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Section A :
Physique théorique.

Gel'fand-Kirillov Dimension for Algebras Associated with the Weyl Algebra (*)

by

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ABSTRACT. — Let K be a commutative field of characteristic zero. The Weyl algebra of degree n over K is defined as the associative algebra A_n (K) whose generators satisfy the canonical commutation relations for a system with n degrees of freedom. It is shown that if B lies in a certain class of subalgebras of the quotient field of A_n (K), then

$$\mathrm{Dim}_K\, A + \mathrm{Dim}_K\, C\,(A, B) \leqq 2\,n,$$

where A is a subalgebra of B, C (A, B) the commutant of A in B and Dim_K denotes a dimension introduced by Gel'fand and Kirillov [1]. Applications and possible generalizations of this result are discussed.

RÉSUMÉ. — Soit K un corps commutatif de caractéristique 0. L'algèbre de Weyl d'indice n sur K est définie comme l'algèbre associative A_n (K) à générateurs satisfaisant les relations de commutation à n degrés de liberté. Au cas où B est continue dans une certaine classe de subalgèbres du corps des fractions de A_n (K), il est démontré que

$$\mathrm{Dim}_K\, A + \mathrm{Dim}_K\, C\,(A, B) \leqq 2\,n,$$

où A est une subalgèbre de B, C (A, B) le commutant de A dans B, et Dim_K est une dimension définie par Gel'fand et Kirillov [1]. On examine les utilisations et les généralisations éventuelles de ce résultat.

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1. INTRODUCTION

Let K be a commutative field of characteristic zero. For example the complex numbers **C**. By the Weyl algebra of degree n over K, we mean the associative algebra $A_n(K)$ (or simply A_n) on K defined by generators q_i, p_j $(i, j = 1, 2, \ldots, n)$, with relations

$$(1.1)\quad \begin{cases} [q_i, p_j] = \delta_{ij} 1, \\ [q_i, q_j] = [p_i, p_j] = 0. \end{cases}$$

When $n = 1$, we drop the subscripts on q and p.

We may recognize (1.1) as being the defining relation of the canonical commutation relations or of the nilpotent Lie algebra whose enveloping algebra, quotient the ideal generated by $1^2 - 1$, is A_n. Again (1.1) is recovered if on the algebra of smooth functions in the variables p_1, $p_2, \ldots, p_n$, we make the identification : $q_j = \frac{\partial}{\partial p_j}$ $(j = 1, 2, \ldots, n)$. The Weyl algebra is important in all of these roles.

Let A be a subalgebra of A_n over K. Let **a** denote a finite collection of elements of A. Let $d(\mathbf{a}, m)$ denote the dimension of the space of polynomials over K of degree at most m, taken from elements of **a**. Following Gel'fand and Kirillov [1], we define

$$(1.2)\quad \mathrm{Dim}_K A = \mathrm{Sup} \varlimsup_{m \to \infty} \frac{\log d(\mathbf{a}, m)}{\log m},$$

where $\varlimsup$ denotes the least upper bound.

These authors have shown that

$$(1.3)\quad \mathrm{Dim}_K A_n = 2n.$$

More generally we have established a result : [2], Theorem 3.5, which can be expressed in the form.

THEOREM 1.1. — *Let* A, B *be subalgebras of* A_n *such that* $[a, b] = 0$, *for all* $a \in A$, $b \in B$. *Then*

$$(1.4)\quad \mathrm{Dim}_K A + \mathrm{Dim}_K B \leqq \mathrm{Dim}_K A_n = 2n.$$

Actually the theorem referred to does not imply Theorem 1.1, though in its proof (1.4) is implicit. In any case here we establish a more general result than Theorem 1.1.

Theorem 1.1 has a number of weaker forms. Firstly we may omit $2n$ from the right hand side of (1.4). Secondly it implies $\mathrm{Dim}_K A \leqq n$, when A is commutative. Finally it asserts the existence of certain polynomial relations between the elements of the given subalgebras. Indeed it was in this form that our results were originally stated.

Theorem 1.1 has a number of applications. It implies [1] the non-isomorphism of A_m with A_n for $m \neq n$. [In fact (1.3) suffices for this purpose.] It is useful in the construction of generators of a Lie algebra from elements of A_n ([1]-[5]). It applies to the study of endomorphisms [6] (i. e. canonical transformations) of A_n.

Our present objective is to obtain generalizations of Theorem 1.1 by extending A_n to a larger algebra. We could do this by taking formal power series; but we believe this choice to be inappropriate, for then independent of n, there is a sense in which the resulting algebra is isomorphic to the algebra of all bounded operators on some infinite dimensional Hilbert space. Consequently n loses its meaning. Instead recall that A_n has no zero divisors [7] and define the quotient field R_n of A_n through

$$R_n = \{ a^{-1} b : a, b \in A_n \}.$$

with the identification, $x = y$ $(x, y \in R_n)$, given nonzero $a \in A_n$, such that $ax, ay \in A_n$, and $ax = ay$.

That R_n is a field is a consequence of the fact that A_n satisfies the left Ore condition. That is for all a, $b \in A_n$, there exist $c, d \in A_n$ such that

$$ca = db. \tag{1.5}$$

Then, for example,

$$(a^{-1} b)^2 \in R_n \qquad \text{since} \quad (a^{-1} b)^2 = a^{-1} d^{-1} dba^{-1} b = (da)^{-1} cb.$$

The validity of (1.5) is generally demonstrated ([1], [7]) by appealing to the Notherian property of A_n. In a more elementary fashion it can also be verified by monomial counting [4], a technique which extends to the enveloping algebra of any Lie algebra.

In attempting to generalize Theorem 1.1 to R_n, we might question whether (1.2) is still an appropriate definition of dimension. Indeed for R_n, Gel'fand and Kirillov [1] introduce a modification of (1.2) which essentially allows for the introduction of a common divisor b of the components of **a**. With this change they show that (1.3) remains valid when R_n replaces A_n. For our purposes this modification is inappropriate since the introduction of b upsets the commutativity hypothesis of Theorem 1.1.

For R_n itself, Theorem 1.1 fails. Yet we believe (*cf.* Section 4) that it may still be possible to establish one of its weaker forms. For the present we restrict our attention to a subalgebra of R_n for which Theorem 1.1 can be demonstrated. For this purpose let ad a $(a \in R_n)$, denote the map of R_n into R_n given by $(\text{ad } a)\, b = [a, b]$.

DEFINITION 1.2. — *A subalgebra* B *of* R_n *is said to have nilpotent quotient if for every finite collection of elements* $b_1, b_2, \ldots, b_n \in B$, *there exists* $a \in A_n$, *such that* $ab_i \in A_n$ *and for some positive integer k,* $(ad^k\, a)\, b_i = 0$ *for all i.*

Example 1. — Let P_n denote the algebra over K generated by $q_1, q_2, \ldots, q_n$. Define

$$B_n = \{ a^{-1}\, b : a \in P_n,\ b \in A_n \}.$$

Then B is a subalgebra of R_n with nilpotent quotient. Indeed in the definition we may choose $a \in P_n$ and then the hypothesis is an easy consequence of (1.1). B_n may be considered as an algebra of differential operators over rational functions.

Example 2. — Define $B \subset R_1$, through

$$B = \{ xa : x \text{ rational in } q^2 p,\ a \text{ polynomial in } p \}.$$

Since $(\mathrm{ad}^3\, q^2 p)\, p = 0$, B has nilpotent quotient.

For $n = 1$, we give a partial characterization (Theorem 3.6) of algebras having nilpotent quotient.

Theorem 1.1 can be demonstrated for algebras with nilpotent quotient since it is straightforward to control the degree of the common divisor of given elements. In general for R_n, this problem reduces to finding good estimates for the degrees of *c*, *d* appearing in the Ore condition (1.5) in terms of the degrees of *a*, *b*. Neither existence proofs mentioned above are useful for this purpose. We have been able to obtain an explicit formula for *c*, *d* which is more useful. Then, for example, given A a commutative subalgebra of R_1, the relation : $\mathrm{Dim}_K\, A \leqq 1$, reduces to establishing certain determinantal identities. Unfortunately we have been unable to verify these except in the simplest cases and it remains a question for further analysis.

2. THE POISSON BRACKET AND THE EXTERIOR PRODUCT

Let $R(X^m)$ (respectively $D(X^m)$) denote the algebra of rational (respectively analytic) functions over K in the *m* variables : $X_1, X_2, \ldots, X_m$. Given $F, G \in R(X^{2n})$, we define their Poisson bracket through the formal differentiation

$$(2.1) \qquad \{ F, G \} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial X_i} \frac{\partial G}{\partial X_{i+n}} - \frac{\partial G}{\partial X_i} \frac{\partial F}{\partial X_{i+n}} \right).$$

Let $\wedge$ denote the exterior product [8]. This defines a distributive, associative, antysimmetric, non-degenerate product on the $2n$ dimensional vector space over $R(X^{2n})$ spanned by dX_i $(i = 1, 2, \ldots, 2n)$.

LEMMA 2.1. — *Let r be an integer : $0 < r < 2n + 1$. Given F_j, $G_k \in R(X^{2n})$ $(j = 1, 2, \ldots, r;\ k = 1, 2, \ldots, 2n + 1 - r)$, with $\{F_j, G_k\} = 0$, for all j, k. Then either $\bigwedge_{j=1}^{r} dF_j = 0$, or $\bigwedge_{k=1}^{2n+1-r} dG_k = 0$.*

Proof. — Assume neither conclusion holds. Since F_j, G_k are rational and the determinant is a polynomial function, there must be at least one point in the X space where neither product vanishes. This observation reduces the proof to the case when the given functions are linear homogeneous, which is a straightforward computation.

Let M_r denote a linear subspace of $R(X^{2n})$ closed under Poisson bracket, where : $r = \dim dM_r$ and dM_r denotes the linear space of differentials dF $(F \in M_r)$. Let N_s denote a second such subspace. M_r and N_s are said to be in involution [9] given $\{F, G\} = 0$, for all $F \in M_r$, $G \in N_s$. Then through the above lemma : $r + s \leq 2n$. This inequality lies at the basis of Theorem 1.1. To examine the condition for equality, we remark that given $M_r \subset D(X^{2n})$, there exists [9] an $N_s \subset D(X^{2n})$ in involution with M_r, such that $r + s = 2n$. (No assertion is made here concerning global definition of the elements of N_s, *see* also example below).

Yet the corresponding statement for $R(X^{2n})$ is false unless $n = 1$. For example, take $K = \mathbf{C}$, $n = 2$, $r = 1$, and let M_1 be complex multiples of the function

$$F = X_1^2 + X_2^2 + X_3^2 + \alpha^2 X_4^2 \qquad (\alpha \text{ irrational}).$$

Then $M_1 \subset R(X^4) \cap D(X^4)$ and is a one dimensional linear space closed under Poisson bracket. We remark that F may be identified with the classical Hamiltonian for the two dimensional harmonic oscillator with incommensurable frequencies. Set

$$Y_1 = X_1^2 + X_3^2, \qquad Y_2 = X_2^2 + \alpha^2 X_4^2,$$
$$Y_3 = \tan^{-1}\frac{X_3}{X_1}, \qquad Y_4 = \frac{1}{\alpha}\tan^{-1}\alpha\frac{X_4}{X_2}.$$

This change of variables is a contact transformation and $F = Y_1 + Y_2$. It follows that $N = \{G \in D(X^4) : \{F, G\} = 0\}$ is the set of all analytic functions of Y_1, Y_2, $Y_3 - Y_4$. When α is rational, say $\alpha = \frac{m}{n}$ $(m. n$ integers), then $\tan m(Y_3 - Y_4) \in R(X^4)$ and there exists an $N_3 \subset R(X^4)$ in involution with M_1. Otherwise this fails to hold. Indeed choose a linear contact transformation to reduce F to the form $X_1 X_3 + \alpha X_2 X_4$ and observe.

LEMMA 2.2. — *Define* $N = \{ F \in R(X^4) : \{ F, X_1 X_3 + \alpha X_2 X_4 \} = 0 \}$. *Given* α *irrational, then* N *is generated by* $X_1 X_3$ *and* $X_2 X_4$.

Proof. — Choose $F \in N$. We may write $F = G^{-1} H$ (G, H polynomial). Through the hypothesis of the lemma

$$(2.2) \qquad G \{ H, X_1 X_3 + \alpha X_2 X_4 \} = H \{ G, X_1 X_3 + \alpha X_2 X_4 \}.$$

Write

$$G = \sum_{m=r_0}^{r} X_1^m G_m (X_2, X_1 X_3, X_2 X_4) \qquad (G_r \neq 0).$$

and

$$G_r = \sum_{m=s_0}^{s} X_2^m G_{rm} (X_1 X_3, X_2 X_4) \qquad (G_{rs} \neq 0).$$

Define $X_1^r X_2^s G_{rs}$ to be the leading term of G. Let $X_1^k X_2^l H_{kl}$ be the leading term of H. Through (2.2) we obtain

$$G_{rs} H_{kl} \{ (r + \alpha s) - (k + \alpha l) \} = 0,$$

which implies : $k = r, s = l$, since α is irrational. Set $H' = HG_{rs} - GH_{rs}$ and $F' = G^{-1} H'$. Then $F' \in N$, and so if $H' \neq 0$, its leading term must be of the form $X_1^r X_2^s H'_{rs}$, which impossible. Hence $H' = 0$ and $F = G_{rs}^{-1} H_{rs}$, which proves the lemma.

Define $F \in R(X^m)$ to be homogeneous of degree s given

$$(2.3) \qquad \sum_{i=1}^{m} X_i \frac{\partial F}{\partial X_i} = s F.$$

LEMMA 2.3. — *Let* S *be a subset of* $R(X^m)$. *Let* r *be an integer* : $0 < r < m$. *Given* $G \in R(X^m)$, $dG \neq 0$, *such that* $dG \wedge \left(\bigwedge_{i=1}^{r} dF_i \right) = 0$, *for all* $F_i \in S$. *Then*

(1) $\bigwedge_{j=1}^{r+1} dF_j = 0$, *for all* $F_j \in S$.

(2) *Given that the elements of* S *are homogeneous of degree* $s \neq 0$, *then*

$$\sum_{j=1}^{r+1} (-1)^j F_j \left(\bigwedge_{i(\neq j)=1}^{r+1} dF_i \right) = 0.$$

Proof. — (1) Let dS denote the subspace over K spanned by dF ($F \in S$). It suffices to shown that dS is at most r dimensional. Assume dim $dS > r$. Let $\{ e_i \} : i = 1, 2, \ldots, 2n$, be a basis for $dR(X^{2n})$ such that

$e_i = dF_i : i = 1, 2, \ldots, r.$ Write

$$dG = \sum_{i=1}^{2n} h_i\, e_i, \qquad h_i \in R(X^{2n}).$$

Through the hypothesis of the lemma and the non-degeneracy of $\wedge$ a simple calculation shows that each h_i must vanish. Hence $dG = 0$, a contradiction which proves the assertion.

(2) Identification of the component of dX_k in $\bigwedge_{j=1}^{r+1} dF_j$, multiplication by X_k, summation over k, use of (1) and (2.3) gives (2).

LEMMA 2.4. — *Let* $S \subset R(X^m)$ *be a subset of homogeneous polynomials of degree* k *and let* $G \in R(X^m)$ *be a homogeneous polynomial of degree* l. *Given* $\bigwedge_{i=1}^{r} d\frac{F_i}{G} = 0$, *for all* $F_i \in S$, $0 < r \leq m$. *Then either*

(1) $k \neq l$, *and* $\bigwedge_{i=1}^{r} dF_i = 0$, *for all* $F_i \in S$, *or,*

(2) $k = l$, *and* $\bigwedge_{i=1}^{r+1} dH_i = 0$, *for all* $H_i \in T$, *where*

$$T = \{ H \in R(X^{m+1}) : H = X_{m+1}\, G \text{ or } H = X_{m+1}\, F : F \in S \}.$$

Proof. — Through the hypothesis

$$\frac{1}{G^r} \bigwedge_{i=1}^{r} dF_i + \frac{1}{G} dG \wedge \left(\sum_{j=1}^{r} \frac{(-1)^j F_j}{G} \bigwedge_{i(\neq j)=1}^{r} d\frac{F_i}{G} \right) = 0. \tag{2.4}$$

Given $k - l \neq 0$, Lemma 2.3.2 applied to the hypothesis shows that the second term in the right hand side of (2.4) vanishes. Hence (1). Otherwise, exterior multiplication of (2.4) by $X_{m+1}^{r+1}\, G^{r+1}\, dX_{m+1}$, gives after a little rearrangement

$$d(X_{m+1}\, G) \wedge \left(\bigwedge_{i=1}^{r} d(X_{m+1}\, F_i) \right) = 0,$$

for all $F_i \in S$. Taking Lemma 2.3.1 into account we obtain (2).

We can now state and prove the central dimensionality estimate required for Theorem 1.1.

THEOREM 2.5. — *Let* m, n, r, s *be non-negative integers. Let* M_{mnrs} *be a linear subspace of* $R(X^n)$ *of homogeneous rational functions of degree* m *having as common divisor a polynomial* $H \in R(X^n)$ *of degree* s. *Given* $\bigwedge_{i=1}^{r+1} dG_i = 0$, *for all* $G_i \in M_{mnrs}$, *then*

$$\dim M_{mnrs} \leq \begin{cases} \dbinom{m+s+r-1}{r-1} & (m \neq 0), \\[2mm] \dbinom{s+r}{r} & (m = 0). \end{cases} \tag{2.5}$$

Proof. — Given $G \in M_{mnrs}$, set $F = GH$. By hypothesis, F is a homogeneous polynomial of degree $m + s$. When $m \neq 0$, $m + s \neq s$, so by the hypothesis and Lemma 2.4.1, it follows that $\bigwedge_{i=1}^{r+1} dF_i = 0$, for all F_i. Then [2], Lemma 3.3 applies and we obtain the top line of (2.5).

When $m = 0$, similar use of [2], Lemma 3.3, taking Lemma 2.4.2 into account, gives the bottom line of (2.5).

3. MAIN THEOREMS

Through (1.1) it is immediate that the monomials in the q_i and p_j $(i, j = 1, 2, \ldots, n)$, with the q_i to the left form a basis for A_n. With respect to this basis, each $a \in A_n$ may be considered as a polynomial in the $2n$ variables : X_i $(i = 1, 2, \ldots, 2n)$, identifying : $X_i = q_i$; $X_{i+n} = p_i$ $(i = 1, 2, \ldots, n)$. Let F_a denote the leading term of this polynomial. Then F_a is a homogeneous polynomial of the X_i. Given a, $b \in A_n$, the following formulae derive from (1.2).

For all α, $\beta \in K$,

$$F_{\alpha a + \beta b} = \alpha F_a + \beta F_b, \tag{3.1}$$

given that the right hand side does not vanish and $\deg F_a = \deg F_b$;

$$F_{\alpha a + \beta b} = \alpha F_a, \tag{3.2}$$

given $\deg F_a > \deg F_b$ and $\alpha \neq 0$;

$$F_{ab} = F_a F_b. \tag{3.3}$$

Given $a, b \neq 0$, either

$$\deg F_{[a,b]} = \deg F_{ab} - 2 \quad \text{and} \quad F_{[a,b]} = \{ F_a, F_b \} \tag{3.4}$$

or,

$$\deg F_{[a,b]} < \deg F_{ab} - 2 \quad \text{and} \quad \{ F_a, F_b \} = 0. \tag{3.5}$$

Let $x \in R_n$. We may write $x = a^{-1} b$ $(a, b \in A_n)$. Then (*cf.* [1], Lemma 4) $F_a^{-1} F_b$ is independent of the particular choice of a, b for which $x = a^{-1} b$. Indeed set $x = c^{-1} d$ $(c, d \in A_n)$. Through the left Ore condition (1.5) there exists e, $f \in A_n$, such that $ea = fc$. Right multiplication by x gives $eb = fd$. Taking (3.3) into account we obtain from these two relations that $F_a^{-1} F_b = F_c^{-1} F_d$, as required. Set $F_x = F_a^{-1} F_b$. A similar use of the left Ore condition and (3.1)-(3.5) establishes the lemma below.

LEMMA 3.1. — *Formulae* (3.1)-(3.5) *extend to the elements of* R_n.

COROLLARY. — *Given* x, $y \in R_n$ *with* $[x, y] = 0$, *then* $\{F_x, F_y\} = 0$. The above result allows the application of the results of Section 2 to R_n. Thus we show :

THEOREM 3.3. — *Let* A *be a subalgebra of* R_n *with nilpotent quotient. Set* $M_r = \{F_x : x \in A\}$, *where* $r = \dim dM_r$. *Then* $\mathrm{Dim}_K A = r$.

Remark. — Through Lemma 3.1, M_r is closed under Poisson bracket.

Proof. — Establish $\mathrm{Dim}_K A \leq r$. Let $\mathbf{x}$ denote an s-tuple of elements $x_1, x_2, \ldots, x_s \in A$. By hypothesis and Definition 1.2, there exists a, $b_1, b_2, \ldots, b_s \in A_n$, such that : $x_i = a^{-1} b_i$; $(\mathrm{ad}^k a)\, b_i = 0$ $(i = 1, 2, \ldots, s)$, for some positive integer k. Set $u = \operatorname{Sup}_i \deg F_{b_i}$; $v = \deg F_a$. Let m be a positive integer. Let z be a polynomial of degree at most m in the x_i. Let l be an integer, $l \geq k$. Denote by L_a, R_a respectively, left and right multiplication by a. Then

$$a^l x_i = a^{l-1} b_i = L_a^{l-1} b_i = (\mathrm{ad}\, a + R_a)^{l-1} b_i$$
$$= \left(\sum_{j=0}^{l-1} \binom{l-1}{j} \mathrm{ad}^j a\, R_a^{l-1-j} \right) b_i;$$

From this formula, it follows that there exists $b'_i \in A_n$, such that $a^l x_i = b'_i a^{l-k}$. Hence $a^{km} z \in A_n$. Set $c = a^{km} z$. Through (3.3) :

$$\deg F_c \leq m\,(u + (k-1)\,v). \tag{3.6}$$

Then applying Theorem 2.5 to the hypothesis of the theorem, taking (3.6) into account, gives

$$d(\mathbf{x}, m) \leq \left\{ \sum_{t (\neq mkv) = 0}^{m(u+(k-1)v)} \binom{t+r-1}{r-1} \right\} + \binom{mkv + r}{r}$$
$$< m^r \left[(u + (k-1)\,v + r)^r + (kv + r)^r \right].$$

Since the square bracketed term is independent of m and $\mathbf{x}$ is arbitrary we obtain, recalling (1.2), that $\mathrm{Dim}_K A \leq r$.

Establish $\mathrm{Dim}_K A \geq r$. Through the hypothesis of the theorem, there exists $x_1, x_2, \ldots, x_r \in A$, such that : $\bigwedge_{i=1}^{r} dF_{x_i} \neq 0$. Suppose that there exists $a \in A$, such that $\deg F_a > 0$. Then for some positive integer l, $\deg F_{a^l x_i} > 0$, for all i. Furthermore $F_{a^l x_i} = F_{a^l} F_{x_i}$, by (3.3), and since $\deg F_{a^l} \neq 0$, we obtain $\bigwedge_{i=1}^{r} dF_{a^l x_i} \neq 0$, through Lemma 2.4.1. Choose positive integers t_i, so that $t_i \deg F_{a^l x_i}$ is independent of i. Set $y_i = (a^l x_i)^{t_i}$. Then $y_i \in A$, and

$$\bigwedge_{i=1}^{r} dF_{y_i} \neq 0 \qquad \text{and} \qquad \deg F_{y_i} = \deg F_{y_j}, \tag{3.7}$$

for all i, j. Through (3.1)-(3.3), (3.7) and Lemma 3.1, it is easily established that the monomials $\{ y_1^{k_1}, y_2^{k_2}, \ldots, y_n^{k_n} : k_i$ non-negative integers $\}$ are linearly independent. Set $\mathbf{y} = (y_1, y_2, \ldots, y_r)$. Then $d(\mathbf{y}, m) \geqq \binom{m+r}{r} > m^r$, which implies : $\mathrm{Dim}_K A \geqq r$.

A similar argument applies given $a \in A$, with $\deg F_a < 0$. Otherwise we may assume $\deg F_{x_i} = 0$, for all i. Set $\mathbf{x} = (x_1, x_2, \ldots, x_r)$. Then by (3.1)-(3.3) and Lemma 3.1, it follows that $d(\mathbf{x}, m) \geqq \binom{m+r}{r} > m^r$. Hence $\mathrm{Dim}_K A \geqq r$, and the theorem is proved.

Let A, B be subalgebras of R_n, with $A \subset B$. Define

$$C(A, B) = \{ x \in B : xy = yx, \text{ for all } y \in A \}.$$

Combining Lemma 2.1, Corollary 3.2 and Theorem 3.3, we obtain the following generalization of Theorem 1.1.

THEOREM 3.4. — *Let* B *be a subalgebra of* R_n *with nilpotent quotient. Let* A *be a subalgebra of* B. *Then*

$$\mathrm{Dim}_K A + \mathrm{Dim}_K C(A, B) \leqq 2n. \tag{3.8}$$

It is false in general that the right hand side of (3.8) can be replaced by $\mathrm{Dim}_K B$. For example, let A, B be generated by q_1, On the other hand, it is a simple consequence of Theorem 3.3, that $\mathrm{Dim}_K B_n = 2n$ (*cf.* Introduction, Example 1), so when $B = B_n$, $\mathrm{Dim}_K B$ may replace the right hand side of (3.8).

It is false in general that we can assert equality in (3.8), except when $n = 1$ and $B = A_1$ or B_1. Indeed through Theorem 3.3, a necessary (though not obviously sufficient) condition for this to hold is the following. Namely that there exist a subspace N_s of $R(X^{2n})$ in involution with the M_r of the hypothesis of Theorem 3.3, such that $s + r = 2n$. When $K = \mathbf{C}$, $n = 2$, $B = A_2$ or B_2 and A is generated by $q_1 p_1 + \alpha q_2 p_2$ (α irrational), this condition is excluded by Lemma 2.2. Indeed it is not hard to show that C (A, B) is generated by $q_1 p_1$, $q_2 p_2$, by direct computation, so that $\mathrm{Dim}_K C(A, B) = 2$. When α is rational, say $\alpha = \frac{m}{n}$, then $q_1^m p_2^n \in C(A, B)$. Then by Theorems 3.3 and 3.4, $\mathrm{Dim}_K C(A, B) = 3$.

We close this section with a partial characterization of algebras with nilpotent quotient.

An element $a \in A_n$ is said to be strictly nilpotent if for each $b \in A_n$, there exists a positive integer k such that $(\mathrm{ad}^k a)\, b = 0$. Let G_n denote the group of all automorphisms of A_n. When $n = 1$, we have [10], Theorem 9.1.

THEOREM 3.5 (Dixmier). — *Let* $a \in A_1$. *Given a strictly nilpotent, there exists* $\varphi \in G_1$, *such that* $\varphi(a) \in P_1$.

Let φ be an endomorphism of A_1. We have : $\ker \varphi = \{0\}$ ([6], Prop. 2.1). Let φ^* denote the extension of φ to an endomorphism of R_1 defined through :

$$\varphi^*(z) = \varphi(a)^{-1}\varphi(b) \quad (z \in R_1), \qquad z = a^{-1}b \quad (a, b \in A_1).$$

THEOREM 3.6. — *Let* $B \subset R_1$ *have nilpotent quotient. Given* $B \supset A_1$, *there exists* $\varphi \in G_1$, *such that* $\varphi^*(B) \subset B_1$.

Proof. — We may suppose that $B \neq A_1$, so there exists $x \in B$, $x \notin A_1$. Consider the 3-tuple (x, q, p). Through Definition 1.2, there exists $a \in A_1$, such that $ax \in A_1$ and $(\mathrm{ad}^k a)\,q = (\mathrm{ad}^k a)\,p = 0$, for some positive integer k. Since q, p generate A_1, it follows that a is strictly nilpotent. Hence by Theorem 3.5, there exists $\varphi \in G_1$, such that $\varphi(a) \in P_1$. We may assume $\varphi(a) = q$. Of $\varphi^*(B) \not\subset B_1$, there exists $y \in B$, such that $\varphi^*(y) \notin B_1$. Consider the 4-tuple (x, y, q, p). As before there exists $b \in A_1$, such that bx, $by \in A_1$ and $\theta(b) \in P_1$ for some $\theta \in G_1$. We show that $\varphi(b) \in P_1$, which contradicts the choice of y.

By choice of φ we may write

$$\varphi^*(x) = \sum_{m=-1}^{r} q^m F_m(p) \qquad (F_{-1} \neq 0) \tag{3.9}$$

where the F_m are polynomial. Since b, $bx \in A_1$, we have

$$\varphi(b)\,\varphi^*(x) = \varphi(bx) \in A_1,$$

which on substitution in (3.9) gives

$$\varphi(b)\left(\sum_{m=-1}^{r} q^m F_m(p)\right) \in A_1. \tag{3.10}$$

Recalling (1.1) it follows, by step-wise removal of top order terms from the left hand side of (3.10), that $\varphi(b)\,q^{-1} \in A_1$. Set $\varphi(b)\,q^{-1} = c$ and $\theta\varphi^{-1} = \psi$. Then $\psi \in G_1$ and by choice of θ,

$$\psi(c)\,\psi(q) = \psi(cq) = \theta(b) \in P_1.$$

Hence $\psi(c)$, $\psi(q) \in P_1$. Through knowledge [10] of G_1, it is readily verified that $\psi(q) \in P_1$ implies that $\psi^{-1}(q) \in P_1$. Hence $c \in P_1$, which implies that $\varphi(b) \in P_1$ as required.

This theorem does not provide a complete characterization of subalgebras of R_1 with nilpotent quotient. Thus Example 2 of the introduction violates its conclusion.

4. FURTHER REMARKS

Let C_n denote the subalgebra of R_n over K generated by q_i, q_i^{-1}, p_j, p_j^{-1} $(i, j, = 1, 2, \ldots, n)$. C_n does not have nilpotent quotient, nor does C_n contain B_n. We have shown that $\mathrm{Dim}_K C_n = 3n$. We conjecture that Theorem 1.1 applies to C_n with $2n$ replaced by $3n$ in the right hand side of (1.4), and that $\mathrm{Dim}_K A \leqq n$, given A a commutative subalgebra of C_n. We have verified these statements for $n = 1$. For $n > 1$, we have not yet been able to resolve an additional technicality which arises and the best estimate we can so far make for the right hand side of (1.4) is $4n - 2$ $(n > 1)$.

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