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On the problem of relativistically moving frames of reference

by

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ABSTRACT. — A method of derivation of relativistic formulae of transformation to accelerated frames of reference is developed in the case of one-dimensional motion. Special examples of uni-accelerated frame and of oscillating frame are considered. The problem of calculation of electromagnetic field of uni-accelerated electron is solved employing the non-inertial frame of reference.

1. INTRODUCTION

A problem of finding the relativistic formulae which describe the transformations to accelerated frames of reference is not a new one. Now and again it becomes a subject of discussions in the literature ([1] à [6]).

The employment of relativistic non-inertial frames seems to be very useful in the attempts to solve certain electrodynamic problems. For example, when the particle motion in different types of accelerators is studied, the introduction of a non-inertial frame of reference makes the calculations of electromagnetic fields and particle trajectories substantially easier, and also may significantly simplify the problem of stability of trajectories (*cf.* [7], [8]).

The simplest of the non-inertial frames are rectilinearly moving non-rotating ones. The uniformly accelerated motion is one particular case of such motions. In relativistic kinematics the uniformly accelerated motion is defined as one in which the magnitude of acceleration a remains constant with respect to proper (instantly co-moving inertial) frame.

The relativistic formulae of transformation to uni-accelerated frames were first indicated (without derivation) by Kottler [1].

These are written as follows :

$$(1) \quad \left\{ \begin{array}{l} X = x, \quad Y = y, \\ Z = \frac{c^2}{a_0} \left[\left(1 + \frac{a_0 z}{c^2} \right) \operatorname{ch} \frac{a_0 t}{c} - 1 \right], \\ T = \frac{c}{a_0} \left(1 + \frac{a_0 z}{c^2} \right) \operatorname{sh} \frac{a_0 t}{c}, \end{array} \right.$$

where X, Y, Z, T are defined with respect to inertial frame of referens.

Subject to condition $\frac{a_0 t}{c} \ll 1$ formulae (1) take the form of

$$X = x, \quad Y = y, \quad Z = z + \frac{a_0 t^2}{2}; \quad T = t,$$

that if of non-relativistic formulae of transformation to uni-accelerated frame. Moreover, if, for example, the motion of the origin of non-inertial frame is considered, then it follows from (1) that

$$\frac{dZ}{dT} = c \operatorname{th} \frac{a_0 t}{c} \leq c,$$

which indicates that the velocity of the point under consideration does not increase indefinitely, but approaches asymptotically that of light. Some time later, Møller [2] made an attempt to derive transformation formulae (1) from the principle of equivalence, his concern being the quantitative analysis of the twin-paradox.

In the present paper the method of derivation of formulae of transformation to relativistically accelerated frames of reference is considered in the case of one-dimensional motion.

2. ONE-DIMENSIONAL RELATIVISTIC MOTION OF FRAME OF REFERENCE

Consider some inertial frame (X, Y, Z, T) , designated as K . Another frame (x, y, z, t) relativistically accelerated with respect to K will be designated as K' . Assume the initial moment $T = t = 0$, when the origins of K and K' , designated as O and O' coincide, and let the frame K' move in the direction of Z -axis.

In this case only the co-ordinates Z and T are evidently subject to transformation,

$$(2) \quad X = x, \quad Y = y, \quad Z = F(z, t), \quad T = G(z, t).$$

Under transformations (2) the space-time metric

$$(3) \quad ds^2 = - dX^2 - dY^2 - dZ^2 + c^2 dT^2$$

will acquire the form

$$(4) \quad ds^2 = - dx^2 - dy^2 - A(z, t) dz^2 + c^2 B(z, t) dt^2 + C(z, t) dt dz.$$

We may further transform the z and t so that two conditions will be fulfilled :

First the factor $C(z, t)$ in the term with $dz dt$ of expression (4) will vanish, and second — the factor $A(z, t)$ will be equal to unity.

So, when the one-dimensional motion of frame K' is considered, the metric (4) may without the loss of generality be anticipated in the form :

$$(5) \quad ds^2 = - dx^2 - dy^2 - dz^2 + c^2 \omega(z, t) dt^2.$$

The condition of quadratic form (5) transformation into the shape (3) may be written as a requirement, that all components of the curvature tensor of space-time vanish :

$$(6) \quad R_{iklm} = 0,$$

which leads to the following equation for metric coefficient $\omega(z, t)$:

$$(7) \quad \frac{\partial}{\partial z} \left(\frac{1}{2\omega} \frac{\partial \omega}{\partial z} \right) + \frac{1}{4\omega^2} \left(\frac{\partial \omega}{\partial z} \right)^2 = 0.$$

The solution of (7) is the function

$$(8) \quad \omega(z, t) = \left[1 + \frac{a(t)z}{c^2} \right]^2,$$

where $a(t)$ is arbitrary function of argument t and has a dimensionality of acceleration. As will be seen below, to every particular choice of function $a(t)$ there corresponds some definite non-uniform motion of frame K' with respect to K .

Thus it is seen that the functions under question F and G, which transform (3) into (5) must satisfy the equations :

$$(9) \quad \left[1 + \frac{a(t)z}{c^2} \right]^2 \left[1 - \left(\frac{\partial F}{\partial z} \right)^2 \right] = - \left(\frac{\partial F}{c \partial t} \right)^2,$$

$$(10) \quad \left[1 + \frac{a(t)z}{c^2} \right]^2 \left[1 + \left(\frac{c \partial G}{\partial t} \right)^2 \right] = \left(\frac{\partial G}{\partial t} \right)^2.$$

Let us turn to somewhat more detailed consideration of the motion of the origin O' of the frame K' . Assume, that at the point O' the electric charge e is present and is acted upon by static non-uniform electric field $\vec{E}(Z)$. Let the Z -axis be directed along the field vector $\vec{E}(Z)$. In this case it follows from the equations of charge motion

$$\frac{d^2 X^i}{ds^2} = \frac{e}{m_0 c^2} F_k^i u^k$$

that

$$(11) \quad \begin{aligned} X = Y = 0, \\ \frac{d}{dT} \left(\frac{\frac{dZ}{dT}}{\sqrt{1 - \frac{1}{c^2} \left(\frac{dZ}{dT} \right)^2}} \right) = \frac{e}{m_0} E(Z). \end{aligned}$$

If we introduce the proper time accelerated particle as

$$(12) \quad t = \int_0^T \sqrt{1 - \frac{1}{c^2} \left(\frac{dZ}{dT} \right)^2} dT,$$

then the equation of trajectory of the particle found by integration of the equation (11) may be written in a parametric form

$$(13) \quad Z = Z(t), \quad T = T(t),$$

while the equations (13) may be viewed as conditions to which the functions F and G are subject at the point $z = 0$.

In this manner, the solution of the problem posed by the derivation of formulae of transformation to some particular accelerated frame of reference may be comprised in the following sequence of operations:

(a) From equations (11) find the law of motion of the origin O' of non-inertial frame.

(b) Write the trajectory of point O' in parametric form

$$(14) \quad Z = Z(t) = F(0, t), \quad T = T(t) = G(0, t),$$

where t is proper time of point O' .

(c) Express the acceleration of the point O' in terms of proper time

$$(15) \quad a = \frac{\frac{d^2 X}{dT^2}}{\left[1 - \frac{1}{c^2} \left(\frac{dZ}{dT} \right)^2 \right]^{3/2}} = a(t).$$

(d) Formulate the Cauchy-problem for equations (9) and (10) with initial conditions (14).

The functions $Z(z, t)$ and $T(z, t)$ then represent the solution of the problem under question.

3. UNI-ACCELERATED FRAME OF REFERENCE

The above advocated procedure for derivation of the formulae of transformation to non-inertial frames may be illustrated by some particular examples.

The integration of the equation of motion (11) in the case of constant uniform field $E(Z) = E_0 = \text{Const.}$ gives Z as

$$(16) \quad Z = \frac{c^2}{a_0} \left[\sqrt{1 + \left(\frac{a_0 T}{c} + \text{sh } \psi_0 \right)^2} - \text{ch } \psi_0 \right],$$

where

$$a_0 = \frac{e E_0}{m_0}, \quad \text{sh } \psi_0 = \frac{V_0}{c} \left(1 - \frac{V_0^2}{c^2} \right)^{-1/2},$$

$$\text{ch } \psi_0 = \left(1 - \frac{V_0^2}{c^2} \right)^{-1/2}, \quad V_0 = \text{Const.}$$

[The motion of the material point following the law (16) is usually referred to as "hiperbolic motion".]

Write the law (16) in a parametric form

$$(17) \quad \begin{cases} Z(t) = F(0, t) = \frac{c^2}{a_0} \left[\text{ch} \left(\frac{a_0 t}{c} + \psi_0 \right) - \text{ch } \psi_0 \right], \\ T(t) = G(0, t) = \frac{c}{a_0} \left[\text{sh} \left(\frac{a_0 t}{c} + \psi_0 \right) - \text{sh } \psi_0 \right]. \end{cases}$$

Since $a(t) = a_0 = \text{Const.}$, the equations (9) and (10) are easily integrated. The integral surfaces of interest embedding correspondently given curves (17) are defined by

$$(18) \quad \begin{cases} Z = \frac{c^2}{a_0} \left[\left(1 + \frac{a_0 Z}{c^2} \right) \text{ch} \left(\frac{a_0 t}{c} + \psi_0 \right) - \text{ch } \psi_0 \right], \\ T = \frac{c}{a_0} \left[\left(1 + \frac{a_0 Z}{c^2} \right) \text{sh} \left(\frac{a_0 t}{c} + \psi_0 \right) - \text{sh } \psi_0 \right], \end{cases}$$

that is we have re-established (with $V_0 = \psi_0 = 0$) the transformation formulae of Kottler-Møller. When $a_0 = 0$ the formulae (18) become the Lorentz transformations.

4. OSCILLATING FRAME OF REFERENCE

Assume, that the motion of origin O' follows the law imposed by equation (11) with $e E(Z) = -x^2 Z$, where x^2 is some positive constant.

In this case the first integral of equation (11) is

$$(19) \quad \frac{dZ}{dT} = c \left[1 - m_0^2 c^4 \left(W_0 - \frac{x^2 Z^2}{2} \right)^{-2} \right]^{1/2},$$

where W_0 is energy constant. If the particle reaches the state of rest at $Z = A_0$, then $W_0 = m_0 c^2 + \frac{x^2 A_0^2}{2}$.

With the use of equations (12) and (19) it is not difficult to find the trajectory of point O' in a parametric form :

$$(20) \quad \begin{cases} Z(t) = F(0, t) = A_0 \operatorname{sn}(u, k), \\ T(t) = G(0, t) = \frac{A_0}{ck} E(u, k) - t. \end{cases}$$

Here we denoted

$$u = \omega t = \left[\frac{x^2}{m_0} \left(1 + \frac{x^2 A_0^2}{4 m_0 c^2} \right) \right]^{1/2} t,$$

$$k = \frac{x A_0}{2 c \sqrt{m_0}} \left(1 + \frac{x^2 A_0^2}{4 m_0 c^2} \right)^{-1/2}, \quad E(u, k) = \int_0^u dn^2(u, k) du,$$

$\operatorname{sn}(u, k)$ and $dn(u, k)$ being the elliptic functions of Jacoby.

In the case considered

$$a(t) = \frac{\frac{d^2 Z}{dT^2}}{\left[1 - \frac{1}{c^2} \left(\frac{dZ}{dT} \right)^2 \right]^{3/2}} = - \frac{x^2 A_0}{m_0} \operatorname{sn}(u, k),$$

so, that after integration of equations (9) and (10) with conditions (20) we shall have finally

$$(21) \quad \begin{cases} Z(z, t) = A_0 \operatorname{sn}(u, k) + \left[\frac{A_0 \omega}{ck} dn^2(u, k) - 1 \right] z, \\ T(z, t) = \frac{A_0}{ck} E(u, k) - t + \frac{A_0 \omega}{c^2} z \operatorname{cn}(u, k) dn(u, k). \end{cases}$$

If $c \rightarrow \infty$ and consequently $k \rightarrow 0$, $\omega \rightarrow \frac{x}{\sqrt{m_0}}$ there follow from equation (21)

$$(22) \quad \begin{cases} Z = z + A_0 \sin\left(\frac{x}{\sqrt{m_0}} t\right), \\ T = t, \end{cases}$$

i. e. non-relativistic formulae of transformation to the oscillating frame of reference.

5. APPLICATION OF NON-INERTIAL FRAME OF REFERENCE TO CALCULATIONS OF THE ELECTROMAGNETIC FIELD OF ACCELERATED ELECTRON

Consider the problem of calculation of the electromagnetic field of point electron in hiperbolic motion using the transformation to uni-accelerated frame of reference.

As is well known, the Maxwells equations are written in arbitrary curvilinear co-ordinates as

$$(23) \quad \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^k} \left[\sqrt{-g} g^{il} g^{km} \left(\frac{\partial A'_m}{\partial x^l} - \frac{\partial A'_l}{\partial x^m} \right) \right] = \frac{4 \pi}{c} j^i,$$

whily Lorentz conditions are

$$(24) \quad \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^i} \left[\sqrt{-g} g^{ik} A'_k \right] = 0.$$

Using uni-accelerated frame of reference described by metric (5) with $\varphi(z) = \left(1 + \frac{a_0 z}{c^2}\right)^2$ we may search for a solution of equations (23) and (24) in the form of

$$j_1 = j_2 = j_3 = 0, \quad A'_1 = A'_2 = 0, \quad A'_3 = \frac{e a_0}{c^2} \left(1 + \frac{a_0 z}{c^2}\right),$$

whily A'_0 satisfies the equation :

$$(25) \quad \frac{\partial^2 A'_0}{\partial x^2} + \frac{\partial^2 A'_0}{\partial y^2} + \left(1 + \frac{a_0 z}{c^2}\right) \frac{\partial}{\partial z} \left[\frac{1}{\left(1 + \frac{a_0 z}{c^2}\right)} \frac{\partial A'_0}{\partial z} \right] = -4 \pi e \delta(x) \delta(y) \delta(z),$$

its solution being the expression

$$(26) \quad A'_0 = \frac{e a_0}{c^2} \frac{x^2 + y^2 + \left(z + \frac{c^2}{a_0}\right)^2 + \frac{c^4}{a_0^2}}{\sqrt{\left[x^2 + y^2 + \left(z + \frac{c^2}{a_0}\right)^2 - \frac{c^4}{a_0^2}\right]^2 + 4 \frac{c^4}{a_0^2} (x^2 + y^2)}}.$$

The components of covariant vector A_i with respect to the inertial frame are equal to

$$(27) \quad A_i = \frac{\partial X^k}{\partial X^i} A'_k.$$

Employing the transformation formulae (1) we get

$$(28) \quad \left\{ \begin{array}{l} \frac{\partial z}{\partial T} = -c \operatorname{sh} \frac{a_0 t}{c}, \quad \frac{\partial z}{\partial Z} = \operatorname{ch} \frac{a_0 t}{c}, \\ \frac{\partial t}{\partial T} = \frac{\operatorname{ch} \frac{a_0 t}{c}}{1 + \frac{a_0 z}{c^2}}, \quad \frac{\partial t}{\partial Z} = -\frac{\operatorname{sh} \frac{a_0 t}{c}}{c \left(1 + \frac{a_0 z}{c^2}\right)}. \end{array} \right.$$

So that the relation (27) with expressions (28) inserted yields the final result :

$$(29) \quad \left\{ \begin{array}{l} A_X = A_Y = 0, \\ A_Z = e \left\{ \frac{c T \left[X^2 + Y^2 + \left(Z + \frac{c^2}{a_0} \right)^2 - c^2 T^2 + \frac{c^4}{a_0^2} \right]}{\left[\left(Z + \frac{c^2}{a_0} \right)^2 - c^2 T^2 \right] R} - \frac{Z + \frac{c^2}{a_0}}{\left(Z + \frac{c^2}{a_0} \right)^2 - c^2 T^2} \right\}, \\ A_0 = e \left\{ \frac{\left(Z + \frac{c^2}{a_0} \right) \left[X^2 + Y^2 + \left(Z + \frac{c^2}{a_0} \right)^2 - c^2 T^2 + \frac{c^4}{a_0^2} \right]}{\left[\left(Z + \frac{c^2}{a_0} \right)^2 - c^2 T^2 \right] R} - \frac{c T}{\left(Z + \frac{c^2}{a_0} \right)^2 - c^2 T^2} \right\}, \\ R = \sqrt{\left[X^2 + Y^2 + \left(Z + \frac{c^2}{a_0} \right)^2 - c^2 T^2 - \frac{c^4}{a_0^2} \right]^2 + 4 \frac{c^4}{a_0^2} (X^2 + Y^2)}. \end{array} \right.$$

The solution (29) was first obtained by Born [9] and later, Schott [10] who made use of retarded potentials technique.

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