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## Charged dust spheres in general relativity

by

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RÉSUMÉ. — Les équations d'Einstein-Maxwell sont étudiées afin d'en déduire une solution pour une distribution non-statique, en symétrie sphérique, de poussière chargée (à conductivité nulle), et la solution générale est présentée sous une formulation implicite. Les solutions sont raccordées avec la solution de Reissner-Nordström, et les équations gouvernant l'effondrement sont examinées. On utilise alors les résultats afin de déduire une solution intérieure générale pour une sphère statique de poussière chargée, en coordonnées de courbure.

### 1. INTRODUCTION

In 1968, A. Hamoui [1] presented two particular new solutions of the Einstein-Maxwell equations corresponding to a non-static spherically symmetric distribution of charged dust. In this paper the general solution is presented in an implicit form.

In section 2 Maxwell's equations are used to obtain an expression for the ratio of charge to mass densities. This ratio is found to be a function of the comoving coordinate  $r$  only. The ratio is then used in section 3 to express the metric coefficients in terms of the curvature distance  $R$  and three arbitrary functions of  $r$ . The equation of motion of matter is obtained and is shown to reduce to Tolman's equation [2] in the absence of charge. In section 4 the solutions are matched over a boundary to the Reissner-Nordström solution. The mass defect of a charged sphere is shown to be given by an arbitrary function of  $r$  as in the case of pure dust.

The analysis of the external gravitational field of a charged spherical body by J. Graves and D. Brill [3] showed that the Reissner-Nordström metric has an oscillatory character. The collapse of uniformly charged spheres has been investigated by V. de la Cruz and W. Israël [4] and

by I. Novikov [5]. They found that after each shell of matter crossed its inner horizon it can avoid a central singularity by re-expanding into a region of space-time different from the one in which the collapse originated. In section 5, I examine the general solution to find the conditions on the parameters that will allow a gravitational bounce for a particular comoving layer.

Finally, in section 6, I use the results to obtain a general static pressure-free interior solution (*in* curvature coordinates) for a Reissner-Nordström particle. The solution includes arbitrary  $\frac{e}{m}$  ratio although only solutions with  $e^2 = m^2$  are free of singularities.

## 2. THE CHARGE TO MASS DENSITY RATIO

In comoving coordinates the most general form for a non-static line element representing a spherically symmetric distribution of matter is

$$(2.1) \quad ds^2 = e^\gamma dt^2 - e^\alpha dr^2 - R^2 (d\theta^2 + \sin^2 \theta d\varphi^2),$$

where  $\alpha$ ,  $\gamma$  and  $R$  are functions only of  $t$  and of the comoving coordinate  $r$ . It will be assumed that  $R(r, t) > 0$  for all  $r \neq 0$ .

The four velocity of matter is the unit timelike vector

$$(2.2) \quad u^a = e^{-\frac{\gamma}{2}} \delta_4^a.$$

The energy-momentum tensor for charged dust is

$$(2.3) \quad T^a_b = \rho u^a u_b + E^a_b,$$

where  $\rho$  is the energy density and the Maxwell tensor  $E_{ab}$  is defined in terms of the skew tensor  $F_{ab}$  by

$$(2.4) \quad 4 \pi E_{ab} = \frac{1}{4} g_{ab} F_{hk} F^{hk} + F_{ak} F^k_b,$$

while  $F_{ab}$  satisfies the equations <sup>(1)</sup> :

$$(2.5) \quad F_{ab,c} + F_{ca,b} + F_{bc,a} = 0$$

and

$$(2.6) \quad F^{ka}_{;a} \equiv \frac{1}{\sqrt{-g}} (\sqrt{-g} F^{ka})_{;a} = 4 \pi J^k,$$

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<sup>(1)</sup> Commas and semi-colons denote partial and covariant differentiation, while dots and dashes denote partial differentiation with respect to  $t$  and  $r$  respectively.

where  $J^k = \varepsilon u^k$  is the convection current,  $\varepsilon$  being the density of electric charge. In the case of spherical symmetry the only non-zero component of  $F_{ab}$  is  $F_{14} = F_{14}(r, t)$ .

$k = 1$  in equation (2.6) gives

$$(2.7) \quad F^{41} = E(r) R^{-2} e^{-\frac{\alpha+\gamma}{2}},$$

where  $E$  is an arbitrary function of  $r$ , while  $k = 4$  implies that

$$(2.8) \quad E' = 4 \pi \varepsilon R^2 e^{\frac{\alpha}{2}},$$

which expresses the conservation of charge inside a sphere comoving with the fluid.

The conservation of the energy-momentum tensor and the relationship  $E^{ab}{}_{;a} = F^{bc} J_c$  give [6] :

$$(2.9) \quad (\rho u^a)_{;a} = 0$$

and

$$(2.10) \quad \rho u^b{}_{;a} u^a = -\varepsilon F^{bc} u_c.$$

Equation (2.10) with  $b = 1$  gives

$$(2.11) \quad \rho R^2 \gamma' = 2 \varepsilon E(r) e^{\frac{\alpha}{2}}$$

and (2.9) gives, on integration,

$$(2.12) \quad 4 \pi \rho R^2 e^{\frac{\alpha}{2}} = M'.$$

$M'$  being an arbitrary function of  $r$ .  $M$  is then, by definition, the invariant mass contained within coordinate radius  $r$ . We now have by (2.8) and (2.12) :

$$(2.13) \quad \frac{\varepsilon}{\rho} = \frac{E'}{M'} = K(r),$$

so that the ratio of charge to mass densities is a function of  $r$  only.

### 3. THE FIELD EQUATIONS

The non-trivial field equations for the metric (2.1) are :

$$(3.1) \quad 8 \pi T_1^1 = \frac{E^2}{R^4} = \left\{ \frac{2\ddot{R}}{R} - \frac{\dot{\gamma}\dot{R}}{R} + \left( \frac{\dot{R}}{R} \right)^2 \right\} \\ \times e^{-\gamma} - \left\{ \frac{\gamma' R'}{R} + \left( \frac{R'}{R} \right)^2 \right\} e^{-\alpha} + \frac{1}{R^2},$$

$$(3.2) \quad 8 \pi T_2^2 = 8 \pi T_3^3 = -\frac{E^2}{R^4} = \left\{ \frac{\ddot{\alpha}}{2} + \frac{\dot{\alpha}}{4}(\dot{\alpha} - \dot{\gamma}) + \frac{\ddot{R}}{R} + \frac{\dot{R}}{2R}(\dot{\alpha} - \dot{\gamma}) \right\} e^{-\gamma} \\ - \left\{ \frac{\gamma''}{2} + \frac{\gamma'}{4}(\gamma' - \alpha') + \frac{R''}{R} + \frac{R'}{2R}(\gamma' - \alpha') \right\} e^{-\gamma},$$

$$(3.3) \quad 8 \pi T_4^4 = 8 \pi \rho + \frac{E_2}{R^4} \\ = \left\{ \frac{\dot{\alpha} \dot{R}}{R} + \left( \frac{\dot{R}}{R} \right)^2 \right\} e^{-\gamma} - \left\{ \frac{2R''}{R} - \frac{\alpha' R'}{R} + \left( \frac{R'}{R} \right) \right\} e^{-\alpha} + \frac{1}{R^2},$$

$$(3.4) \quad 8 \pi T_4^1 = -8 \pi T_1^4 = \left\{ \frac{2\dot{R}'}{R} - \frac{\gamma' \dot{R} + \dot{\alpha} R'}{R} \right\} e^{-\alpha} = 0.$$

The last equation may be rewritten in the form

$$(3.5) \quad 2\dot{R}' = \gamma' \dot{R} + \dot{\alpha} R'.$$

Now, using (2.11) and (2.13), this field equation becomes

$$(3.6) \quad 2\dot{R}' = \frac{2KE}{R^2} e^{\frac{\alpha}{2}} \dot{R} + \dot{\alpha} R',$$

which may be integrated to give

$$(3.7) \quad e^{\frac{\alpha}{2}} = \frac{RR'}{\Gamma R - KE},$$

where  $\Gamma(r)$  is an arbitrary function of integration.  $\gamma$  is then obtained from (3.7), (2.11) and (2.13) :

$$(3.8) \quad e^{\gamma} = \frac{e^{2\chi}}{R^2},$$

where

$$(3.9) \quad \chi = \int \frac{\Gamma R'}{\Gamma R - KE} dr.$$

Equation (3.1) can now be integrated to give

$$(3.10) \quad e^{-\gamma} \dot{R}^2 = \Gamma^2 - 1 + \frac{2F}{R} - \frac{(1 - K^2) E^2}{R^2},$$

where  $F = F(r)$  is again a constant of integration. This equation reduces to Tolman's equation when no electric forces are present ( $E = 0$ ) [2].

The density is given by (3.3) :

$$(3.11) \quad 4 \pi \rho = \frac{(F + KE \Gamma)'}{R^2 R'} - \frac{EE'}{R^3 R'}.$$

An alternative expression for  $\rho$  may be obtained from (2.12) and (3.7) :

$$(3.12) \quad 4 \pi \rho = \frac{\Gamma M'}{R^2 R'} - \frac{EE'}{R^3 R'}$$

Comparison of these equations shows that the five functions of  $r$  ( $F, K, E, \Gamma, M$ ) are related by

$$(3.13) \quad (F + KE \Gamma)' = \Gamma M',$$

as well as (2.13). Thus the solutions depend on only three independent arbitrary functions of  $r$ .

The remaining field equation (3.2) is identically satisfied. In fact it can be shown in general that, given spherical symmetry and comoving coordinates, the  $T^2_2$  equation will be automatically satisfied as a consequence of the Bianchi identities if the  $T^1_1, T^4_4$  and  $T^4_4$  equations are.

The cosmological constant  $\Lambda$  could have been included in the analysis. This would only have changed (3.10) — the term  $+\frac{\Lambda R^2}{3}$  would have to be added to the right hand side of this equation.

#### 4. THE BOUNDARY CONDITIONS

If we consider a charged dust sphere of radius  $r = r_b = \text{const.}$  then the exterior field is given by the Reissner-Nordström metric

$$(4.1) \quad ds^2 = h d\bar{t}^2 - h^{-1} d\bar{r}^2 - \bar{r}^2 (d\theta^2 + \sin^2 \theta d\varphi^2)$$

with

$$(4.2) \quad h = 1 - \frac{2m}{\bar{r}} + \frac{e^2}{\bar{r}^2},$$

where  $m$  and  $e$  are the gravitational mass and total charge of the sphere.

I use Darmois' conditions [7] at the boundary B which require that the first and second fundamental forms should be the same whether obtained from the interior or exterior metrics. Equivalence of the  $g_{22}$ 's on B imply that on B  $\bar{r} = R(r_b, t)$  while the  $g_{44}$ 's give, on B,

$$(4.3) \quad h \dot{\bar{t}}^2 - h^{-1} \dot{\bar{r}}^2 = e^\gamma.$$

The equivalence of the second fundamental forms on B then gives the equation of motion of the boundary :

$$(4.4) \quad e^{-\gamma} \dot{R}^2 = - \left( 1 - \frac{2m}{R} + \frac{e^2}{R^2} \right) + R'^2 e^{-\alpha},$$

where all functions are calculated at  $r = r_b$ . With the help of (3.7), this can be rewritten as

$$(4.5) \quad e^{-\gamma} \dot{R}^2 = - \left( 1 - \frac{2m}{R} + \frac{e^2}{R^2} \right) + \left( \Gamma - \frac{KE}{R} \right)^2.$$

Using (3.10) this implies that

$$(4.6) \quad e = E_{(r_b)}$$

and

$$(4.7) \quad m = (F + KE \Gamma)_{(r_b)}.$$

Since the gravitational mass reduces to  $F_{(r_b)}$  in the cases where the dust is uncharged ( $K \equiv 0$ ) and when the total electric charge is zero, it seems reasonable to assume that  $F$  is nonnegative as is done in section 5.

We are now in a position to interpret equation (3.13) — it gives the relationship between the active gravitational mass  $m$  and the invariant mass  $M$  :

$$(4.8) \quad m' = \Gamma M'$$

and only in the case where  $\Gamma = 1$  can the two be equal. This expression also holds for uncharged dust where  $\Gamma$  determines not only the mass defect but also the total energy of the system and the geometry of 3-spaces  $t = \text{const.}$  [8]. These interpretations are not possible here.

It can be shown that regularity [9] at the centre  $r = R(0, t) = 0$  requires that  $F(0) = E(0) = 0$ ,  $\Gamma^2(0) = 1$  and  $K(0) = \text{const.}$

## 5. GRAVITATIONAL COLLAPSE

In this section the solutions are examined to find the conditions necessary for a particular comoving particle (with  $F \geq 0$ ) to avoid a central singularity. The differential operator  $D_t \equiv e^{-\frac{\gamma}{2}} \frac{\partial}{\partial t}$  is introduced so that  $D_t R$  is then the proper velocity of the fluid [10]. The equation of motion for the interior (3.10) can now be rewritten as

$$(5.1) \quad R^2 (D_t R)^2 = f R^2 + 2FR - (1 - K^2) E^2,$$

where  $f = \Gamma^2 - 1$ . When the proper velocity is zero (5.1) will then give two positive finite roots if and only if

$$(5.2) \quad f < 0, \quad K^2 < 1 \quad \text{and} \quad F^2 < f(K^2 - 1) E^2.$$

The comoving particle will then oscillate between these two values of the curvature distance. The only circumstances other than (5.2) under which the collapse of a comoving particle will not result in a singu-

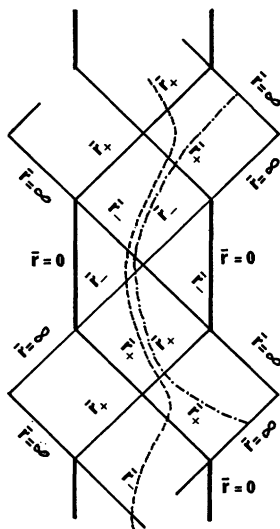
larity at  $R = 0$  is when either

$$(5.3) \quad f = 0 \quad \text{and} \quad K^2 < 1,$$

or

$$(5.4) \quad F = 0, \quad f > 0 \quad \text{and} \quad K^2 < 1.$$

These particles, falling inward, come to rest at a certain minimum radius and then rebound out to infinity.



Cross-sections  $\theta = \text{const.}, \varphi = \text{const.}$  of the extension of the Reissner-Nordström metric in the case  $0 < e^2 < m^2$ .

Heavy lines represent irremovable singularities.

----- and - - - - represent the history of oscillating and "bouncing" comoving particles respectively.

It is interesting to note that I. Novikov [5] has shown that the matter density of any uncharged layers ( $K = 0$ ) will become infinite ( $R' = 0$ ) during the collapse due to the crossing of dust particles.

It is well known, however, that if  $e^2 \leq m^2$  an external observer sees the surface of the sphere collapse asymptotically on to the gravitational radius  $\bar{r}_+ = m + (m^2 - e^2)^{1/2}$ , thus an external observer never sees the re-expanding sphere. The paradox is resolved by the extension of the Reissner-Nordström manifold given by J. Graves and D. Brill [3] for  $e^2 < m^2$  and by Carter [11] for  $e^2 = m^2$ . After crossing its inner horizon,  $\bar{r}_- = m - (m^2 - e^2)^{1/2}$ , the sphere bounces and re-expands into another asymptotically flat region of space-time different from the one in which the collapse originated. This can be shown using (4.5)



if  $e^2 \leq m^2$ . It then follows from this equation that the comoving particle (5.2) oscillates between a maximum greater than  $\bar{r}_+$  and a minimum smaller than  $\bar{r}_-$  while the minimum radius attained by the particles (5.2) (5.4) is less than  $\bar{r}_-$  (see *fig.*). Normal oscillations are possible if  $e^2 > m^2$  for then the metric (4.1) can be used throughout the whole of the exterior space time.

## 6. STATIC SOLUTIONS

If we consider the case in which  $R$  is a function of  $r$  only, the metric will then be static.  $\ddot{R} = \dot{R} = 0$  implies that the arbitrary functions of  $r$  must further satisfy the equation  $F^2 = f(K^2 - 1)E^2$ . However, regularity now requires that  $F \equiv 0$  and it then follows from  $\ddot{R} = \dot{R} = 0$  that  $K^2 = \Gamma^2 = 1$ . We now have, from (4.6) and (4.7), that  $\frac{e}{m} = \pm 1$  as it should be for a static solution free of singularities ([12], [13]).

A change of coordinates ( $R = r$ ) can now be made so that the singularity-free interior solution becomes, from (3.7) and (3.9),

$$(6.1) \quad ds^2 = e^\gamma dt^2 - \left(1 - \frac{m(r)}{r}\right)^{-2} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2),$$

where

$$(6.2) \quad \gamma = 2 \int \frac{m(r) dr}{r^2 \left(1 - \frac{m}{r}\right)},$$

$m(r)$  being an arbitrary function of  $r$  such that the Reissner-Nordström parameter is  $m(r_b)$ . The density of matter is given by

$$(6.3) \quad 4 \pi \rho = \frac{m'}{r^2} \left(1 - \frac{m}{r}\right).$$

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