mathbf{e}')\mathbf{v})\, \theta(\mathbf{e}).
\end{aligned}
$$

For each $(x, \mathbf{v}) \in X \times V$, a subspace $N_{(x,\mathbf{v})}$ of $E \times \operatorname{Ker} \theta$, defined by

$$N_{(x,\mathbf{v})} = \{ (e, \mathbf{u}) \in E \times \operatorname{Ker} \theta \mid \rho_{(x,\mathbf{v})} ((\mathbf{e}, \mathbf{u}) \wedge (\mathbf{e}', \mathbf{u}')) = 0, \text{ for all } (\mathbf{e}', \mathbf{u}') \in E \times \operatorname{Ker} \theta \}$$

is called the characteristic space of ρ at $(x, \mathbf{v})$. The restriction of $\rho_{(x,\mathbf{v})}$ to $\operatorname{Ker} \theta \times \operatorname{Ker} \theta$ is proportional to the restriction of $\omega_{(x,\mathbf{v})}$ to $\operatorname{Ker} \theta \times \operatorname{Ker} \theta$, and therefore it is non-singular. Since dimension of $E \times \operatorname{Ker} \theta$ is 7 and the rank of a 2-form is even, $\dim N_{(x,\mathbf{v})} = 1$.

We denote by N the 1-dimensional distribution in $X \times V$, associating to each point $(x, \mathbf{v}) \in X \times V$ the subspace $N_{(x,\mathbf{v})}$ of $E \times \operatorname{Ker} \theta$. It can be easily shown that N is a differentiable distribution in $X \times V$. The distribution N is called the characteristic distribution of a 2-form ρ.

THEOREM. — *Let* $\chi : J \to X \times V$ *be a curve in* $X \times V$ *such that* $\mathrm{pr}_1 \circ \chi : J \to X$ *is a motion; here* pr_1 *is the first projection from the Cartesian product* $X \times V$. *The following conditions are equivalent :*

(i) $\mathrm{pr}_1 \circ \chi$ *is a dynamically admissible motion for a body of mass* m, *under the action of a force* f, *and* $\mathrm{pr}_2 \circ \chi : J \to V$ *is the velocity of* $\mathrm{pr}_1 \circ \chi$;

(ii) $\chi(J)$ *is an integral manifold of* N.

Proof. — For each $t \in J$, we denote by $(v(t), w(t))$ the tangent vector to χ at $\chi(t)$. Since $\mathrm{pr}_1 \circ \chi$ is a motion, $\theta(v(t)) = 1$. Let $(\mathbf{e}, \mathbf{u}) \in E \times \operatorname{Ker} \theta$ be arbitrary, then

$$\begin{aligned}
&\rho_{\chi(t)} ((v(t), w(t)) \wedge (\mathbf{e}, \mathbf{u})) \\
&\quad = mg(v(t) - \mathrm{pr}_2 \circ \chi(t), \mathbf{u}) - mg(\mathbf{e} - \theta(\mathbf{e})\, \mathrm{pr}_2 \circ \chi(t), w(t)) \\
&\qquad - g(f \circ \chi(t), v(t) - \mathrm{pr}_2 \circ \chi(t))\, \theta(\mathbf{e}) + g(f \circ \chi(t), \mathbf{e} - \theta(\mathbf{e})\, \mathrm{pr}_2 \circ \chi(t)) \\
&\quad = g(\mathbf{e} - \theta(\mathbf{e})\, \mathrm{pr}_2 \circ \chi(t), f \circ \chi(t) - mw(t)) \\
&\qquad + g(v(t) - \mathrm{pr}_2 \circ \chi(t), m\,\mathbf{u} - \theta(\mathbf{e})\, f \circ \chi(t)).
\end{aligned}$$

If $\chi(J)$ is an integral manifold of N, then $\rho_{\chi(t)} ((v(t), w(t)) \wedge (\mathbf{e}, \mathbf{u})) = 0$, for each $t \in J$ and each $(\mathbf{e}, \mathbf{u}) \in E \times \operatorname{Ker} \theta$. This is equivalent to

$$(\star) \qquad mw(t) = f \circ \chi(t),$$

and

$$(\star\star) \qquad v(t) = \mathrm{pr}_2 \circ \chi(t).$$

The equality $(\star\star)$ means that $\mathrm{pr}_2 \circ \chi$ is the velocity of the motion $\mathrm{pr}_1 \circ \chi$. In this case, the acceleration a of the motion $\mathrm{pr}_1 \circ \chi$ satisfies

$$(\star\star\star) \qquad a(t) = w(t),$$

for each $t \in J$. Hence, $pr_1 \circ \chi$ is a dynamically admissible motion for a body of mass m, under the action of a force f.

Conversly, if $pr_1 \circ \chi$ is a dynamically admissible motion for a body of mass m, under the action of a force f, and $pr_2 \circ \chi$ is the velocity of the motion $pr_1 \circ \chi$, then the equations (★), (★★), and (★★★) are satisfied for each $t \in J$. Therefore $\chi(J)$ is an integral manifold of N.

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