KRYSTYNA BUGAJSKA
SLAWOMIR BUGAJSKI

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# The lattice structure of quantum logics

by

**Krystyna BUGAJSKA and Slawomir BUGAJSKI**
Katowice, Poland

ABSTRACT. — A solution of the question of the lattice structure for quantum logics is proposed. It is achieved by a construction of a natural embedding of the logic into a complete lattice, preserving all essential features of the logic. Special cases of classical mechanics and the standard form of quantum mechanics are investigated.

## 1. INTRODUCTION

The lattice property for quantum logics is one of puzzling questions of the logical approach. Usually we assume the logic to be an orthomodular, complete ortholattice without any further substantiations. However, a deeper analysis suggest, that this property is neither obvious nor natural. The problem is important, because of the basic role played by the lattice property in the logical approach. Two possible ways out arises: (*i*) to give a clear phenomenological interpretation to the lattice joins and meets in quantum logics, or (*ii*) to derive the lattice property from others, more plausible assumptions.

Concerning the first possibility, it is probable, that the property in question has no satisfactory interpretation in terms of experimental operations (*see* Mac Laren's discussion in [*9*], p. 11-12); the recent attempt of Jauch and Piron [*7*] in this direction is not entirely conclusive [*11*]. The second way is preferred by some authors (for example [*2*], [*9*], [*6*], [*12*]. Newertheless, usually assumed postulates to imply the desired structure seem to be not more convincing than the derived property itself.

In this paper we propose a solution of the problem in question not by forcing the logic to possess a complete lattice structure, but by constructing a natural extension of the logic into complete lattice. Such extended logic appears to be a good substitute for original one and possesses all its essential features. Our « critical » assumption if any, is the von Neumann's projection postulate—unquestionable in the common opinion, but criticized by some prominent philosophers and theorists (e. g. [*5*]). The somewhat similar extension was previously discussed by the authors [*3*] on a quite different axiomatic basis.

In the next section we collect our assumptions. They are formulated in the technical language of the logical approach. Definitions of the used terms one can find in some other papers (e. g. [*10*], [*12*]).

## 2. ASSUMPTIONS

Let $\mathscr{L}$ denote the logic of given physical system. The fundamental properties of $\mathscr{L}$ are discussed in many papers; the reader is referred to the book of Mackey [*8*] for a justification of the first postulate:

POSTULATE 1. — *$\mathscr{L}$ is an orthomodular $\sigma$-ortho-poset.*

This hypothesis has a good tradition in the quantum logic approach and is not questioned.

The next assumption states a connection between the set of atoms $\mathscr{A}$ of $\mathscr{L}$ and the set of pure states of the system.

POSTULATE 2. — *There exists a set $\mathscr{P}$ of probability measures on $\mathscr{L}$ (the set of pure states), such that:* (i) *$\mathscr{P}$ is separating;* (ii) *for any $b \in \mathscr{L}$ there exists $\alpha \in \mathscr{P}$ such that $\alpha(b) = 1$;* (iii) *there exists an one-to-one mapping* car: $\mathscr{P} \to \mathscr{A}$ *with property: $\alpha(b) = 1$, $b \in \mathscr{L}$, implies* car $(\alpha) \leqslant b$.

The following supposition looks very natural: every state is a mixture (countable or not) of pure states. The first assertion of postulate 2 may be treated as a consequence of this not precise, but plausible statement. As regards (*iii*), this property essentially states, that one can unambiguously identify the pure state in a single yes-no measurement.

Observe, that postulate 2 implies the atomicity of $\mathscr{L}$ and the existence, for any $a \in \mathscr{A}$, of one and only one pure state $\alpha$, such that $a = \text{car}(\alpha)$.

Our next assumption is essentially the usual projection postulate of quantum mechanics:

POSTULATE 3. — *Let $b \in \mathscr{L}$ and $\alpha \in \mathscr{P}$. If $\alpha(b) \neq 0$ then there exists one and only one pure state $\beta$ such that $\beta(b) = 1$ and $\alpha(b) = \alpha$* [car $(\beta)$].

The projection postulate (in many, seemingly non-related formulations) is one of the most popular axioms in the quantum logics approach, *see* [*4*] for further remarks and examples.

Postulate 3 completes our list of needed assumptions.

## 3. THE EMBEDDING

Let us introduce some notations. If M is a subset of $\mathscr{L}$, then

$$M^{\triangle} = \{ a \in \mathscr{L} \mid a \leqslant b,\ \forall b \in M \}, \qquad M^{\triangledown} = \{ a \in \mathscr{L} \mid a \geqslant b,\ \forall b \in M \},$$
$$M' = \{ a \in \mathscr{L} \mid a' \in M \}$$

with $a'$—the orthocomplement of $a \in \mathscr{L}$. In the sequel we shall use some simple properties of operations$^{\triangle}$, $^{\triangledown}$, $'$ collected in

LEMMA 1. — *Let* M, N $\subseteq \mathscr{L}$, *then:*
(*i*) $M \subseteq N$ *implies* $M^{\triangledown} \supseteq N^{\triangledown}, M^{\triangle} \supseteq N^{\triangle}$;
(*ii*) $M^{\triangledown\triangle\triangledown} = M^{\triangledown}$;
(*iii*) $(M \cup N)^{\triangledown} = M^{\triangledown} \cap N^{\triangledown}$;
(*iv*) $M'^{\triangle} = M^{\triangledown\prime}$, $M'^{\triangledown} = M^{\triangle\prime}$. □

The application of the operation$^{\triangle}$ afte the operation$^{\triangledown}$ gives a kind of closure operation—the one of H. C. Moore [*1*]. It is the content of

LEMMA 2. — *If* M, N $\subseteq \mathscr{L}$, *then:*
(*i*) $M \subseteq M^{\triangledown\triangle}$;
(*ii*) $(M^{\triangledown\triangle})^{\triangledown\triangle} = M^{\triangledown\triangle}$;
(*iii*) $M \supseteq N$ *implies* $M^{\triangledown\triangle} \supseteq N^{\triangledown\triangle}$. □

We define the extended logic $\tilde{\mathscr{L}}$ as the set of all « closed » (in the sense of above closure operation) subsets of $\mathscr{L}$. It is easy to see (the reader is referred to [*1*] for the general theorem), that $\tilde{\mathscr{L}}$ is a complete lattice under the partial ordering by inclusion, with lattice joins and meets defined $(M, N \in \tilde{\mathscr{L}})$ as $(M \cup N)^{\triangledown\triangle}$ and $M \cap N$ resp.

One can introduce a natural orthocomplementation in $\tilde{\mathscr{L}}$. Indeed, the operation $M \to M'^{\triangle}$ for $M \in \tilde{\mathscr{L}}$ possesses needed properties:

LEMMA 3. — *If* M, N $\in \tilde{\mathscr{L}}$, *then:*
(*i*) $M'^{\triangle} \cap M = \emptyset$, $(M'^{\triangle} \cup M)^{\triangledown\triangle} = \mathscr{L}$;
(*ii*) $M \subseteq N$ *implies* $M'^{\triangle} \supseteq N'^{\triangle}$;
(*iii*) $(M'^{\triangle})'^{\triangle} = M$.

*Proof:* (*i*) Obviously, there does not exist any element $a$ of $\mathscr{L}$ such that $a \in M'^{\triangle\triangledown}$ and $a \in M^{\triangledown}$, except $e$—the greatest element of $\mathscr{L}$. Thus

$$M'^{\triangle\triangledown} \cap M^{\triangledown} = e \qquad \text{and} \qquad (M'^{\triangle} \cup M)^{\triangledown\triangle} = (M'^{\triangle\triangledown} \cap M^{\triangledown})^{\triangle} = e^{\triangle} = \mathscr{L}.$$

(*ii*) Is a consequence of lemma 1. (*iii*) $(M'^{\triangle})'^{\triangle} = (M'')^{\triangledown\triangle} = M^{\triangledown\triangle} = M$. □

We summarize the obtained results in

COROLLARY 1. — *The extended logic* $\tilde{\mathscr{L}}$ *is a complete, atomic ortholattice.*

The relation between $\mathscr{L}$ and $\tilde{\mathscr{L}}$ is clear. Namely, the set $a^{\triangle}$ for $a \in \mathscr{L}$ is a close subset of $\mathscr{L}$, hence is a member of $\tilde{\mathscr{L}}$. The mapping of $\mathscr{L}$ into $\tilde{\mathscr{L}}$:

$a \to a^{\triangle}$ preserves all essential features of $\mathscr{L}$, thus that is the mentioned extension of $\mathscr{L}$.

LEMMA 4. — *Let $a, b, a_i \in \mathscr{L}$, $i \in \mathrm{I}$, I—some set of indices. Then:*

*(i)* $a \leqslant b$ *implies* $a^{\triangle} \subseteq b^{\triangle}$;

*(ii)* $(a')^{\triangle} = (a^{\triangle})'^{\triangle}$;

*(iii)* $\left(\bigvee_{i \in \mathrm{I}} a_i\right)^{\triangle} = \left(\bigcup_{i \in \mathrm{I}} a_i^{\triangle}\right)^{\nabla\triangle}$ *(provided left hand side exists).*

*Proof.* — The first assertion as well as the second one is obvious. As regards *(iii)*, $b^{\triangle} = b^{\nabla\triangle}$ for any $b \in \mathscr{L}$. The equality

$$\left(\bigvee_{i \in \mathrm{I}} a_i\right)^{\nabla} = \left(\bigcap_{i \in \mathrm{I}} a_i^{\nabla}\right)$$

is easy to prove. Thus

$$\left(\bigvee_{i \in \mathrm{I}} a_i\right)^{\triangle} = \left(\bigvee_{i \in \mathrm{I}} a_i\right)^{\nabla\triangle} = \left(\bigcap_{i \in \mathrm{I}} a_i^{\nabla}\right)^{\triangle} = \left(\bigcap_{i \in \mathrm{I}} a_i^{\triangle\nabla}\right)^{\triangle} = \left(\bigcup_{i \in \mathrm{I}} a_i^{\triangle}\right)^{\nabla\triangle}. \quad \square$$

So the above constructed embedding $\mathscr{L} \to \tilde{\mathscr{L}}$ preserves the partial ordering, the orthocomplementation and all existing joins (it also preserves all meets, of course, as a result of this). The extended logic would be therefore a good substitute for the original one if only it were orthomodular. The orthomodularity is commonly accepted as a basic feature of quantum logics and an eventual lack of it makes our construction unuseful. The next section is devoted to a demonstration of the orthomodularity of $\tilde{\mathscr{L}}$.

## 4. THE QUESTION OF ORTHOMODULARITY

Before we come near to the problem, we examine some properties of $\mathscr{L}$.

LEMMA 5. — *If $a \in \mathscr{A}$ and $b \in \mathscr{L}$ then there exists $a \vee b \in \mathscr{L}$.*

*Proof.* — The projection postulate implies the existence of the atom $a_1$, such that

$$a_1 \leqslant b', \ \alpha(a_1) = \alpha(b') \qquad \text{with} \qquad a = \operatorname{car}(\alpha).$$

Moreover,

$$\alpha(a_1 \vee b) = \alpha(a_1) + \alpha(b) = 1, \qquad \text{thus} \qquad a \leqslant (a_1 \vee b).$$

Let $c \geqslant a, b$. We would like to demonstrate $c \geqslant b \vee a_1$. Indeed, by the orthomodularity, there is in $\mathscr{L}$ an element, say $b_1$, such that $b_1 \perp b$ and $c = b_1 \vee b$. If one applies the projection postulate to $a$ and $b$, then one can find the atom $\bar{a}_1$ with the properties:

$$\bar{a}_1 \leqslant b_1, \qquad \alpha(\bar{a}_1) = \alpha(b_1) = 1 - \alpha(b) = \alpha(b').$$

Thus $\bar{a}_1 = a_1$ and $c \geqslant a_1, b$. It means, that $b \vee a_1$ is the least element in the set $\{a, b\}^{\nabla}$, i. e. $b \vee a_1 = b \vee a$. □

LEMMA 6. — *If* $b \in \mathscr{L}$, *then* $b = \bigvee_{a \in b^{\triangle} \cap \mathscr{A}} a$.

*Proof.* — Obviously, $b \in (b^{\triangle} \cap \mathscr{A})^{\nabla}$. If $c \in (b^{\triangle} \cap \mathscr{A})^{\nabla}$, then two following cases are possible: $c^{\triangle} \cap \mathscr{A} = b^{\triangle} \cap \mathscr{A}$ or $c^{\triangle} \cap \mathscr{A} \supset b^{\triangle} \cap \mathscr{A}$. Let us consider the first one. By the projection postulate, there exist, for given $\alpha \in \mathscr{P}$ $[\alpha(b) \neq 0, \alpha(c) \neq 0]$, atoms $c_1$ and $b_1$ such that $a(c_1) = \alpha(c), \alpha(b_1) = \alpha(b)$. But $\alpha(b) \geqslant \alpha(c_1) = \alpha(c)$ and $\alpha(c) \geqslant \alpha(b_1) = \alpha(b)$. Thus $\alpha(b) = \alpha(c)$ for all $\alpha \in \mathscr{P}$, and the separating property of $\mathscr{P}$ assures $b = c$. If $c^{\triangle} \cap \mathscr{A} \supset b^{\triangle} \cap \mathscr{A}$, then $c > b$ by lemma 5. Thus $c \in (b^{\triangle} \cap \mathscr{A})^{\nabla}$ implies $c \geqslant b$, i. e. $b = \bigvee_{a \in b^{\triangle} \cap \mathscr{A}} a$. □

The proved lemma states, that $\mathscr{L}$ is atomistic (atomic in terms of [9]). Moreover,

COROLLARY 2. — $\tilde{\mathscr{L}}$ *is atomistic too.*

*Proof.* — It suffices to note, that if $M \in \tilde{\mathscr{L}}$, then $b \in M$ is equivalent to

$$b^{\triangle} \cap \mathscr{A} \subset M \cap \mathscr{A}. \quad \square$$

Let $B_M$ denote a maximal set of pairwise orthogonal atoms, contained in $M \in \tilde{\mathscr{L}}$ (an orthobasis of M). We demonstrate M to be determined by $B_M$, namely.

LEMMA 7. — *If* $M \in \tilde{\mathscr{L}}$ *and* $B_M$ *is an orthobasis of* M, *then* $B_M^{\nabla\triangle} = M$.

*Proof.* — $B_M \subset M$, then $B_M^{\nabla\triangle} \subset M^{\nabla\triangle} = M$. Let a be an atom belonging to $M \setminus B_M^{\nabla\triangle}$, and let $a = \text{car}(\alpha)$. One can prove, that if $b \in B_M$ then $\alpha(b) \neq 0$ only for some countable subset of $B_M$, say $\{b_1, b_2 \dots\}$. By lemma 1 there is an atom $c$, such that

$$\bigvee_{n=1}^{\infty} b_n \vee a = \bigvee_{n=1}^{\infty} b_n \vee c, \qquad \text{and} \qquad c \perp \bigvee_{n=1}^{\infty} b_n.$$

Atom $a$ is orthogonal to all $b \in B_M \setminus \{b_1, b_2 \dots\}$, thus $c$ is orthogonal to whole $B_M$. Hence, it is no atom in $M \setminus B_M^{\nabla\triangle}$ and, by the atomicity of $\tilde{\mathscr{L}}$, $M = B_M^{\nabla\triangle}$. □

COROLLARY 3. — $\tilde{\mathscr{L}}$ *is orthomodular.*

*Proof.* — Let $M_1 \subset M_2$ and $B_1$ be an orthobasis of $M_1$. One can extend $B_1$ to some orthobasis $B_2$ of $M_2$. Obviously, $B_1 \neq B_2$. Let

$B_3 = B_2 \setminus B_1$ and $M_3 = B_3^{\nabla\triangle}$. All elements of $B_3$ are orthogonal to $B_1$, i. e. $B_3 \subset B_1'^{\triangle}$. By applying lemmas 1 and 2 we obtain

$$M_3 = B_3^{\nabla\triangle} \subset B_1'^{\triangle\nabla\triangle} = B_1^{\nabla\triangle\prime\triangle} = M_1'^{\triangle},$$

thus $M_3$ is orthogonal to $M_1$. Moreover,

$$\begin{aligned}(M_1 \cup M_3)^{\nabla\triangle} &= (B_1^{\nabla\triangle} \cup B_3^{\nabla\triangle})^{\nabla\triangle} \\ &= (B_1^{\nabla\triangle\nabla} \cap B_3^{\nabla\triangle\nabla})^{\triangle} = (B_1^{\nabla} \cap B_3^{\nabla})^{\nabla} = (B_1 \cup B_3)^{\nabla\triangle} = B_2^{\nabla\triangle} = M_2 .\end{aligned}$$

This proves the orthomodularity of $\tilde{\mathscr{L}}$. □

The above corollary solves the question of orthomodularity for $\tilde{\mathscr{L}}$.

## 5. CONCLUSION

Thus we have proved the following.

THEOREM 1. — *If $\mathscr{L}$, $\mathscr{P}$ satisfy postulates* 1-3, *then there exists an orthomodular, complete, atomic ortholattice $\tilde{\mathscr{L}}$ and a natural embedding of $\mathscr{L}$ into $\tilde{\mathscr{L}}$, preserving the partial ordering, the orthocomplementation and all joins of $\mathscr{L}$.*

One can treat $\tilde{\mathscr{L}}$ as a new, extended logic of the system, satisfying all regularity conditions usually assumed for $\mathscr{L}$. The described extension procedure takes, however, into considerations some new elements, with no counterparts in « experimental questions » concerning the system. It is the usual price paid for an application of a regular mathematical structure to a description of some features of reality. Theoretical physics provide numerous examples of that.

An application of the obtained results to the problem of linear representation of quantum logic will be a subject of subsequent paper of the authors.

## 6. TWO SPECIAL CASES

The problem of lattice structure for $\mathscr{L}$ becomes remarkably simpler in two special cases of interest: (*i*) any two elements of $\mathscr{L}$ split (the case of classical mechanics); (*ii*) $\mathscr{L}$ is separable, i. e. any set of pairwise orthogonal elements of $\mathscr{L}$ is countable (the case of the standard quantum mechanics).

In the first case, $\mathscr{L}$ appears to be a Boole'an algebra, hence it is a lattice. If $\mathscr{L}$ is assumed to be not complete, then the described extension is not trivial. Our embedding theorem holds in this case with weaker assumptions:

THEOREM 1 (i). — *If $\mathscr{L}$ is an atomic, orthomodular orthoposet and any two elements of $\mathscr{L}$ split, then the thesis of theorem* 1 *holds.*

*Proof.* — The construction of $\tilde{\mathscr{L}}$ and the embedding are quite analogous to the above ones. We must prove the orthomodularity of $\tilde{\mathscr{L}}$ only. Any $b, c \in \mathscr{L}$ split, i. e. there exist pairwise orthogonal elements $b_1, c_1, d$ of $\mathscr{L}$, such that $b = b_1 \vee d, c = c_1 \vee d$. Let $c \in (b^{\triangle} \cap \mathscr{A})^{\nabla}$, then $b_1 = 0$ and $b \leqslant c$. Thus $\mathscr{L}$ is atomistic. Now the orthomodularity of $\tilde{\mathscr{L}}$ readily follows. □

In the second case we also do not need the set $\mathscr{P}$. Observe, that the projection postulate may be formulated without making any reference to transition probabilities:

POSTULATE 3′. — *If $b \in \mathscr{L}$ and $a \in \mathscr{A}$, then there exist unique atoms $a_1, a_2$ such that $a_1 \leqslant b'$, $a_2 \leqslant b$ and $a \leqslant a_1 \vee a_2$.*

This is the Varadarajan's version [*13*]; for further discussion see [*4*]. The embedding theorem takes now the form:

THEOREM 1 (ii). — *If $\mathscr{L}$ is separable, orthomodular, atomic $\sigma$-orthoposet and the projection postulate 3′ holds in $\mathscr{L}$, then $\mathscr{L}$ is isomorphic to $\tilde{\mathscr{L}}$, i. e. $\mathscr{L}$ is in itself a complete lattice.*

*Proof.* — One can prove, in an analogous manner as for lemma 5, that if $a$ is an atom, then $a \vee b$ with any $b \in \mathscr{L}$, exists in $\mathscr{L}$. Moreover, let $\{ c_1, c_2 \dots \}$ be an orthobasis of $c \in \mathscr{L}$. By the orthomodularity, $c = c_1 \vee c_2 \vee \dots$. If

$$d \in (c^{\triangle} \cap \mathscr{A})^{\nabla}, \qquad \text{then} \qquad d \in \{ c_1, c_2 \dots \}^{\nabla} = c^{\nabla}$$

and $\mathscr{L}$ is atomistic. Now let $\{ a_1, a_2 \dots \}$ be an orthobasis of $\mathrm{M} \in \tilde{\mathscr{L}}$. The existence of such $a_0 \in \mathrm{M}$ that $a_0 \notin (a_1 \vee a_2 \vee \dots)^{\triangle}$ is contrary to the just demonstrated properties of $\mathscr{L}$. Hence $\mathrm{M} = (a_1 \vee a_2 \vee \dots)^{\triangle}$. □

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