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Section A :
Physique théorique.

Local perturbations and approach to equilibrium

by

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ABSTRACT. — For the Fermi Lattice system locally perturbed equilibrium states are studied and conditions are found for approach to equilibrium within an order $t^{-\alpha}$ where $\alpha = n$, or $\frac{1}{n}$; n is a positive integer.

1. INTRODUCTION

We study the approach to equilibrium of a perturbed state by a local perturbation. We start with a solvable model (bilinear Hamiltonian) of a Fermi system. Its equilibrium state is well-defined. Then we apply a local perturbation. The first problem is to establish the perturbed equilibrium state (theorem 1 below). Then for those states which tend to equilibrium (in particular for ergodic states) we study how fast they tend to equilibrium. Sufficient conditions are given for an approach to equilibrium faster then $t^{-\alpha}$ where $\alpha = \frac{1}{n}$ or $\alpha = n$; n is a positive integer.

Technically we use the algebraic set-up and start with the C*-Clifford algebra $\mathcal{A} \equiv \overline{\mathcal{A}(\mathrm{H}, s)}$ (*see* e. g. [1]) built on a real Hilbert space (H, s); it is also the smallest C*-algebra generated by the set of elements $\{ \mathrm{B}(\Psi) \mid \Psi \in \mathrm{H} \}$, where B is a real linear map of H into $\mathcal{A}$ satisfying the anticommutation relations

$$[\mathrm{B}(\Psi), \mathrm{B}(\varphi)]_+ = 2\, s(\Psi \mid \varphi).\mathbf{1}$$

$\mathbf{1}$ is the unit element of $\mathcal{A}$.

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In particular we are interested in the description of the Fermi lattice system (F. L. S.). Let $\mathbf{Z}$ be the set of all positive and negative integers, and $\{ \varphi_n \}_{n \in \mathbf{Z}}$ an orthonormal basis of H. For any finite subset Λ of $\mathbf{Z}$ consider the C*-subalgebra $\mathcal{A}(\Lambda)$ of $\mathcal{A}$ generated by the set $\{ B_k \equiv B(\varphi_k) \mid k \in \Lambda \}$; then $\mathcal{A}$ is the norm closure of $\bigcup_{\Lambda \in \mathbf{Z}} \mathcal{A}(\Lambda)$ and can be considered as the algebra of quasi-local observables of the lattice.

Let $\sigma : a \in \mathbf{Z} \to \sigma_a$ be a mapping of the group $\mathbf{Z}$ of lattice translations into the $\bigstar$-automorphisms σ_a of $\mathcal{A}$ defined by $\sigma_a B_k = B_{k+a}$.

On the other hand, consider a time translations induced by the following locally defined bilinear Hamiltonians : for each $\Lambda \subset \mathbf{Z}$, the hamiltonian $H_{0\Lambda}$ is given by

$$H_{0\Lambda} = \sum_{i, j \in \Lambda} v_{ij} B_i B_j \tag{1}$$

the c-numbers v_{ij} satisfy :

(i) $v_{ij} = \bar{v}_{ji}$; $i, j \in \mathbf{Z}$;

(ii) $\sum_{i \in \mathbf{Z}} | v_{ij} | < \infty$; $j \in \mathbf{Z}$

(iii) $v_{i+k, j+k} = v_{i, j}$ for $i, j \in \mathbf{Z}$, $k \in \mathbf{Z}$.

The details of the time evolution automorphisms will be given in the next section.

2. THE EQUILIBRIUM STATES

Define $\alpha_z^{0\Lambda}$ by $\alpha_z^{0\Lambda}(x) = e^{izH_{0\Lambda}} x e^{-izH_{0\Lambda}}$ for $x \in \mathcal{A}$ $(z \in \mathbf{C})$ (complex numbers). The set $\{ \alpha_z^{0\Lambda} \mid z \in \mathbf{C} \}$ is a group of automorphisms of $\mathcal{A}$ ($\bigstar$-automorphisms if $\operatorname{Im} z = 0$).

If $\{ \Lambda_n \}_n$ is a sequence of finite subsets of $\mathbf{Z}$ tending to infinity in the sense that is contains every finite subset then for each $t \in R$ (real numbers), the sequence $\{ \alpha_t^{0\Lambda_n} \}_n$ tends strongly to a $\bigstar$-automorphism α_t^0 of $\mathcal{A}$ defining a strongly continuous one-parameter group of automorphisms $t \in R \to$ aut. $\mathcal{A}$ ([2], Th. 7.6.2).

LEMMA 1. — *The set of automorphisms* $\{ \alpha_z^{0\Lambda} \mid z \in \mathbf{C} \}$ *tends to a group of automorphisms* $\{ \alpha_z^0 \mid z \in \mathbf{C} \}$ *densily defined on* $\mathcal{A}$ *as* Λ *tends to infinity and*

$$\| \alpha_z^0 (B_k) \| \leq \exp (| z | C),$$

where

$$C = \sum_{l \in \mathbf{Z}} | v_{l0} - v_{0l} |.$$

Proof. — For $k \in \Lambda$, where Λ is some finite subset of **Z**, consider

$$\alpha_z^{0\Lambda}(B_k) = B_k + \frac{-z}{1!} B(D_\Lambda \varphi_k) + \ldots + \frac{(-z)^n}{n!} B(D_\Lambda^n \varphi_k) + \ldots,$$

where D_Λ is an antisymmetric operator on H defined by

$$(2) \qquad \begin{aligned} s(\varphi_k \mid D_\Lambda \varphi_l) &= i(v_{lk} - v_{kl}) && \text{for } k, l \in \Lambda, \\ &= 0 && \text{for } k \text{ or } l \notin \Lambda. \end{aligned}$$

Note $D \equiv D_Z$.

An immediate calculation shows that for all Λ,

$$\| \alpha_z^{0\Lambda}(B_k) \| \leq e^{|z|C}, \qquad \text{with} \quad C = \sum_{l \in Z} v_{l0} - v_{0l}.$$

The existence of the limit : $\lim_{\Lambda \to \infty} \alpha_z^{0\Lambda}(B_k) \equiv \alpha_z^0(B_k)$ follows from the fact that $B(D_\Lambda^n \varphi_k)$ tends in norm to $B(D^n \varphi_k)$ as Λ tends to infinity because $\| B(D_\Lambda^n \varphi_k) \|^2 = s(D_\Lambda^n \varphi_k \mid D_\Lambda^n \varphi_k)$ and that $\alpha_z^{0\Lambda}(B_k)$ is the limit, uniform in Λ, of the sequence $\{ f_\Lambda^N(z) \}_N$ with

$$f_\Lambda^N(z) = B_k + \sum_{l=1}^{N} \frac{(-z)^2}{l!} B(D_\Lambda^l \varphi_k).$$

Q. E. D.

Now we consider perturbations of the free (bilinear) Hamiltonian defined in (1). In particular consider the locally defined Hamiltonian :

$$(3) \qquad H_\Lambda = H_{0\Lambda} + V_n,$$

where $H_{0\Lambda}$ is defined in (1) and V_n is any monomial of order n in the B_i :

$$V_n = B_{i_1} B_{i_2} \ldots B_{i_n}.$$

From [2] (Th. 7.6.2) it is clear that also the map

$$\alpha_t^\Lambda : \quad x \in \mathcal{A} \; e^{itH_\Lambda} x e^{-itH_\Lambda}$$

tends to a $\star$-automorphism α_t of $\mathcal{A}$ as Λ tends to infinity.

LEMMA 2 :

(*a*) *For each finite subset* $\Lambda \subset Z$ *and any* $\beta \in R$, $\exp(-\beta H_\Lambda)$ *is an element of* $\mathcal{A}$ *and can be written in the form*

$$e^{-\beta H_\Lambda} = e^{-\beta H_{0\Lambda}} T_{\Lambda,\beta},$$

where

$$T_{\Lambda,\beta} = 1 + \sum_{k=1}^{\infty} \int_0^\beta d\beta_1 \int_0^{\beta_1} d\beta_2 \ldots \int_0^{\beta_{k-1}} d\beta_k \prod_{l=1}^{k} \alpha_{i\beta_l}^{0\Lambda}(V_n).$$

(*b*) *If* Λ *tends to infinity,* $T_{\Lambda,\beta}$ $(T_{\Lambda,\beta}^{-1})$ *tends to an element* T_β (T_β^{-1}) *of the algebra* $\mathcal{A}$, *such taht*

$$T_\beta = 1 + \sum_{k=1}^{\infty} \int_0^{\beta} d\beta_1 \int_0^{\beta_2} d\beta_2 \ldots \int_0^{\beta_{k-1}} d\beta_k \prod_{l=1}^{k} \alpha_{i\beta_l}^0 (V_n).$$

(*c*) *The group of* $\star$*-automorphisms* $\{ \alpha_t \mid t \in \mathbb{R} \}$ *extends to a group of automorphism (not* $\star$*-automorphisms).* $\{ \alpha_{i\beta} \mid \beta \in \mathbb{R} \}$ *densily defined on* $\mathcal{A}$ *such that*

$$\alpha_{i\beta}(.) = \alpha_{i\beta}^0 (T_\beta \, . \, T_\beta^{-1}).$$

Proof. — (*a*) is proved in [4], Appendix I.

To prove (*b*) remark first that

$$\left\| \int_0^{\beta} d\beta_1 \int_0^{\beta_2} d\beta_2 \ldots \int_0^{\beta_{k-1}} d\beta_{k-1} \prod_{l=1}^{k} \alpha_{i\beta_l}^{0\,\Lambda} (V_n) \right\| \leq \frac{|\beta|^k}{k!} (e^{|\beta| \mathbf{C}})^{nk}$$

and

$$T_{\Lambda,\beta}^{-1} = T_{\Lambda,\beta}^{*}.$$

Hence for all Λ :

$$\| T_{\Lambda,\beta} \| \leq \exp [\, |\beta| \, (e^{|\beta| \mathbf{C}})^n],$$
$$\| T_{\Lambda,\beta}^{-1} \| \leq \exp [\, |\beta| \, (e^{|\beta| \mathbf{C}})^n].$$

To prove the existence of the limit T_β in $\mathcal{A}$ it is sufficient to remark that $T_{\Lambda,\beta}$ is the norm limit, uniform in Λ of the sequence $\{ g_\Lambda^N \}_N$,

$$g_\Lambda^N = 1 + \sum_{k=1}^{N} \int_0^{\beta} d\beta_1 \int_0^{\beta_2} d\beta_2 \ldots \int_0^{\beta_{k-1}} d\beta_k \prod_{l=1}^{k} \alpha_{i\beta_l}^{0\,\Lambda} (V_n)$$

and that by lemma 1, each of the elements g_Λ^N has a limit if Λ tends to infinity. An analogous argument establishes the limit T_β^{-1} in $\mathcal{A}$.

Using analogous arguments to derive (*a*) (*see* [4]) with an imaginary variable (— *it*) instead of the real variable β, we get

$$\alpha_t^\Lambda (x) = \alpha_t^{0\,\Lambda} (T_{\Lambda,-it} \, x \, T_{\Lambda,-it}^{-1}), \qquad x \in \mathcal{A}.$$

As in (*b*), $T_{\Lambda,-it}$ tends to an element T_{-it} of the algebra if Λ tends to infinity. Then trivially

$$\alpha_t (x) = \alpha_t^0 (T_{-it} \, x \, T_{-it}^{-1}), \qquad x \in \mathcal{A}.$$

The proof of (*c*) is complete if we prove that this expression can be extended to a purely imaginary variable. For this it follows from (*b*) that $\alpha^0_{i\beta}(T_\beta)$ exists, hence on all elements $x \in \mathfrak{A}$ in the domain of $\alpha^0_{i\beta}$, $\alpha_{i\beta}$ is defined and given by

$$\alpha_{i\beta}(x) = \alpha^0_{i\beta}(T_\beta \, x \, T_\beta^{-1})), \qquad x \in \mathfrak{A}.$$

Q. E. D.

In the F. L. S. for any evolution $t \in R \to \alpha^0_t : \alpha^0_t(B(\varphi_k)) = B(e^{Dt}\varphi_k)$ there exists a unique quasi-free state on $\mathfrak{A}$, satisfying the Kubo-Martin-Schwinger boundary condition with respect to the evolution α^0_t [5]. The quasi-free state is uniquely determined as is well known *see* e. g. [1] by an antisymmetric operator A on H given by

$$A = J \frac{e^{\beta|D|} - e^{-\beta|D|}}{e^{\beta|D|} + e^{-\beta|D|}},$$

$$D = J\,|D| \qquad \text{(polar decomposition of D).}$$

This state will be called the equilibrium state for that evolution. Note by ω_0 the equilibrium state linked to the evolution α^0_t induced by the bilinear Hamiltonian given above in (1) and let H_0, $\mathcal{H}_0$, Ω_0 be the G. N. S.-representation, respectively representation space an cyclic vector induced by ω_0.

Now we look for the equilibrium state ω_β linked t_0 the evolution $t \to \alpha_t$, induced by the Hamiltonian (3). We prove that

$$\omega_\beta(.) = \frac{\omega_0(T_\beta\,.)}{\omega_0(T_\beta)} = \frac{(\Omega_0 \mid \Pi_0(T_\beta)\,\Pi_0(.)\,\Omega_0)}{(\Omega_0 \mid \Pi_0(T_\beta)\,\Omega_0)}. \tag{4}$$

LEMMA 3. — *With the notations of above and* T_β, $\beta \in R$ *defined in Lemma* 3, *we have*

$$\Pi_0(T_\beta) = \lambda\, \Pi_0(\alpha^0_{-i\beta/2}(T_{\beta/2})\, T_{\beta/2}),$$

where λ *is a constant.*

Proof. — We prove that :

$$\Pi_0(T_\beta)^{-1}\, \Pi_0(\alpha^0_{-i\beta/2}(T_{\beta/2})\, T_{\beta/2})$$

commutes with a weakly dense subset of $\Pi_0(\mathfrak{A})''$, hence it belongs to the commutant $\Pi_0(\mathfrak{A})'$. Otherwise it follows from Lemma 2 (*b*) that it also belongs to the algebra $\Pi_0(\mathfrak{A})''$, and ω_0 being a factor state [6] the lemma follows.

By lemma 2 (*c*) there exists a dense set of elements $x \in \mathfrak{A}$ such that (for convenience $\beta = 1$) :

$$\Pi_0(\alpha_{i/2}(x)) = \Pi_0(\alpha^0_{i/2}(T_{1/2}\, x\, T_{1/2}^{-1})).$$

Also

$$\alpha_i = \alpha_{i/2} \circ \alpha_{i/2}, \qquad \alpha^0_i = \alpha^0_{i/2} \circ \alpha^0_{i/2},$$

hence

$$\Pi_0(\alpha_i(x)) = \Pi_0(\alpha_i^0(\alpha_{-i/2}^0 T_{1/2}) T_{1/2}\, x\, T_{1/2}^{-1}\, \alpha_{-i/2}^0(T_{i/2}^{-1}))$$

and

$$\begin{aligned}\Pi_0(T_1)\,\Pi_0(x)\,\Pi_0(T_1^{-1}) &= \Pi_0(\alpha_{-i}^0 \circ \alpha_i(x))\\ &= \Pi_0(\alpha_{-i/2}^0(T_{1/2}))\,\Pi_0(T_{1/2})\,\Pi_0(x)\\ &\quad\times \Pi_0(T_{1/2}^{-1})\,\Pi_0(\alpha_{-i/2}^0(T_{1/2}^{-1})).\end{aligned}$$

Therefore

$$\begin{aligned}&\Pi_0(T_1)^{-1}\,\Pi_0(\alpha_{-i/2}^0(T_{1/2}))\,\Pi_0(T_{1/2})\,\Pi_0(x)\\ &\quad = \Pi_0(x)\,\Pi_0(T_1)^{-1}\,\Pi_0(\alpha_{-i/2}^0(T_{1/2}))\,\Pi_0(T_{1/2})\end{aligned}$$

for a dense set of elements $\Pi_0(x)$ of $\Pi_0(\mathfrak{A})''$.

Q. E. D.

LEMMA 4. — *With the same notations as in lemma* 3,

$$\Pi_0(\alpha_{i\beta/2}^0(T_{+\beta/2})) = \mu\,\Pi_0(T_{\beta/2}^*) = \mu\,\Pi_0(T_{-\beta/2})^{-1},$$

where μ *is a constant.*

Proof. — As in the proof of lemma 3 for a dense set of elements $x \in \mathfrak{A}$ (put $\beta = 1$) :

$$\Pi_0(\alpha_{-i/2}(x)) = \Pi_0(\alpha_{-i/2}^0(T_{-1/2}\, x\, T_{-1/2}^{-1}))$$

and

$$\alpha_{i/2} \circ \alpha_{-i/2}(x) = x.$$

Hence

$$\begin{aligned}\Pi_0(x) &= \Pi_0(\alpha_{i/2}^0(T_{1/2}[\alpha_{-i/2}^0(T_{-1/2}\, x\, T_{-1/2}^{-1})]\, T_{1/2}^{-1}))\\ &= \Pi_0(\alpha_{i/2}^0(T_{1/2}))\,\Pi_0(T_{-1/2})\,\Pi_0(x)\,\Pi_0(T_{-1/2})^{-1}\,\Pi_0(\alpha_{i/2}^0(T_{1/2}))^{-1}\end{aligned}$$

and

$$\Pi_0(\alpha_{i/2}^0(T_{1/2}))\,\Pi_0(T_{-1/2}) \in \Pi_0(\mathfrak{A})'' \cap \Pi_0(\mathfrak{A})'.$$

By the factoriality of Π_0 :

$$\mu\,\Pi_0(T_{-1/2})^{-1} = \Pi_0(\alpha_{i/2}^0(T_{1/2})) \qquad \text{with} \quad \mu \in \mathbf{C}.$$

Q. E. D.

THEOREM 1. — *The functional* ω_β *defined by* (4) *on* $\mathfrak{A}$ *is a state satisfying the K. M. S. boundary condition with respect to the evolution* $t \to \alpha_t$ *linked to the Hamiltonian defined in* (3).

Proof. — First we prove that it is a state. By lemma 3 for a dense set of elements $x \in \mathfrak{A}$,

$$\omega_\beta(x) = \frac{\omega_0(\alpha_{-i\beta/2}^0(T_{\beta/2})\, T_{\beta/2}\, x)}{\omega_0(\alpha_{-i\beta/2}^0(T_{\beta/2})\, T_{\beta/2})}.$$

By the invariance of ω_0 for α_t^0 :

$$\omega_\beta(x) = \frac{\omega_0(\alpha^0_{-i\beta/2}(T_{\beta/2})\,\alpha^0_i(T_{\beta/2}\,x))}{\omega_0(T_{\beta/2}\,\alpha^0_{i\beta/2}(T_{\beta/2}))}.$$

By the fact that ω_0 is a K. M. S. state for the evolution $t \to \alpha_t^0$:

$$\omega_\beta(x) = \frac{\omega_0(T_{\beta/2}\,x\,\alpha^0_{i\beta/2}(T_{\beta/2}))}{\omega_0(T_{\beta/2}\,\alpha^0_{i\beta/2}(T_{\beta/2}))}$$

and by lemma 4 :

$$\omega_\beta(x) = \frac{\omega_0(T_{\beta/2}\,x\,T^*_{\beta/2})}{\omega_0(T_{\beta/2}\,T^*_{\beta/2})}$$

hence ω_β is a state.

That ω_β satisfies the K. M. S. condition with respect to the evolution $t \to \alpha_t$ follows from the fact ω_0 is a K. M. S. state for the evolution $t \to \alpha_t^0$ as follows :

$$\begin{aligned}\omega_0(T_\beta\,y\,\alpha_i(x)) &= \omega_0(T_\beta\,y\,\alpha^0_{i\beta}(T_\beta\,x\,T_\beta^{-1}))\\ &= \omega_0(T_\beta\,x\,T_\beta^{-1}\,T_\beta\,y) = \omega_0(T_\beta\,xy)\end{aligned}$$

for all $y \in \mathfrak{A}$ and a dense set of elements $x \in \mathfrak{A}$.

Q. E. D.

3. APPROACH TO EQUILIBRIUM

In this section we consider the evolution and behaviour at infinity of the state

(5) $$\rho_t = \omega_\beta \circ \alpha_t^0$$

this is the free evolution of the perturbated state. Because of the particular form of the equilibrium state ω_β in terms of the state ω_0 (*see* theorem 1), an easy argument shows that the state ρ_t tends pointwise to the original state ω_0 if the latter one is ergodic in the sense that it is an extremal time invariant state.

A necessary and sufficient condition for ergodicity is that the spectrum of the operator D is purely continuous ([3], Th. 1). In what follows we suppose that the evolution α_t^0 satisfies this condition, and we are interested in the question, how fast the state ρ_t tends to ω_0 as t tends to infinity. We do not give a general answer to this question, but only consider the situation that the convergence is faster than any rational power of t.

LEMMA 5. — *For ergodic bilinear interactions as given in* (1), *the approach in the weak sense, for local observables of the perturbed state* ρ_t (5) *to the equilibrium state* ω_0 *is majorized by homogeneous polynomials in the functions*

$$F_{ij}(t) = s(\varphi_i\, e^{Dt}\, \varphi_j)$$

in particular, if $X = B_{i_1} \dots B_{i_{2n}}$, *then*

$$| \rho_t (X) - \omega_0 (X) |$$

tends to zero if t tends to infinity, at least as fast as $F_{ij}(t)^{2m}$.

Proof. — We prove the lemma explicitly for observables which are monomials of order two, i. e. $m = 1$ in the lemma. The generalization to arbitrary order is trivial.

By lemma 2 :

$$\rho_t (B_i B_j) - \omega_0 (B_i B_j) = \frac{1}{\omega_0 (T_\beta)} [\omega_0 (T_\beta B (e^{Dt} \varphi_i) B (e^{Dt} \varphi_j)) - \omega_0 (T_\beta) \omega_0 (B_i B_j)].$$

Using the time invariance of ω_0; in particular :

$$\omega_0 (B (e^{Dt} \varphi_j) B (e^{Dt} \varphi_j)) = \omega_0 (B_i B_j)$$

we have

$$\rho_t (B_i B_j) - \omega_0 (B_i B_j) = \frac{1}{\omega_0 (T_\beta)} \sum_{k=1}^{\infty} \int_0^{-\beta} d\beta_1 \int_0^{\beta_1} d\beta_2 \dots \int_0^{\beta_{k-1}} d\beta_k \times \left[\omega_0 \left(\prod_{l=1}^{k} \alpha^0_{i\beta_l} (V_n) B (e^{Dt} \varphi_i) B (e^{Dt} \varphi_j) \right) - \omega_0 \left(\prod_{l=1}^{k} \alpha^0_{\beta_l} (V_n) \right) \omega_0 (B_i B_j) \right].$$

But

$$\alpha^0_{i\beta_l} (V_n) = \prod_{p=1}^{n} \alpha^0_{i\beta_l} (B_{i_p})$$

hence

$$\rho_t (B_i B_j) - \omega_0 (B_i B_j) = \frac{1}{\omega_0 (T_\beta)} \sum_{k=1}^{\infty} \int_0^{-\beta} d\beta_1 \int_0^{\beta_1} d\beta_2 \dots \int_0^{\beta_{k-1}} d\beta_k \times \left[\omega_0 \left(\prod_{l=1}^{k} \prod_{p=1}^{n} \alpha^0_{i\beta_l} (B_{i_p}) B (e^{Dt} \varphi_i) B (e^{Dt} \varphi_j) \right) - \omega_0 \left(\prod_{l=1}^{k} \prod_{p=1}^{n} \alpha^0_{i\beta_l} (B_{i_p}) \right) \omega_0 (B_i B_j) \right].$$

From lemma 1 :

$$\| \alpha^0_{i\beta_l}(B_{i_p}) \| \leq e^{\beta C}$$

for all l and p.

Suppose for the moment $V_n = B_1 B_2$ and put $\alpha^0_{i\beta_l}(B_s) = G_{2(l-1)+s}$ for $l = 1, \ldots, k$ and $s = 1, 2$, then using the recursion formula for quasi-free states (*see* e. g. [5], Appendix) :

$$\Bigg| \omega_0 \left(\prod_{l=1}^{k} \alpha^0_{i\beta_l}(B_1)\, \alpha^0_{i\beta_l}(B_2)\, B(e^{Dt}\varphi_i)\, B(e^{Dt}\varphi_j) \right) - \omega_0 \left(\prod_{l=1}^{k} \alpha^0_{i\beta_l}(B_1)\, \alpha^0_{i\beta_l}(B_2) \right) \omega_0(B_i B_j) \Bigg|$$

$$\leq \sum_{m=1}^{2k} \sum_{m \neq q} | \omega_0 (G_1 \ldots \hat{G}_m \ldots \hat{G}_q \ldots G_{2k})\, \omega_0(G_m B(e^{D\prime}))\, \omega_0(G_q B(e^{Dt}\varphi_j)) |,$$

where $\hat{G}_m$ and $\hat{G}_q$ stand for G_m and G_q omitted in the product.

The right hand side is majorized by

$$2k(2k-1)\, e^{(2k-2)\beta C} f(t),$$

with

$$f(t) = \max_{m, q \in Z} | \omega_0(G_m B(e^{Dt}\varphi_i))\, \omega_0(G_q B(e^{Dt}\varphi_j)) | \leq e^{2\beta C}.$$

Hence

$$|(\rho_t - \omega_0)(B_t B_j)| \leq \frac{1}{|\omega_0(T_\beta)|} \left(\sum_{k=1}^{\infty} \frac{\beta^k}{k!} 2k(2k-1)\, e^{(2k-2)\beta C} \right) f(t)$$

$$= \frac{1}{|\omega_0(T_\beta)|} \left(\frac{\partial^2}{\partial^2 P} (e^{\beta P^2}) \right) f(t),$$

where $P = e^{2\beta C}$.

In general, for V_n a monomial of order n as in (3) :

$$|(\rho_t - \omega_0)(B_i B_j)| \leq \frac{1}{|\omega_0(T_\beta)|} \left(\frac{\partial^2}{\partial^2 P} (e^{\beta P^n}) \right) f(t).$$

Hence the majorization is dominated by the behaviour of the function

$$t \to \omega_0(\alpha^0_{i\beta}(B_p)\, B(e^{Dt}\varphi_i)).$$

By straight forward calculation, analogous to that in the proof of lemma 1, we get

$$|\omega_0(\alpha^0_{i\beta}(B_p)\, B(e^{Dt}\varphi_j))| \leq |F_{p_1 j}(t)| + \beta \frac{C}{1!} |F_{p_1 j}(t)| + \ldots + \frac{\beta^n C^n}{n!} |F_{p_n j}(t)| + \ldots$$

where $p_1 \neq p$, $p_2 \neq p$, p_1, ...

Remarking that the right hand side is convergent, the lemma follows.

Q. E. D.

From lemma 5, it follows that the approach to equilibrium is reduced to the study of the asymptotic behaviour of the function

$$F_{ij}(t) = s(\varphi_i\, e^{Dt}\, \varphi_j) \qquad (i, j \in Z) \tag{6}$$

as t tends to infinity.

Remark first that because of the translation invariance $F_{ij}(t)$ depends only on the difference $\delta = |j - i|$. It will become clear that the asymptotic behaviour of $F_{ij}(t)$ for large t does even not depend on δ.

Let $a_n = i(v_{n0} - v_{on})$ and

$$g(x) = \sum_{n<0} a_n\, e^{inx} - \sum_{n\geq 0} a_n\, e^{inx} = 2i \sum_{n\geq 0} a_n \sin nx, \tag{7}$$

then for all φ_k elements of the orthonormal basis

$$\{\varphi_j(x) = e^{ijx},\ x \in [0, 2\pi)\}_{j\in Z}$$

of $\mathcal{L}^2([0, 2\pi])$:

$$(D\varphi_j)(x) = g(x)\, \varphi_j(x)$$

and

$$s(\varphi_i\, e^{Dt}\, \varphi_j) = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\delta x}\, e^{tg(x)}\, dx. \tag{8}$$

THEOREM 2. — *Let* H_0 *be a bilinear Hamiltonian* [*see* (1)] *such that the corresponding function* $g(x)$ [*see* (7)] *is at least* K*-times differentiable and also* K *is the maximum order of the stationnary points in the interval* $[0, 2\pi]$, *then the function* $F_{ij}(t)$ [*see* (6)] *tends to zero as* $\frac{1}{t^{1/(1+k)}}$ *if* t *tends to infinity.*

Proof. — The theorem follows immediatly from [7] (p. 52) applied to the integral (8). The result is independent of δ. We remark that the function $g(x)$ being odd, the contribution of the term $\frac{1}{t^{1/(1+k)}}$ does not vanish if $g(x)$ has a stationnary point of order K.

Q. E. D.

THEOREM 3. — *Let* H_0 *be as theorem* 1 *and the function* $g(x)$ *such that there exists a finite sequence* $(x_0, \ldots, x_l)$:

$$0 = x_0 < x_1 < \ldots < x_l = \pi$$

such that :

(i) *The function* $g(x)$ *is monotone and* $(K + 1)$*-times continuously differentiable in the open intervals* (x_i, x_{i+1}) *with* $i = 0, 1, \ldots, l - 1$ *and* $g'(x) \neq 0$ *and* g *continuous.*

(ii) *$g(x)$ has left and right points of contact of order* K *in each x_i, i. e. if g^{-1} is the inverse function of g, then :*

$$\lim_{y \to g(x_i) \pm 0} (g^{-1})'(y) = \lim_{y \to g(x_i) \pm 0} (g^{-1})'' = \ldots = \lim_{y \to g(x_i) \pm 0} (g^{-1})^{(K)}(y) = 0.$$

Then $F_{ij}(t)$ *tends to zero if t tends to infinity as* $\frac{1}{t^{1+K}}$.

Proof. — In each interval $[x_k, x_{k+1}]$ introduce the variable u by $iu = g(x)$ then

$$F_{ij}(t) = \frac{1}{2\pi} \int \Phi(u)\, e^{itu}\, du,$$

where

$$\Phi(u) = e^{-i\delta(g^{-1})(iu)} \frac{dg^{-1}(iu)}{du}.$$

If $iu_j = g(x_j)$; $j = 0, 1, \ldots, l$ then

$$\Phi^{(n)}(u_i) = 0 \qquad \text{for} \quad n = 0, 1, \ldots, K-1; \quad i = 0, \ldots, l.$$

Now the result follows from [7] (p. 49).

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Remarks. — If in theorem 3, the function $g(x)$ satisfies condition (ii) for all integers K, then the function $F_{ij}(t)$ is a " rapidly decreasing " function of t as t tends to infinity.

Although we only stated the results for the Fermi lattice systems, the treatment can easily be generalized to the continuous Fermi system, where $H_{0\Lambda}$ is the free Hamiltonian for the volume Λ; for the interaction we take a finite linear combination of monomials

$$B(\Psi_{i_1}) \ldots B(\Psi_{i_n})$$

such that all Ψ_{i_k} belong to the domain of the operator D induced by the free Hamiltonian.

The independence of the results on the local perturbation suggests that the results remain valid for some strictly quasi-local perturbations. The characterization of the most general perturbation such that the perturbed equilibrium state is a K. M. S. state for the same von Neumann algebra is an open question.

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APPENDIX

As an example we treat the perturbed XY-model given locally by

(A 1) $$H_N = -\sum_{j=0}^{N} \{ J_x \sigma_j^x \sigma_{j+1}^x + J_y \sigma_j^y \sigma_{j+1}^y + \mu h \sigma_j^z \},$$

σ_j^x, σ_j^y, σ_j^z being the ordinary Pauli matrices.

The Hamiltonian (A 1) is a particular case of (1). An explicit calculation of the operator D [*see* formula (2)] yields :

$$(D^2 \varphi)(z) = h(z) \varphi(z); \qquad z = e^{ix}, \qquad x \in [0, 2\pi].$$

where $\varphi \in L^2([0, 2\pi])$:

$$-h(z) = J_x^2 + J_y^2 + (\mu h)^2 + \mu h (J_x + J_y)(z^{-2} + z^2) + J_x J_y (z^{-4} + z^4).$$

By a straightforward calculation, the function $s(\varphi_i\, e^{Dt} \varphi_j)$ is obtained as the sum of integrals of the type (8), where $g(x)$ is replaced by the function $h(z)$; then we can apply theorem 2; the equation for the stationnary point is

$$\frac{dh(z)}{dx} = \frac{dz}{dx}\frac{d}{dz} h(z) = 0, \qquad \text{where} \quad z = e^{ix}.$$

The solutions of this equation are

$$z_n = e^{in\pi/2} \qquad (n = 0, 1, 2, 3).$$

These stationnary points are of order one (K = 1), because

$$\left.\frac{d^2 h(z)}{dx^2}\right|_{z=z_i} = -z_i^2 . 8 \mu h (J_x + J_y) - 32 J_x J_y \neq 0.$$

The asymptotic behaviour of $s(\varphi_i\, e^{Dt} \varphi_j)$ is therefore of order $t^{-1/2}$ as t tends to infinity.

In particular for $X = B_k B_l$; it follows that

$$\rho_t(B_k B_l) - \omega_0(B_k B_l)$$

tends to zero proportional to t^{-1} as t tends to infinity, a result found in [8]. Remark that in [8] only a perturbation of the type $V_2 = B_{i_1} B_{i_2}$ is considered. In lemma 5 we proved furthermore that the results are independent of the perturbation as far as the perturbation remains strictly local.

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