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Section A :
Physique théorique.

Existence of a lower bound of the compressibility in quantum lattice gases

by

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ABSTRACT. — The existence of a positive lower bound of the compressibility is proved in the case of general quantum lattice systems at high temperature.

1. The compressibility has been shown to be strictly positive for certain classical [*1*], [*4*] and quantum systems [*2*] of interacting particles in equilibrium. This implies that the pressure is a continuous function of the density, as it is expected in an empirical basis. In the general case of quantum lattice systems the continuity of the pressure has been proved in [*3*]. By an argument based on the Tomita-Takesaki Theory of K. M. S. states, but the possibility of the existence of points of vanishing compressibility has not been ruled out by this method. Here we will also be concerned with general lattice systems, but our argument will follow as in [*2*] a method inspired from the classical case. With it we are able to find a positive lower bound on the compressibility. However the application of the method appears to be restricted to interactions verifying the condition $\| \Phi \|_\alpha < 1$ which will be introduced in the following. We can say, because the temperature appears as a multiplicative constant $\frac{1}{kT}\Phi$, that the result remains valid only at high temperature (Although in a large range that the one in which

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the analyticity of the pressure has been proved). In order to get a result valid for every T, some new techniques, restricted to less general systems, as the Wiener integral, have to be introduced (see [2]).

2. Let Λ be a finite subset of a ν-dimensional lattice $\mathbb{Z}^\nu$ and denote by $|\Lambda|$ the number of points of Λ.

The Hilbert space $\mathcal{H}_\Lambda$ of the states of the system inside the region Λ is a $2^{|\Lambda|}$-dimensional Hilbert space. We choose in $\mathcal{H}_\Lambda$ an orthonormal basis whose elements $|X\rangle$ are indexed by the subsets of Λ. If $X \subset \Lambda_1 \subset \Lambda_2$ we identify the element $|X\rangle$ of $\mathcal{H}_{\Lambda_1}$ and $\mathcal{H}_{\Lambda_2}$ and we get $\mathcal{H}_{\Lambda_1} \subset \mathcal{H}_{\Lambda_2}$. A number operator $N = N^*$ is defined through $N|X\rangle = |X|\,|X\rangle$.

The interaction potentiel $\Phi(.)$ is a function applying every finite subset $X \neq \emptyset$ of $\mathbb{Z}^\nu$ into a self-adjoint operator in $\mathcal{H}_X$. Then the Hamiltonian of the system in Λ is the following operator

$$H_\Lambda = H_\Lambda^* = \sum_{X \subset \Lambda} \Phi(X)$$

Q being an operator on $\mathcal{H}_\Lambda$ the equilibrium expectation value of Q is given by

$$\langle Q \rangle = \mathrm{Tr}\,(e^{-H_\Lambda} Q).\ \mathrm{Tr}\,(e^{-H_\Lambda})^{-1}$$

We assume that the interaction $\Phi(X)$ for all $X \subset \Lambda$ commutes with the number operator N, and that it verifies

$$\|\Phi\|_\alpha = \frac{1}{\alpha} \sum_{\substack{n=0 \\ |X|=n}}^{\infty} \sup_{x \in X} \|\Phi(X)\| e^{\alpha(n-1)} < +\infty,$$

for some $\alpha > 0$.

THEOREM. — *Suppose that* $\|\Phi\|_\alpha < 1, \alpha > 0$, *then the following inequality is verified*

$$\frac{\langle (N - \langle N \rangle)^2 \rangle}{\langle N \rangle} \geqslant \frac{\rho}{\rho + A}$$

where A *is a certain positive constant independent of* Λ *and* $\rho = \dfrac{\langle N \rangle}{|\Lambda|}$ *the density.*

Remark. — The right hand side of the inequality is the expression of the compressibility for the system in a region Λ, or in more precise terms, the compressibility χ is as follows:

$$\chi = \frac{1}{\rho} \frac{d\rho}{dP}, \qquad \beta^{-1} \frac{d\rho}{dP} = \frac{(N - \langle N \rangle)^2}{\langle N \rangle}$$

Since A is independent of Λ, a rigorous lower bound on the compressibility follows from the theorem.

In order to prove the theorem we will adapt some methods used in the classical case.

First we introduce the creation and annihilation operators by means of

$$\begin{array}{lll} a^*(x)\,|\,X\rangle = 0 & a(x)\,|\,X\rangle = |\,X/\{x\}\rangle & \text{if } x \in X \\ a^*(x)\,|\,X\rangle = |\,X \cup \{x\}\rangle & a(x)\,|\,X\rangle = 0 & \text{if } x \notin X \end{array}$$

the number operator can then be written as

$$N = \sum_{x\in\Lambda} a^*(x)a(x)$$

We introduce also an auxiliary operator R by

$$R = R^* = e^{\frac{1}{2}H}\Big(\sum_{x\in\Lambda} a(x)e^{-H}a^*(x)\Big)e^{\frac{1}{2}H}$$

for which the following lemma can be proved.

LEMMA. — *If* $|\langle R^2 - NR \rangle| \leqslant A\,|\,\Lambda\,|$ *then,*

$$\frac{\langle (N - \langle N \rangle)^2 \rangle}{\langle N \rangle} \geqslant \frac{\rho}{\rho + A}$$

Proof of the lemma. — The lemma follows from the Schwartz inequality in the form

$$|\langle (N - \langle N \rangle)(N - R) \rangle|^2 \leqslant \langle (N - \langle N \rangle)^2 \rangle \langle (N - R)^2 \rangle$$

It suffices to verify the following expressions

$$\langle R \rangle = \langle N \rangle \geqslant 0$$
$$\langle NR \rangle = \langle RN \rangle = \langle N(N-1) \rangle$$

since then

$$\langle (N - \langle N \rangle)(N - R) \rangle = \langle N \rangle$$
$$\langle (N - R)^2 \rangle = \langle N \rangle + \langle R^2 - NR \rangle \geqslant 0$$

and finally

$$\frac{\langle (N - \langle N \rangle)^2 \rangle}{\langle N \rangle} \geqslant \frac{\langle N \rangle}{\langle N \rangle + \langle R^2 - NR \rangle} \geqslant \frac{\rho}{\rho + A}$$

Proof of the theorem. — Using this lemma, in order to prove the theorem, it suffices to prove that

$$|\langle R^2 - NR \rangle| \leqslant A\,|\,\Lambda\,|$$

where by computation

$$\langle R^2 - NR \rangle = \sum_{x\in\Lambda}\sum_{y\in\Lambda} \langle [e^{-\frac{1}{2}H}a^*(x)e^{\frac{1}{2}H}, e^{\frac{1}{2}H}a(y)e^{-\frac{1}{2}H}]e^{-\frac{1}{2}H}a^*(y)a(x)e^{\frac{1}{2}H} \rangle$$

We use the formulae

$$e^{-\frac{1}{2}H}Qe^{\frac{1}{2}H} = \sum_{n=0}^{\infty} \frac{1}{(-2)^n n!}[H, Q]^{(n)}$$

where

$$[H, Q]^{(0)} = Q, \qquad [H, Q]^{(n)} = [H, [H, Q]^{(n-1)}]$$

and therefore

$$[H, Q]^{(n)} = \sum_{X_1 \subset \Lambda} \dots \sum_{X_n \subset \Lambda} [\Phi(X_n), [\dots, [\Phi(X_1), Q] \dots] \tag{1}$$

We write

$$\sum_{X_i} = \sum_{K_i=0}^{\infty} \sum_{X_i, |X_i| = K_i + 1}$$

and we remark that we may restrict the summations in (1) by the conditions (defining also S_i):

$$X_i \cap S_i \neq \emptyset$$
$$S_1 = \Lambda_1 \qquad \text{if} \quad Q \text{ is an operator on } \mathscr{H}_{\Lambda_1}$$
$$S_{i+1} = X_i \cup S_i$$

Notice that S_i contains at most $|\Lambda_1| + K_1 + \dots + K_{i-1}$ elements, therefore

$$\frac{1}{n!} \| [H, Q]^{(n)} \|$$
$$\leqslant \frac{2^n \| Q \|}{n!} \sum_{K_1 \dots K_n} \prod_{i=1}^{n} \left((|\Lambda_1| + K_1 + \dots + K_{i-1}) \sup_x \sum_{\substack{X_i \ni x \\ |X_i| = K_i + 1}} \| \Phi(X_i) \| \right)$$
$$\leqslant 2^n \| Q \| \sum_{K_1 \dots K_n} \exp(|\Lambda_1| + K_1 + \dots + K_n) \prod_{i=1}^{n} \sup_x \sum_{\substack{X_i \ni x \\ |X_i| = K_i + 1}} \| \Phi(X_i) \|$$
$$\leqslant \| Q \| e^{|\Lambda_1|} (2 \| \Phi \|_\alpha)^n$$

with

$$\| \Phi \|_\alpha = \sum_{K=1}^{\infty} \sup_x \sum_{\substack{X \ni x \\ |X| = K}} \| \Phi(X) \| e^{K-1}$$

Let us suppose $\| \Phi \|_\alpha < 1$, then

$$\| e^{-\frac{1}{2}H} a^*(y) a(x) e^{\frac{1}{2}H} \| \leqslant \sum_{n=0}^{\infty} \| a^*(y) a(x) \| e^2 \| \Phi \|_\alpha^n = \frac{e^2}{1 - \| \Phi \|_\alpha}$$

On the other hand, if we developp both terms in the commutator

$$[e^{-\frac{1}{2}H}a^*(x)e^{\frac{1}{2}H}, e^{\frac{1}{2}H}a(y)e^{-\frac{1}{2}H}] = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(-\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^m \sum_{K_1=0}^{\infty}\sum_{\substack{X_1\subset\Lambda\\|X_1|=K_1+1}} \cdots \sum_{K_n=0}^{\infty}\sum_{\substack{X_n\subset\Lambda\\|X_n|=K_n+1}}\sum_{K'_1=0}^{\infty}\sum_{\substack{Y_1\subset\Lambda\\|Y_1|=K'_1+1}}\cdots\sum_{K'_m=0}^{\infty}\sum_{\substack{Y_m\subset\Lambda\\|Y_m|=K'_m+1}}$$

$$[[\Phi(X_1), \ldots [\Phi(X_n), a^*(x)], \ldots], [\Phi(Y_1), \ldots [\Phi(Y_m), a(y)], \ldots]]$$

We may restrict the summations by

$$X_i \cap S_i \neq \emptyset, \qquad S_1 = \{x\}, \qquad S_{i+1} = X_i \cup S_i$$
$$Y_i \cap S'_i \neq \emptyset, \qquad S'_1 = \{y\}, \qquad S'_{i+1} = Y_i \cup S'_i$$
$$S_{n+1} \cap S'_{m+1} \neq \emptyset$$

We see that there are in each term at most $(1 + K_1 + \ldots + K_n)$ possibilities for points x to give non zero terms, coupled to $(1 + K'_1 + \ldots + K'_m)$ possibilities for y. But we have also

$$\frac{1}{n!}\prod_{i=1}^{n}(1 + K_1 + \ldots + K_{i-1})(1 + K_1 + \ldots + K_n) \leqslant e\prod_{i=1}^{n}e^{K_i}$$

and we can similarly bound, provided that $||\Phi||_\alpha < 1$,

$$\sum_{x\in\Lambda}\sum_{y\in\Lambda} ||[e^{-\frac{1}{2}H}a^*(x)e^{\frac{1}{2}H}, e^{\frac{1}{2}H}a(y)e^{-\frac{1}{2}H}]|| \leqslant \sum_{x\in\Lambda}\frac{e^2}{(1-||\Phi||_\alpha)^2} = |\Lambda|\frac{e^2}{(1-||\Phi||_\alpha)^2}$$

We conclude then at the inequality required in the hypothesis of the lemma with

$$A = \frac{e^4}{(1-||\Phi||_\alpha)^3}$$

and the theorem is proved.

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