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# Definite magnetofluid scheme in general relativity

by

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ABSTRACT. — The treatment of « definite material schemes » in relativistic hydrodynamics by Lichnerowicz (1955) is extended to magnetohydrodynamics. The new field equations imply that the energy density is conserved along a Born rigid flow and the magnitude of the magnetic field vector is constant along an essentially rotational flow. In class one space-times the scheme degenerates to definite magnetofluid scheme (DMS) i. e., a self-gravitating definite material scheme in magnetohydrodynamics with the magnetic field vector as an eigenvector of the stress-energy tensor $T_{ab}$. With conformally flat class one space-times the DMS is compatible while the perfect magnetofluid scheme (Lichenerowicz, 1967) is not compatible. When the space-time admits a motion, $T_{ab}$ for DMS has Segre characteristic [3 1] or the eigenvectors of $T_{ab}$ are invariants of the group. When $T_{ab}$ of DMS has the Segre characteristic [3 1], Ricci collineation along the stream lines implies motion but Ricci collineation along magnetic lines implies motion only if the magnitude of the magnetic field is an invariant of the group.

## 2. INTRODUCTION

Modern astronomical discoveries have stimulated keen interest in the applications of relativistic gravitational fields to astrophysics. Since 1962, the belief that strong gravitational fields may provide the clue for quasars, for violent events in nuclei of galaxies, for death-by-collapse of very massive stars and for the periodic burst of radio sources has gained momentum (Thorne, 1968). Indeed the occurrence of magnetic field in solar winds (Parker, 1964), spiral arms (Hewish, 1963) and sunspots (Wilson, 1968) emphasize the necessity of relativistic magnetohydrodynamics for the development of astrophysics.

Perfect fluid schemes are too ideal to describe the astounding complexity of astrophysical systems. It is necessary to study more realistic schemes. In fact self-gravitating non-perfect fluid schemes were initiated by Lichenerowicz (1955) for hydrodynamics. It is imperative to extend this treatment to magnetohydrodynamics for application to astrophysical systems. The aim of this paper is to propose such a scheme.

## 3. FIELD EQUATIONS FOR DEFINITE MATERIAL SCHEMES IN RELATIVISTIC MAGNETOHYDRODYNAMICS

The field of a symmetric stress-energy tensor $T_{ab}$ in a domain of space-time is known as an *energy scheme*. When $T_{ab}$ admits a time-like eigenvector then it is called a *normal tensor* and the scheme is known as a *normal scheme*. A normal scheme with positive eigenvalue corresponding to time-like eigenvector is said to be a *material scheme*. When the quadratic form associated with it is positive definite, the tensor $T_{ab}$ is said to be positive definite. In this case the energy scheme is called as a *definite scheme*. The eigenvalues of $T_{ab}$ with respect to the metric tensor $g_{ab}$ (with signature -2) are given by the equation

$$\det (T_{ab} - S g_{ab}) = 0 ,$$

i. e. $$T_{ab} = S_4 U_a U_b - S_1 V_a V_b - S_2 W_a W_b - S_3 N_a N_b ,$$

where

$$\begin{aligned} - U_a U^a = V_a V^a = W_a W^a = N_a N^a = - 1 , \\ U_a V^a = U_a W^a = U_a N^a = V_a W^a = V_a N^a = W_a N^a = 0 , \end{aligned} \tag{3.1}$$

and $S_1$, $S_2$, $S_3$, $S_4$, are the eigenvalues corresponding to the eigenvectors $V^a$, $W^a$, $N^a$, $U^a$ respectively. Now interpreting the time-like eigenvector as a flow vector with density as the corresponding eigenvalue we have the expression for $T_{ab}$ as

$$T_{ab} = \rho U_a U_b + p_1 V_a V_b + p_2 W_a W_b + p_3 N_a N_b ,$$

where $p_\alpha = - S_\alpha$ ($\alpha = 1, 2, 3$) are the partial pressures. This represents *definite material scheme in relativistic hydrodynamics* (Lichnerowicz, 1955).

We define the stress-energy tensor for a self-gravitating thermodynamical arbitrary fluid with infinite conductivity and constant magnetic permeability (i. e., the stress-energy tensor for definite material schemes in relativistic magnetohydrodynamics) as

$$T_{ik} = (\rho + m)U_iU_k + (p_1 + m)V_iV_k + (p_2 + m)W_iW_k + (p_3 + m)N_iN_k - \mu H_iH_k\,, \quad (3.2)$$

where

$$(U^kH^i - U^iH^k)_{;k} = 0\,, \quad (3.3)$$

$$TdS - di + dP/r = 0\,, \quad (3.4)$$

$$3P - p_1 - p_2 - p_3 = 0\,, \quad (3.5)$$

$$\rho - r(1 + \varepsilon) = 0\,, \quad (3.6)$$

$$\varepsilon + \frac{P}{r} - i = 0\,, \quad (3.7)$$

$$U^kH_k = 0\,. \quad (3.8)$$

Here $\rho$ is the energy density, $p_\alpha$ ($\alpha = 1, 2, 3$) are the partial pressures, $H^k$ is the magnetic field vector, $r$ is the matter density, $\varepsilon$ is the internal energy density, $i$ is the enthalpy, S is the entropy, $U^k$ is the 4-velocity and

$$H_kH^k = -/\overline{H}/^2, \qquad m = (1/2)\mu/\overline{H}/^2.$$

If $p_1 = p_2 = p_3 = p$ we recover Lichnerowicz's (1967) expression for the stress-energy tensor for a thermodynamical perfect fluid with infinite electric conductivity and constant magnetic permeability viz.,

$$T'_{ik} = (\rho + p + 2m)U_iU_k - (p + m)g_{ik} - \mu H_iH_k\,.$$

But when $H^i = 0$ we obtain Lichnerowicz's (1955) field equations for definite material scheme in relativistic hydrodynamics as

$$T''_{ik} = \rho U_iU_k + p_1V_iV_k + p_2W_iW_k + p_3N_iN_k\,.$$

The Hawking-Ellis (1968) energy condition which is satisfied by all known forms of matter and all predicted equations of state, is

$$T_{ik}U^iU^k \geqslant (1/2)T\,, \quad (3.9)$$

$$\text{i. e.,} \qquad \rho + 3P + 2m \geqslant 0\,. \quad (3.10)$$

In fact positive definite character of $T_{ik}$ implies

$$\left.\begin{array}{ll} \rho + m > 0\,, & p_1 + m - \mu\alpha^2 > 0\,, \\ p_2 + m - \mu\beta^2 > 0\,, & p_3 + m - \mu\delta^2 > 0\,, \end{array}\right. \quad (3.11)$$

at a point in Minkowski space and hence

$$\rho + 3P + 2m > 0\,. \quad (3.12)$$

Here $\alpha$, $\beta$, $\delta$ are the projections of $H^k$ along $V^k$, $W^k$, $N^k$ respectively.

Incidentally we observe that Einstein spaces are not compatible with definite material schemes in relativistic magnetohydrodynamics (hence forth we call this scheme as *general scheme*), in particular, and with any material schemes in general. For, in Einstein spaces (Eisenhart, 1960)

$$R_{ik} - (1/4)g_{ik}R = 0,$$

$$\text{i. e.,} \quad T_{ik} - (1/4)g_{ik}T = 0.$$

Consequently we have

$$T_{ik}U^iU^k = (1/4)T$$

and this violates the condition (3.9) and hence any material scheme is incompatible with Einstein spaces.

THEOREM. — A necessary condition that definite material scheme in relativistic magnetohydrodynamics be embeddable in a 5-dimensional Minkowskian space-time is that the magnetic field vector is one of the space-like eigenvectors.

*Proof.* — If the Riemannian curvature tensor can be expressed as (Eisenhart, 1960)

$$R_{ijkn} = e(b_{in}b_{jk} - b_{ik}b_{jn}), \qquad e = \pm 1 \tag{3.13}$$

and

$$b_{ij;k} - b_{ik;j} = 0,$$

then the space-time $ds^2 = g_{ik}dx^idx^k$ embedded in a 5-dimensional Minkowski space. Let

$$b_{ik} = AU_iU_k - BV_iV_k - DW_iW_k - EN_iN_k. \tag{3.14}$$

By virtue of (3.13), (3.14) and Einstein's field equations we have

$$\begin{aligned} \int T_{ik} = e\{&(BD + DE + BE)U_iU_k - (AD + DE + AE)V_iV_k \\ &- (AB + BE + AE)W_iW_k - (AB + BD + AD)N_iN_k\}, \end{aligned} \tag{3.15}$$

where $\int = 8\pi G/c^4$. From equation (3.2) we get

$$\begin{aligned} T_{ik} = (\rho + m)U_iU_k &+ (p_1 + m - \mu\alpha^2)V_iV_k + (p_2 + m - \mu\beta^2)W_iW_k \\ &+ (p_3 + m - \mu\delta^2)N_iN_k - \mu\{\alpha\beta(V_iW_k + W_iV_k) + \beta\delta(W_iN_k \\ &+ N_iW_k) + \delta\alpha(N_iV_k + V_iN_k)\}. \end{aligned} \tag{3.16}$$

Comparing (3.15), (3.16) we have

$$\alpha\beta = \beta\delta = \delta\alpha = 0,$$

$$\text{i. e.,} \quad H_k = \alpha V_k, \quad \text{or} \quad H_k = \beta W_k, \quad \text{or} \quad H_k = \delta N_k, \tag{3.17}$$

which completes the proof. This theorem leads us to the consideration of special schemes described in the next section.

## 4. DEFINITE MAGNETOFLUID SCHEMES

DEF. — *A definite magnetofluid scheme is a definite material scheme in relativistic magnetohydrodynamics when the magnetic field vector is along* *one of the space-like vectors of the tetrad.* It follows that the stress-energy tensor for definite magnetofluid scheme (DMS) is

$$T_{ik} = (\rho + m)U_iU_k + (p_1 - m)V_iV_k + (p_2 + m)W_iW_k + (p_3 + m)N_iN_k\,, \quad (4.1)$$

when $H^k$ is along $V^k$. The form of Ricci tensor is

$$R_{ik} = \kappa\{ (\rho + m)U_iU_k + (p_1 - m)V_iV_k + (p_2 + m)W_iW_k + (p_3 + m)N_iN_k \} - (1/2)g_{ik}(\rho - 3P)\,. \quad (4.2)$$

For the sake of simplicity we use the following notations:

$$\begin{aligned} a^2 &= \kappa(\rho + m), \qquad a^2n^2 = 2\kappa m, \qquad n^2 < 1\,, \\ b &= (1/2)\kappa(\rho - 3P + 2m), \qquad p'_\alpha = \kappa p_\alpha\,, \\ 3P' &= 3\kappa P = 3\kappa(p_1 + p_2 + p_3)\,. \end{aligned} \quad (4.3)$$

By using this notation we get an elegant form of $R_{ik}$ as

$$R_{ik} = (a^2 - b)U_iU_k + (p'_1 + b - a^2n^2)V_iV_k + (p'_2 + b)W_iW_k + (p'_3 + b)N_iN_k\,. \quad (4.4)$$

« STRESS BALANCE » EQUATION. — As a consequence of the Bianchi identities we have

$$(\rho U^a)_{;a} - U_{a;b}(p_1V^aV^b + p_2W^aW^b + p_3N^aN^b) = 0\,, \quad (4.5)$$

as the equation of continuity and

$$(\rho + m)U_{a;b}U^b + (\rho U^b)_{;b}U_a + (p_1V_aV^b + p_2W_aW^b + p_3N_aN^b)_{;b} - m_{,b}h^b_a + m_{;b}U^bU_a + 2mU_aU^b{}_{;b} - (2mV_aV^b)_{;b} = 0\,, \quad (4.6)$$

characterising the differential system of stream lines.

MAXWELL EQUATIONS. — For definite magnetofluid scheme Maxwell equations (3.3) reads to

$$/\overline{H}/(V^iU^k - U^iV^k)_{;k} = /\overline{H}/_{,k}(V^kU^i - U^kV^i)\,. \quad (4.7)$$

Consequently we have

$$V_{i;k}W^iU^k = U_{i;k}W^iV^k\,, \quad (4.8)$$

$$V_{i;k}N^iU^k = U_{i;k}N^iV^k\,, \quad (4.9)$$

$$/\overline{H}/(V_{i;k}U^iU^k - V^k{}_{;k}) = /\overline{H}/_{,k}V^k\,, \quad (4.10)$$

$$/\overline{H}/U_{i;k}V^iV^k = /\overline{H}/_{,k}U^k + /\overline{H}/U^k{}_{;k}\,. \quad (4.11)$$

CONFORMALLY FLAT CLASS ONE SPACE-TIME. — From equations (3.15) and (4.1) we get

$$\begin{aligned} \kappa(\rho + m) &= e(BD + DE + BE) \\ \kappa(p_1 - m) &= -\,e(AD + DE + AE)\,, \\ \kappa(p_2 + m) &= -\,e(AB + BE + AE)\,, \\ \kappa(p_3 + m) &= -\,e(AB + BD + AD)\,. \end{aligned} \quad (4.12)$$

It has been shown by Pandey and Gupta (1970) that if the space-time is conformally flat and class one then

$$B = D = E . \tag{4.13}$$

Therefore equations (4.12) give

$$\begin{aligned} p_3 &= p_2 = p_1 - 2m \\ \text{i. e.} \quad p_2 + m &= p_3 + m = p_1 - m . \end{aligned} \tag{4.14}$$

From (4.14) we note that for perfect magnetofluid where $p_1 = p_2 = p_3 = p$ we get $2m = \mu/\overline{H}/^2 = 0$. We conclude that *a perfect magnetofluid scheme is incompatible with conformally flat class one space-times while definite magnetofluid scheme is compatible and has Segre characteristic* [3 1].

## 5. PARAMETERS OF SPACE-LIKE AND TIME-LIKE CONGRUENCES IN DEFINITE MATERIAL SCHEME

In definite material schemes only two types of congruences exist viz., time-like congruence determined by $U^k$ and three space-like congruences represented by $V^k$, $W^k$, $N^k$. Here we describe the parameters associated with $U^k$ and $V^k$ by following Greenberg's (1970*a*, 1970*b*) formalism

| PARAMETERS | Time-like congruence $U^k$ | Space-like congruence $V^k$ | |
|---|---|---|---|
| Expansion | $\theta = U^k{}_{;k}$ | $\theta^* = V^k{}_{;k} - V_{i;k}U^iU^k$ | (5.1) |
| Rotation | $\Omega_{ik} = U_{[i;k]} - U_{[k}\dot{U}_{i]}$ | $\Omega^*_{ik} = P^d_k P^c_i V_{[c;d]}$ | (5.2) |
| Shear | $\sigma_{ik} = U_{(i;k)} - \dot{U}_{(i}U_{k)} - (1/3)h_{ik}\theta$ | $\sigma^*_{ik} = P^c_i P^d_k V_{(c;d)} - (1/2)P_{ik}\theta^*$ | (5.3) |

where

$$\dot{U}_k = U_{k;i}U^i , \tag{5.4}$$

$$h_{ik} = g_{ik} - U_iU_k , \qquad P_{ik} = g_{ik} - U_iU_k + V_iV_k , \tag{5.5}$$

$$g^{ik}h_{ik} = 3 , \qquad g^{ik}P_{ik} = 2 , \qquad h_{ik}U^k = P_{ik}U^k = P_{ik}V^k = 0 . \tag{5.6}$$

The vectors $W^k$, $N^k$, and $U^i$ should satisfy the transport laws

$$\begin{aligned} U_{i;k}V^k &= V_{i;k}U^k - U_iV_{k;n}U^kU^n + V_iV_{k;n}U^kV^n , \\ W_{i;k}V^k &= U_iV_{k;n}W^kU^n - V_iV_{k;n}W^kV^n , \\ N_{i;k}V^k &= U_iV_{k;n}N^kU^n - V_iV_{k;n}N^kV^n . \end{aligned} \tag{5.7}$$

Consequently we observe the following relations

$$\begin{aligned} V_{i;k}W^iU^k &= U_{i;k}W^iV^k , \\ V_{i;k}N^iU^k &= U_{i;k}N^iV^k , \\ V_{i;k}N^kW^i &= 0 . \end{aligned} \tag{5.9}$$

By virtue of Maxwell equations the first two relations are satisfied (vide equations (4.8), (4.9)).

We consider the tetrad vectors $\lambda^a_\alpha$ (the Latin index denotes coordinate suffix and Greek index denotes tetrad suffix).

The identification

$$\lambda^k_\alpha = (V^k, W^k, N^k, U^k) \tag{5.10}$$

provides the relationship between the parameters and

$$\underset{\alpha\beta\delta}{\gamma} = \lambda_{\alpha i;k}\lambda^i_\beta\lambda^k_\delta .$$

We readily get the following results:

(*i*) Born rigid flow is characterised by:

$$\left.\begin{aligned} \underset{4\alpha\beta}{\gamma} &= -\underset{4\beta\alpha}{\gamma} \quad \text{for} \quad \alpha, \beta = 1, 2, 3, \quad \alpha \neq \beta \\ \underset{411}{\gamma} &= \underset{422}{\gamma} = \underset{433}{\gamma} . \end{aligned}\right\} \tag{5.11}$$

(*ii*) An essentially rotational flow is characterised by

$$\underset{4\alpha\beta}{\gamma} = -\underset{4\beta\alpha}{\gamma} \quad \text{for} \quad \alpha, \beta = 1, 2, 3 \quad \text{and} \quad \alpha \neq \beta$$

$$\underset{k44}{\gamma} = \underset{411}{\gamma} = \underset{422}{\gamma} = \underset{433}{\gamma} = 0. \tag{5.12}$$

(*iii*) An essentially shear flow is characterised by:

$$\underset{4\alpha\beta}{\gamma} = \underset{4\beta\alpha}{\gamma} \quad \text{for} \quad \alpha, \beta = 1, 2, 3,$$

$$\underset{k44}{\gamma} = \underset{411}{\gamma} = \underset{422}{\gamma} = \underset{433}{\gamma} = 0 . \tag{5.13}$$

(*iv*) An essentially accelerating flow is characterised by

$$\underset{4\alpha\beta}{\gamma} = 0, \quad \text{for} \quad \alpha, \beta = 1, 2, 3 . \tag{5.14}$$

(*v*) An essentially expanding flow is characterised by

$$\underset{k44}{\gamma} = \underset{4\alpha\beta}{\gamma} = 0 \quad \text{for} \quad \alpha, \beta = 1, 2, 3, \quad \alpha \neq \beta \tag{5.14}$$

$$\underset{411}{\gamma} = \underset{422}{\gamma} = \underset{433}{\gamma} .$$

SPECIAL FLOWS. — In expansion-free flow equation (4.5) leads to

$$U^k\rho_{,k} = (p_1 - p_3)\underset{411}{\gamma} + (p_2 - p_3)\underset{422}{\gamma} . \tag{5.15}$$

Hence in the case of Born rigid flow we have

$$U^k\rho_{,k} = 0 . \tag{5.16}$$

Thus for definite magnetofluid scheme energy density is conserved along a Born rigid flow while in the case of perfect magnetofluid scheme it is conserved along an expansion-free stream lines.

In the case of *essentially rotational* flow we have

$$U_{a;b} + U_{b;a} = 0 ,$$

so that (4.11) leads to

$$/\overline{H}/_{,k}U^k = 0 . \tag{5.17}$$

i. e. the magnitude of the magnetic field vector is constant along an essentially rotational flow.

The *uniform flow* defined by

$$U_{a;b} = 0, \qquad \underset{4ik}{\gamma} = 0 , \tag{5.18}$$

is incompatible with definite magnetofluid scheme. This can be established in the following way: the Ricci identity yields for uniform flow

$$U^kR_{ik} = 0 ,$$

i. e. $\rho + 3P + 2m = 0$ ,

by virtue of (4.3) and (4.4). But this contradicts the condition (3.12).

COVARIANT CONSTANT MAGNETIC FIELD. — When the magnetic field is covariantly constant we have

$$V_{i;k} = 0, \qquad /\overline{H}/_{,k} = 0, \qquad \underset{1ik}{\gamma} = 0 . \tag{5.19}$$

Hence the Ricci identity implies

$$\rho - 3P + 2p_1 = \text{constant} . \tag{5.20}$$

Equation (4.11) produces

$$U^k{}_{;k} = 0 , \tag{5.21}$$

i. e. the flow is expansion-free and hence by (4.6) we get

$$V^k(p_1)_{,k} = 0 , \tag{5.22}$$

on contracting with $V^k$. Thus the field equations for DMS are compatible with covariant constant magnetic field. In this case the flow is expansion-free and the longitudinal partial pressure $p_1$ along the magnetic field vector is constant along magnetic lines.

## 6. GROUP OF MOTIONS AND DEFINITE MAGNETOFLUID SCHEME

In order to comprehend the significance of the field equations governing the self-gravitating magnetofluid scheme a usual technique is to investigate the groups of motions admitted by the corresponding space-times. However « the problem as to which groups of motion correspond to a given type of gravitational field remains unsolved » (Petrov, 1969). Hence we study the converse problem by examining the consequences of certain symmetries on the metric tensor as well as on the Ricci tensor.

THEOREM I. — Let the space-time admits a motion. If the stress-energy tensor $T_{ik}$ has not Segre characteristic [3 1] then tetrad is an invariant of the group.

*Proof.* — A space-time is said to admit motion when the functional form of the metric tensor $g_{ik}$ is invariant under an infinitesimal transformation.

$$x'^k = x^k + \eta^k(x^i)\delta t ,$$
$$\text{i. e.} \qquad \underset{\eta}{£} g_{ik} = \eta_{k;i} + \eta_{i;k} = 0 . \tag{6.1}$$

It is well-known that equation (6.1) implies

$$\underset{\eta}{£} R_{ik} = 0 . \tag{6.2}$$

For DMS the equation reads as

$$\underset{\eta}{£} \{ (a^2 - b)U_iU_k + (p'_1 + b - a^2n^2)V_iV_k + (p'_2 + b)W_iW_k + (p'_3 + b)N_iN_k \} = 0 . \tag{6.3}$$

Hence on contracting with $U^iU^k$ and using (6.1) we get

$$\underset{\eta}{£}(a^2 - b) = 0 . \tag{6.4}$$

Similarly contracting in succession with $V^iV^k$, $W^iW^k$, $N^iN^k$ we produce

$$\underset{\eta}{£}(p'_1 + b - a^2n^2) = \underset{\eta}{£}(p'_2 + b) = \underset{\eta}{£}(p'_3 + b) = 0 . \tag{6.5}$$

On substituting (6.5), (6.4) in (6.3) the equation $U^iV^k \underset{\eta}{£} R_{ik} = 0$ yields

$$(a^2 + p'_1 - a^2n^2)V^k \underset{\eta}{£} U_k = 0 . \tag{6.6}$$

Similarly other contractions produce

$$\begin{aligned} (p'_1 - p'_2 - a^2n^2)W^k \underset{\eta}{£} V_k = 0 , \\ (p'_1 - p'_3 - a^2n^2)N^k \underset{\eta}{£} V_k = 0 , \\ (p'_2 - p'_3)N^k£W_k = (a^2 + p'_2)W^k£U_k = (a^2 + p'_3)N^k£U_k = 0 . \end{aligned} \tag{6.7}$$

For positive pressures and density the equations (6.17) imply either

$$\underset{\eta}{£} U_k = \underset{\eta}{£} V_k = \underset{\eta}{£} W_k = \underset{\eta}{£} N_k = 0 ,$$

or

$$p'_2 = p'_3 = p'_1 - a^2n^2 .$$

In the second case we have

$$p_2 = p_3 = p_1 - 2m ,$$
$$\text{i. e.} \qquad p_2 + m = p_3 + m = p_1 - m ,$$

hence $T_{ik}$ has Segre characteristic [3 1]. Thus we conclude that if the stress-energy tensor has not the Segre characteristic [3 1] the tetrad vectors are invariants of the group.

*Remark.* — In spherically symmetric and plane symmetric space-times the tensor $T_{ik}$ of DMS has Segre characteristic [3 1].

## 7. RICCI COLLINEATION

The infinitesimal transformations which leave invariant the functional form of the Ricci tensor is known as Ricci collineation. Collinson (1970) established the conservation law corresponding to the Ricci collineation which can be given as

$$\underset{\eta}{\pounds} R_{ik} = 0 \Rightarrow (R^i_k \eta^k)_{;i} = 0 , \qquad (7.1)$$

where $\eta^k$ is any vector.

For « non-Zeldovich » magnetofluid Khade (1973) has shown that a Ricci collineation implies motion. In this section we study the Ricci collineation along the principal vectors of the stress-energy tensor for DMS.

THEOREM II. — Ricci collineation along the world line implies that the streamlines are geodesic and expansion-free for DMS,

$$\text{i. e.} \qquad \underset{U}{\pounds} R_{ik} = 0 \Rightarrow \dot{U}_k = U^k{}_{;k} = 0 .$$

*Proof.* — From the definition of Lie derivative we have

$$\begin{gathered} \underset{U}{\pounds} U_k = U_{k;i}U^i = \dot{U}_k , \qquad U^k \underset{U}{\pounds} U_k = 0 , \\ \underset{U}{\pounds} V_k = U^i(V_{i;k} - V_{k;i}) , \qquad V^k \underset{U}{\pounds} V_k = U_{i;k}V^iV^k , \\ U^k \underset{U}{\pounds} V_k = U^k \underset{U}{\pounds} W_k = U^k \underset{U}{\pounds} N_k = 0 . \end{gathered} \qquad (7.2)$$

From equations (4.8) and (4.9) we observe that

$$\begin{aligned} V^k \underset{U}{\pounds} W_k &= V^iU^k(W_{i;k} - W_{k;i}) = 0 , \\ V^k \underset{U}{\pounds} N_k &= V^iU^k(N_{i;k} - N_{k;i}) = 0 . \end{aligned} \qquad (7.3)$$

The Ricci collineation along $U^k$ for DMS is expressed through

$$\underset{U}{\pounds} \{ (a^2 - b)U_iU_k + (p'_1 + b - a^2n^2)V_iV_k + (p'_2 + b)W_iW_k + (p'_3 + b)N_iN_k \} = 0 .$$

By Collinson's conservation law (7.1) we have

$$\{ (a^2 - b)U^k \}_{;k} = 0 . \qquad (7.5)$$

Equation $U^iU^k \underset{U}{\pounds} R_{ik} = 0$ implies

$$U^k(a^2 - b)_{,k} = U^k(\rho + 3P + 2m)_{,k} = 0 ,$$

and hence (7.5) implies

$$U^i_{;i} = 0 . \tag{7.6}$$

Contracting (7.4) successively with $U^iV^k$, $U^iW^k$, $U^iN^k$ we have

$$\dot{U}^kV_k = \dot{U}^kW_k = \dot{U}^kN_k = 0 \Rightarrow \dot{U}^k = 0 , \tag{7.7}$$

since $\rho + 3P + 2m \neq 0$. Thus

$$\underset{U}{£}R_{ik} = 0 \Rightarrow \dot{U}_k = U^k_{;k} = 0 , \tag{7.8}$$

where $\rho + 3P + 2m \neq 0$. This condition is consistent with (3.12).

THEOREM III. — In the case $p'_2 = p'_3 = p'_1 - a^2n^2$, $p'_1 + b - a^2n^2 \neq 0$ the invariance of Ricci tensor along the flow vector implies the invariance of the metric tensor.

*Proof.* — By virtue of (7.3), (7.8) the equation (7.4) on inner multiplication with $V^iV^k$, $W^iW^k$, $N^iN^k$, $V^iW^k$, $V^iN^k$, $W^iN^k$ implies

$$\begin{aligned} &2(p'_1 + b - a^2n^2)\underset{411}{\gamma} - (p'_1 + b - a^2n^2)_{,k}U^k = 0 , \\ &2(p'_2 + b)\underset{422}{\gamma} - (p'_2 + b)_{,k}U^k = 0 , \\ &2(p'_3 + b)\underset{433}{\gamma} - (p'_3 + b)_{,k}U^k = 0 , \\ &(p'_1 + b - a^2n^2)(\underset{421}{\gamma} + \underset{412}{\gamma}) = 0 , \\ &(p'_1 + b - a^2n^2)(\underset{431}{\gamma} + \underset{413}{\gamma}) = 0 , \\ &(p'_2 + b)\underset{423}{\gamma} + (p'_3 + b)\underset{432}{\gamma} + (p'_2 - p'_3)\underset{234}{\gamma} = 0 , \end{aligned} \tag{7.9}$$

We consider the special case:

*The case $p'_2 = p'_3 = p'_1 - a^2n^2$, $p'_1 + b - a^2n^2 \neq 0$.* — In this case first three equations of (7.9) and (7.8) give

$$\underset{411}{\gamma} = \underset{422}{\gamma} = \underset{433}{\gamma} = 0 . \tag{7.10}$$

Similarly last three equations produce

$$\underset{4\alpha\beta}{\gamma} = - \underset{4\beta\alpha}{\gamma} \quad \text{for} \quad \alpha, \beta = 1, 2, 3, \quad \alpha \neq \beta . \tag{7.11}$$

From equations (7.10) and (7.11) we observe that

$$\sigma_{ab} = 0 ,$$

and hence by (7.8)

$$U_{i;k} + U_{k;i} = 0 ,$$

$$\text{i. e.,} \quad \underset{U}{£}g_{ik} = 0 .$$

Thus

$$£R_{ik} = 0 \Leftrightarrow £g_{ik} = 0 .$$

*Remark.* — It may be notted that analogous to Theorem II there exist similar theorems for the vectors $V^k$, $W^k$, $N^k$. For Ricci collineation along magnetic lines we have the conditions $p'_2 = p'_3 = p'_1 - a^2n^2$, $p'_1 + b - a^2n^2 \neq 0$ and $/\overline{H}/_{,k}V^k = 0$.

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## REFERENCES

[*] C. D. Collinson, Conservation Laws in General Relativity Based Upon the Existence of Prefered Collineation. *General Relativity and Gravitation*, t. **1**, 2, 1970, p. 137.
[*1*] L. P. Eisenhart, *Riemannian Geometry*, Princeton University Press, Princeton, 1960, p. 92.
[*2*] P. J. Greenberg, Propagation Equations for the Set of Natural Geometrical Invariants in Relativistic Hydrodynamics. *J. Math. Anal. Appl.*, t. **29**, 1970, p. 647.
[*3*] P. J. Greenberg, The General Theory of Space-like Congruences with an application to Vorticity in Relativistic Hydrodynamics. *J. Math. Anal. Appl.*, t. **30**, 1970, p. 128.
[*4*] S. W. Hawking and G. F. R. Ellis, The cosmic Black-Body Radiation and the Existence of Singularities in Our Universe. *Astrophys. J.*, t. **152**, 1968, p. 25.
[*5*] A. Hewish, Pulsars. *Endeavour*, t. **28**, 104, 1969, p. 55.
[*6*] V. D. Khade, *Studies in Groups of Motions and Ricci Collineations.* Ph. D. Thesis, unpublished, Shivaji University, 1973.
[*7*] A. Lichnerowicz, *Théories relativistes de la Gravitation et de l'Électromagnétisme.* Masson et Cie, Paris, 1955, chap. I.
[*8*] A. Lichnerowicz, *Relativistic Hydrodynamics and Magnetohydrodynamics.* W. A. Benjamin, Inc., New York, chap. IV.
[*9*] S. N. Pandey and Y. K. Gupta, Space-time of Class one and Perfect Fluid Distribution. *University of Roorkee Research Journal*, t. **12**, 1970, p. 9.
[*10*] E. N. Parker, The Solar Wind. *Frontiers in Astronomy.* Readings from Scientific American, W. H. Friedman and Co., San Francisco, 1964, 1970, p. 98.
[*11*] A. Z. Petrov, *Einstein Spaces.* Pergamon Press, 1969, Chap. 4, p. 132.
[*12*] K. S. Thorne, Astrophysical Relativity. *Orange Aid Preprint Series in Nuclear Astrophysics*, 1968, OAP-123.
[*13*] P. R. Wilson, The Structure of a Sunspot. *Solar Physics*, 1968, t. **3**, p. 243.

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