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Unitary equivalence of local algebras in the quasifree representation

by

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ABSTRACT. — We consider the von Neumann algebras $\mathcal{A}(\mathbf{B}, m)$ generated in the free representation by bose fields of mass $m \geq 0$ with test functions supported in a bounded region $\mathbf{B} \subseteq \mathbb{R}^d$. We show that in $d = 2, 3$ space dimensions $\mathcal{A}(\mathbf{B}, m)$ is unitarily equivalent to $\mathcal{A}(\mathbf{B}, 0)$, by proving a more general theorem about equivalence of local algebras.

1. INTRODUCTION

This note deals with a problem arising in the algebraic approach to quantum field theory. We compare different (relativistic) *local* algebras (of observables) generated by neutral bose fields with different two point functions in the *quasi-free representation*. We derive criteria for such algebras to be isomorphic or to be unitarily equivalent.

There is an extensive literature on quasi-free representations (see e. g. [1] [2]), in which necessary and sufficient conditions have been given for such representations to give rise to isomorphic algebras. However, these criteria seem unfortunately *not to apply to local* algebras, due to an initial setup which does not incorporate our case at hand.

Another line of ideas, initiated by Glimm and Jaffe in [3] and discussed in [4] [5] and [6], uses the existence of a local *non-relativistic* number operator as the starting point for the proof of isomorphism of von Neumann

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algebras. We apply this idea to the quasi-free case at hand and we show that if the non-diagonal part of the Bogoliubov transformation connecting two representations is « locally » a Hilbert-Schmidt (H. S.) operator then the two corresponding local algebras are isomorphic (Theorem 4.3). In particular, the local algebras $\mathcal{A}(\mathbf{B}, m_1)$ and $\mathcal{A}(\mathbf{B}, m_2)$, $m_1 \geq 0$, $m_2 \geq 0$ of free scalar bose fields of mass m_1 and m_2 respectively are unitarily equivalent in $d = 2$ and 3 space dimensions if $m_1 \cdot m_2 = 0$ and in $d = 1, 2, 3$ dimensions otherwise (Theorems 5.1, 5.2). The case $m_1 \cdot m_2 = 0$ is an extension of results in [5].

This note is motivated by and has applications in an analysis of scattering states in relativistic field theories without a mass gap [8].

2. NOTATION. STATEMENT OF THE PROBLEM

Let \mathbb{R}^d be the configuration space and let $\mathcal{H} = L^2(\mathbb{R}^d)$ be the Hilbert space of one particle wave functions. As usual, the Fock space \mathcal{F} over \mathcal{H} is

$$\mathcal{F} = \bigoplus_{m=0}^{\infty} L^2(\mathbb{R}^d)^{\otimes_s m} \quad \text{where} \quad L^2(\mathbb{R}^d)^{\otimes_s 0} = \mathbb{C},$$

and \otimes_s is the symmetric tensor product. The vector $\{1, 0, 0, \dots\}$ is called the vacuum and is denoted by Ω_0 .

On \mathcal{F} , the creation and annihilation operators $A^*(f)$ and $A(g)$ obey for $f, g \in \mathcal{H}$ the commutation relations (CCR)

$$[A(f), A^*(g)] = (f, g)_2 = \int_{\mathbb{R}^d} dx \bar{f}(x) g(x), \quad (2.1)$$

$$[A(f), A(g)] = [A^*(f), A^*(g)] = 0$$

and

$$A(f)\Omega_0 = 0 \quad \text{for all} \quad f \in \mathcal{H}. \quad (2.2)$$

Let μ be a real positive selfadjoint operator on $\mathcal{H} = L^2(\mathbb{R}^d)$ such that for some dense $D_\mu \subseteq L^2_{\text{real}}(\mathbb{R}^d)$ we have

$$\mathcal{D}(\mu^{1/2}) \supseteq \mathcal{D}_\mu = D_\mu + iD_\mu, \quad \mathcal{D}(\mu^{-1/2}) \supseteq \mathcal{D}_\mu.$$

We define the field φ_μ « with two point function μ^{-1} » and its conjugate momentum π_μ by

$$\begin{aligned} \varphi_\mu(f) &= 2^{-1/2}(A^*(\mu^{-1/2}f) + A(\mu^{-1/2}f)), \\ \pi_\mu(f) &= i2^{-1/2}(A^*(\mu^{+1/2}f) + A(\mu^{+1/2}f)), \end{aligned} \quad (2.3)$$

for $f \in D_\mu$. They satisfy the CCR in Weyl form,

$$e^{i\varphi_\mu(f)} e^{i\pi_\mu(g)} = e^{-i(f,g)_2} e^{i\pi_\mu(g)} e^{i\varphi_\mu(f)}, \quad f, g \in D_\mu. \quad (2.4)$$

For any bounded open region $B \subseteq \mathbb{R}^d$ we define $\mathcal{A}_\mu^\circ(B)$ as the complex *-algebra generated by

$$\{ e^{i\varphi_\mu(f)}, e^{i\pi_\mu(f)}, \quad f \in D_\mu, \text{supp } f \subset B \}, \tag{2.5}$$

and we let $\mathcal{A}_\mu(B)$ be the von Neumann algebra generated by $\mathcal{A}_\mu^\circ(B)$ with respect to the weak operator topology on \mathcal{F} .

We shall deal with problems where two different two point functions are considered. Call them μ_1^{-1} and μ_2^{-1} . We shall always assume that for some dense $D \subseteq L^2_{\text{real}}(\mathbb{R}^d)$, one has with $\mathcal{D} = D + iD$,

$$\begin{aligned} \mathcal{D}(\mu_1^{+1/2}) \supseteq \mathcal{D} \cup \mu_2^{-1/2}\mathcal{D}, & \quad \mathcal{D}(\mu_2^{+1/2}) \supseteq \mathcal{D} \cup \mu_1^{-1/2}\mathcal{D}, \\ \text{and} & \\ \mathcal{D}(\mu_1^{-1/2}) \supseteq \mathcal{D} \cup \mu_2^{+1/2}\mathcal{D}, & \quad \mathcal{D}(\mu_2^{-1/2}) \supseteq \mathcal{D} \cup \mu_1^{+1/2}\mathcal{D}. \end{aligned} \tag{2.6}$$

With D replacing D_μ in equ. (2.5), we define the algebras $\mathcal{A}_{\mu_i}^\circ(B)$ and $\mathcal{A}_{\mu_i}(B)$, $i = 1, 2$. We want to give sufficient conditions for $\mathcal{A}_{\mu_1}(B)$, and $\mathcal{A}_{\mu_2}(B)$ to be isomorphic or unitarily equivalent.

3. BOGOLIUBOV TRANSFORMATIONS

This section reviews well-known formulae and conditions involving Bogoliubov transformations [1]. We need them for our derivation of a formula for the existence of a local number operator in Section 4. We also show why the usual isomorphism criteria [1] [2] do not apply.

A Bogoliubov transformation β is a pair of real linear maps $\beta = (\beta_+, \beta_-)$ on $L^2(\mathbb{R}^d)$ satisfying

$$\beta_+^* \beta_+ - \beta_-^* \beta_- = 1, \quad \beta_+^* \beta_- = \beta_-^* \beta_+. \tag{3.1}$$

The restriction to real β_+ and β_- is for convenience only and will have the effect of producing only transformations which map fields and conjugate momenta onto themselves. Given β , we define for $f \in \mathcal{D}(\beta_\pm)$,

$$\begin{aligned} A_\beta^*(f) &= A^*(\beta_+ f) + A(\beta_- f), \\ \text{and} & \\ A_\beta(f) &= A^*(\beta_- f) + A(\beta_+ f), \end{aligned} \tag{3.2}$$

so that by (3.1), A_β^* and A_β satisfy the CCR (2.1).

If μ_1 and μ_2 satisfy (2.6), we set

$$\beta_\pm = \frac{1}{2}(\mu_2^{-1/2}\mu_1^{1/2} \pm \mu_2^{1/2}\mu_1^{-1/2}). \tag{3.3}$$

Then the map $(A^*, A) \rightarrow (A_\beta^*, A_\beta)$ defines through equation (2.3) a map $(\varphi_{\mu_1}(f), \pi_{\mu_1}(g)) \rightarrow (\varphi_{\mu_2}(f), \pi_{\mu_2}(g))$ and hence a natural invertible homomorphism τ_β from the algebra $\mathcal{A}_{\mu_1}^\circ(\mathbb{R}^d)$ onto $\mathcal{A}_{\mu_2}^\circ(\mathbb{R}^d)$. It is well known that for irreducible representations τ_β is unitarily implementable (spatial) on \mathcal{F} if and only if β_- is a Hilbert-Schmidt operator [1]. If τ_β is spatial then

obviously for every open $B \subseteq \mathbb{R}^d$ the algebras $\mathcal{A}_{\mu_1}(B)$ and $\mathcal{A}_{\mu_2}(B)$ are unitarily equivalent. This paper deals with situations in which (the global) τ_β is not spatial, but where, for bounded B , the algebras $\mathcal{A}_{\mu_1}(B)$ and $\mathcal{A}_{\mu_2}(B)$ will turn out to be isomorphic or unitarily equivalent. We wish to use the fact that β_- is « locally H. S. » in a sense to be made precise below. Since μ_1 does in general not necessarily map $D_{\mu_1} \cap L^2_{\text{real}}(B)$ into $L^2_{\text{real}}(B)$, it seems that the general criteria of [1] [2] do not apply, because such a condition is always implied by the general setup of the problem in these papers.

4. LOCAL PARTICLE NUMBER OPERATORS

We want to apply the basic philosophy used in [3]: if a representation of the CCR is given by a state ω and if local number operators from another representation can be defined on ω , then the two representations are isomorphic. This motivates our analysis of local particle number operators which now follows.

For a Borel set $\Delta \subseteq \mathbb{R}^d$ we define the local number operator $N(\Delta)$ on \mathcal{F} by

$$N(\Delta) = \int_{\Delta} dx A^*(x) A(x). \tag{4.1}$$

This is a selfadjoint operator on \mathcal{F} whose spectrum is $\{0, 1, 2, \dots\}$. For simplicity we restrict ourselves to the case where Δ is a d -dimensional hypercube of unit volume.

Let β be a Bogoliubov transformation and set

$$\omega_\beta(A) = \omega_0(\tau_\beta(A)) \quad \text{for} \quad A \in \mathcal{A}'_1(\mathbb{R}^d),$$

where $\omega_0(A) = (\Omega_0, A\Omega_0)$ and $\mathcal{A}'_1 = \mathcal{A}'_{\mu=1}$.

The GNS space associated with the algebra $\mathcal{A}'_1(\mathbb{R}^d)$ and the state ω_β is denoted by \mathcal{F}_β and its cyclic vector is denoted by Ω_β . By construction,

$$\begin{aligned} &(\Omega_\beta, \exp i(\varphi_1(f) + \pi_1(g))\Omega_\beta) \\ &= (\Omega_0, \exp i(\varphi_1((\beta_+ + \beta_-)f) + \pi_1((\beta_+ - \beta_-)g))\Omega_0), \end{aligned} \tag{4.2}$$

for all f and g in $\mathcal{D}(\beta_\pm)$.

We assume for convenience that the operator β_- has a kernel $\beta_-(x, y)$ which is a measurable function on $\mathbb{R}^d \times \mathbb{R}^d$. Then we have

THEOREM 4.1. — *If $\mathcal{D}(\beta_+) \cap L^2(\Delta)$ is dense in $L^2(\Delta)$ and*

$$\int_{\Delta} dx \int_{\mathbb{R}^d} dy |\beta_-(x, y)|^2 < \infty, \tag{4.3}$$

then there exists a positive, s. a. (local) number operator for the algebra $\Pi_{\mathcal{F}_\beta}(\mathcal{A}'_1(\Delta))''$ on \mathcal{F}_β (which is formally given by the operator $N(\Delta)$ defined in (4.1)).

COROLLARY 4.2. — *If, in addition, ω_β is a pure state then*

$$\Pi_{\mathcal{F}_\beta}(\mathcal{A}_1(\Delta))'' \cong \mathcal{A}_1(\Delta). \tag{4.4}$$

In particular, if (4.3) holds for $\Delta = \mathbb{R}^d$, then $\mathcal{F}_\beta = \mathcal{F}$,

$$\Pi_{\mathcal{F}_\beta}(\mathcal{A}_1(\mathbb{R}^d))'' = \mathcal{A}_1(\mathbb{R}^d),$$

and the isomorphism τ_β is unitarily implementable on \mathcal{F} (Shale's theorem).

Proof. — Let P_Δ be the orthogonal projection onto $L^2(\Delta)$. It follows from hypothesis (4.3) that $\beta_- P_\Delta$ is a bounded operator and

$$\|\beta_- P_\Delta\|^2 \leq \int_\Delta dx \int_{\mathbb{R}^d} dy |\beta_-(x, y)|^2 < \infty.$$

Let h be a normalized vector in $L^2(\Delta) \cap \mathcal{D}(\beta_+)$. By equations (4.2), (3.2),

$$\begin{aligned} \omega_\beta(A^*(h)A(h)) &= \omega_0(A_\beta^*(h)A_\beta(h)) \\ &= (\beta_- h, \beta_- h)_2 \leq \|\beta_- P_\Delta\|^2 \|h\|_2^2 < \infty. \end{aligned} \tag{4.5}$$

Since τ_β is a homomorphism, $A_\beta^*(h)A_\beta(h)$ is a number operator for one degree of freedom for the algebra spanned by $\{\exp i(\varphi_1((\beta_+ + \beta_-)h)), \exp i(\pi_1((\beta_+ - \beta_-)h))\}$.

Let $\{h_n\}_{n=0}^\infty$ be an arbitrary complete orthonormal system contained in $\mathcal{D}(\beta_+) \cap L^2(\Delta)$. Then, by (4.5),

$$\begin{aligned} \omega_\beta(N(\Delta)) &:= \lim_{M \rightarrow \infty} \omega_\beta\left(\sum_{n=0}^M A^*(h_n)A(h_n)\right) \\ &= \lim_{M \rightarrow \infty} \sum_{n=0}^M (\beta_- h_n, \beta_- h_n)_2 \end{aligned} \tag{4.6}$$

exists and is equal to

$$\int_\Delta dx \int_{\mathbb{R}^d} dy |\beta_-(x, y)|^2$$

and hence independent of the basis $\{h_n\}_{n=0}^\infty$. Therefore $\omega_\beta(N(\Delta))$ is defined for net convergence.

Let \mathcal{L} be the subspace of \mathcal{F}_β spanned by $\exp i(\varphi_1(f) + \pi_1(g))\Omega_\beta$, f and g in $\mathcal{D}(\beta_\pm)$. The subspace \mathcal{L} is dense in \mathcal{F}_β . We use the canonical commutation relations and equation (4.6) and conclude that for all θ in \mathcal{L} ,

$$\lim_{M \rightarrow \infty} \sum_{n=0}^M (A(h_n)\theta, A(h_n)\theta) = (\theta, N(\Delta)\theta)$$

exists and is independent of the basis $\{h_n\}_{n=0}^\infty$.

Therefore $N(\Delta)$ is a densely defined, positive quadratic form. Hence

Theorem 1 of Chaiken [2], applies and proves Theorem 4.1. Theorem 2 of [2] proves Corollary 4.2.

Q. E. D.

The relation (4.4) is only an isomorphism between so called *Newton-Wigner* (type I_∞) local algebras ($\mu = \mathbb{1}$). In this paper we are interested in proving that local algebras with *different*, non trivial *two point functions* are isomorphic. Theorem 4.1 does not apply any more. Nevertheless, the existence of local number operators $\{N(\Delta) \mid \Delta \subset \mathbb{R}^d\}$ is the starting point of the powerful isomorphism theorem of Glimm and Jaffe [3] for relativistic local algebras. Its application yields our main theorem.

Given μ_1 and μ_2 , we need the following conditions:

C1) μ_1 and μ_2 satisfy (2.6).

C2) μ_1 satisfies: let ξ be a C^∞ function which equals one on a hypercube Δ of unit volume and let $\Delta' \supset \text{supp } \xi$ be a cube. If Δ_a is a hypercube of unit volume centered at \mathcal{A} then for some $\alpha > 0$

$$\| \chi_{\Delta_a} \mu_1^{\pm 1/2} \xi \mu_2^{\pm 1/2} \|_{\text{H.S.}} \leq O(1) (e^{-\alpha \text{dist}(\Delta_a, \Delta)})$$

whenever $\Delta_a \cap \Delta' = \emptyset$.

C3) Set $\beta_\pm = \frac{1}{2} (\mu_2^{-1/2} \mu_1^{1/2} \pm \mu_2^{1/2} \mu_1^{-1/2})$. Then

$$\int_{\Delta_a} dx \int_{\mathbb{R}^d} dy |\beta_-(x, y)|^2 < C$$

uniformly in $\mathcal{A} \in \mathbb{R}^d$.

THEOREM 4.3. — *Suppose μ_1 and μ_2 satisfy C1, C2, C3. If the algebras $\mathcal{A}_{\mu_1}(\mathbf{B})$ and $\mathcal{A}_{\mu_2}(\mathbf{B})$ are factors, then they are isomorphic for all bounded open regions \mathbf{B} .*

Proof. — By definition, $\mathcal{A}_{\mu_i}(\mathbf{B}) = \pi_{\omega_0}(\mathring{\mathcal{A}}_{\mu_i}(\mathbf{B}))''$, $i = 1, 2$. Also, for β as in C3, $\pi_{\omega_\beta}(\mathring{\mathcal{A}}_{\mu_1}(\mathbf{B})) = \pi_{\omega_0}(\mathring{\mathcal{A}}_{\mu_2}(\mathbf{B}))$ and we have to show that

$$\pi_{\omega_0}(\mathring{\mathcal{A}}_{\mu_2}(\mathbf{B}))'' = \pi_{\omega_\beta}(\mathring{\mathcal{A}}_{\mu_1}(\mathbf{B}))'' \cong \pi_{\omega_0}(\mathring{\mathcal{A}}_{\mu_1}(\mathbf{B}))'' \tag{4.6}$$

We construct a sequence of states approximating ω_β . Cover \mathbb{R}^d with unit volume hypercubes. Let \mathbf{P}_M be the projection onto fewer than $M(a + 1)^{d+1}$ particles in each hypercube at a distance a of the origin whenever $a < e^M$ and onto zero particles elsewhere (cf. equ. (4.7), (4.44) in [3]). The states $\omega_M(\cdot) = \omega_\beta(\mathbf{P}_M \cdot \mathbf{P}_M)$ are normal on $\mathcal{A}_{\mu_1}(\mathbf{B})$. By C1 and C3 we can apply Theorem 4.1 to conclude that $\omega_\beta(N(\Delta_a))$ and hence $\omega_M(N(\Delta_a))$ is uniformly bounded in a and M . From C2 and [3], we first conclude that

$$\omega_\beta(A) = \lim_{M \rightarrow \infty} \omega_M(A)$$

for all $A \in \mathring{\mathcal{A}}_{\mu_1}(\mathbf{B})$, (equ. (4.14) and equ. (4.47) of [3]). In fact by Theorem 4.1 of [3], the above limit is a *norm limit* on $\mathcal{A}_{\mu_1}(\mathbf{B})$, (but not a norm limit, e. g.

in the dual of $\mathcal{A}_1(\mathbb{R}^d)$). Some modifications of the original proof [3] are necessary so that it will apply in our case and these modifications have been given by Rosen in Appendix B of the preprint version of [5]. Since $\omega_\beta = n - \lim_{M \rightarrow \infty} \omega_M$, on $\mathcal{A}_{\mu_1}(\mathbb{B})$, we conclude that ω_β is ultraweakly continuous on $\mathcal{A}_{\mu_1}(\mathbb{B})$ and this proves (4.6), i. e. $\tau_\beta : \mathcal{A}_{\mu_1}(\mathbb{B}) \rightarrow \mathcal{A}_{\mu_2}(\mathbb{B})$ extends to an isomorphism of factors (see e. g. [3]).

The following theorem deals with a situation which is typical for field theory applications.

THEOREM 4.4 [7]. — *Suppose*

- (1) $\mathcal{A}_{\mu_1}(\mathbb{B})$ is a separable factor for each \mathbb{B} .
- (2) If the closure of \mathbb{B} is contained in the interior of \mathbb{C} then $\mathcal{A}_{\mu_1}(\mathbb{B})' \cap \mathcal{A}_{\mu_1}(\mathbb{C})$ contains a factor of type I_∞ .

Then every isomorphism between $\mathcal{A}_{\mu_1}(\mathbb{B})$ and $\mathcal{A}_{\mu_2}(\mathbb{B})$ is unitarily implementable.

We note that the hypotheses of Theorem 4.4 follow from a mild regularity assumption on μ_1 :

LEMMA 4.5. — *Let $\mathbb{B} \subseteq \mathbb{R}^d$ be bounded open with piecewise smooth boundaries. Let $\{h_n\}$ and $\{h'_n\}$ be bases of $\mathcal{S}(\mathbb{B})$ and $\mathcal{S}(\sim \mathbb{B})$, respectively. Assume that $L_2(\mathbb{R}^d)$ is spanned by*

$$\{\mu^{+1/2}h_n\} \cup \{\mu^{+1/2}h'_n\} \quad \text{and by} \quad \{\mu^{-1/2}h_n\} \cup \{\mu^{-1/2}h'_n\}.$$

Then the hypotheses (1) and (2) of Theorem 4.4 hold.

The proof is easy (see [7] for similar results).

5. APPLICATION

We consider the case where μ_1 is the Fourier transform (F. T.) of multiplication by $(k \cdot k + m^2)^{1/2}$ on $L^2(\mathbb{R}^d, dk)$, $m > 0$ and μ_2 is the F. T. of multiplication by $|k|$. These μ 's are conventionally associated to free Bose fields of mass m and mass zero respectively.

THEOREM 5.1. — *With the above μ 's, in $d = 2$ or $d = 3$ space dimensions the factors $\mathcal{A}_{\mu_1}(\mathbb{B})$ and $\mathcal{A}_{\mu_2}(\mathbb{B})$ are unitarily equivalent for bounded open \mathbb{B} with piecewise smooth boundaries.*

Proof. — We first apply Theorem 4.3 to show isomorphism of $\mathcal{A}_{\mu_1}(\mathbb{B})$ and $\mathcal{A}_{\mu_2}(\mathbb{B})$. C1 is satisfied with $\mathbb{D} = \mathcal{S}(\mathbb{R}^d)_{\text{real}}$. C2 is shown e. g. in [5], equ. (6.4). We show that C3 holds. Since $\mu_i^{\pm 1/2}(x, y) = \mu_i^{\pm 1/2}(x - y)$, $i = 1, 2$, it is advantageous to go into the F. T. representation: β_- has a kernel whose F. T. is

$$\tilde{\beta}_-(k) = \frac{1}{2} \left(\frac{(k^2 + m^2)^{1/4}}{|k|^{1/2}} - \frac{|k|^{1/2}}{(k^2 + m^2)^{1/4}} \right).$$

Hence

$$0 \leq \tilde{\beta}_-(k) = \frac{1}{2} \frac{(k^2 + m^2)^{1/2} - |k|}{|k|^{1/2}(k^2 + m^2)^{1/4}} \leq 0(1)(|k| + 1)^{-3/2} \cdot |k|^{-1/2}$$

Therefore

$$\int_{\mathbb{R}^d} dx |\beta_-(x - y)|^2 = \int_{\mathbb{R}^d} dk |\tilde{\beta}_-(k)|^2 \leq 0(1) \int_{\mathbb{R}^d} dk (|k| + 1)^{-3} |k|^{-1} < \infty$$

if $d = 2$ or $d = 3$.

One can verify that Lemma 4.5 applies for $\mu = \mu_1$ in $d \geq 2$ dimensions (See [9] [10] for results of this type). Hence the von Neumann algebra $\mathcal{A}_{\mu_1}(\mathbf{B})$ is a factor [7]. It is easy to show that $\mathcal{A}_{\mu_1}(\mathbf{B})$ is separable [7]. Hypothesis (2) of Theorem 4.4 is obviously satisfied. Thus we have verified the hypotheses of Theorems 4.3 and 4.4.

Q. E. D.

THEOREM 5.2. — *If $\tilde{\mu}_i(k) = (k^2 + m_i^2)^{1/2}$, $m_i > 0$, $i = 1, 2$, then the factors $\mathcal{A}_{\mu_1}(\mathbf{B})$ and $\mathcal{A}_{\mu_2}(\mathbf{B})$ are unitarily equivalent for \mathbf{B} as in Theorem 5.1 and $d = 1, 2, 3$.*

Proof. — In $d = 2, 3$ dimensions the proof is identical to the one of Theorem 5.1 except for the verification of C3. Condition C3 follows from the inequality

$$\int_{\mathbb{R}^d} dx |\beta_-(x - y)|^2 \leq 0(1) \int_{\mathbb{R}^d} dk (|k| + 1)^{-4},$$

and the R. H. S. is finite for $d = 1, 2, 3$.

In $d = 1$ dimension conditions C1, C2, C3 are verified as before. However, we cannot apply lemma 4.5 to verify that $\mathcal{A}_{\mu_1}(\mathbf{B})$ is a factor. But Glimm and Jaffe [3] [7] (see also [5]) have shown that $\mathcal{A}_{\mu_1}(\mathbf{B})$ is a separable factor. Again, hypothesis (2) of Theorem 4.4 is obviously true.

Q. E. D.

Remark. — Theorem 5.2 has been shown earlier in [5].

COROLLARY. — Let s be a positive number and let $\mathbf{B}_s = \left\{ x \mid \frac{1}{3}x \in \mathbf{B} \right\}$.

Let μ be the Fourier transform of $(k^2 + m^2)^{1/2}$, $m > 0$ and $d = 1, 2, 3$. Then $\mathcal{A}_{\mu}(\mathbf{B})$ and $\mathcal{A}_{\mu}(\mathbf{B}_s)$ are unitarily equivalent.

Proof. — This follows from theorem 5.2 by a dilation argument. We thank Prof. M. Guenin for calling our attention to this problem.

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