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A note on quantising Kolmogorov systems

by

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ABSTRACT. — It is shown that there exists a quantisation of classical systems of a certain mathematical structure. The method gives the usual commutation relations for the electro-magnetic field.

RÉSUMÉ. — Il existe une quantification des systèmes classiques d'une certaine structure mathématique. La méthode donne les relations de commutations ordinaires pour le champ électro-magnétique.

One method of quantising a classical system is to construct a Weyl system on it. A symplectic space (S, σ) is obtained from the classical system and a Weyl map W is defined from S to unitary operators on a Hilbert space \mathcal{H} , such that

$$W(s_1)W(s_2) = W(s_1 + s_2) \exp\left(-\frac{1}{2}\sigma(s_1, s_2)\right). \quad (1)$$

and $R(s)$, where $W(s) = e^{iR(s)}$, exists. The dynamics of the classical system is defined by a 1-parameter symplectic group $U(t)$ on S . If $(S, \sigma, U(t))$ is Hilbertisable-i. e., S can be represented as a complex Hilbert space, σ the imaginary part of the inner product and $U(t)$, a unitary group with positive generator, then Weinless [6] showed there exists a quantisation of the system with a unique vacuum state. This is the case if the classical system is a real Kolmogorov system.

A real Kolmogorov system is a triple $\{H, U(t), D\}$ of a real Hilbert space H , $U(t) = e^{itC}$, a 1-parameter group of orthogonal operators on H , and D a subspace of H , with $U(t)D \subset D$, $t \geq 0$, $\bigcap_t U(t)D = \{0\}$,

$\bigcup_t U(t)D = H$. Sinai [5] proved a representation theorem for such systems.

THEOREM. — For any K-system $\{H, U(t), D\}$ there exists an isometric isomorphism $B : H \rightarrow L_2(-\infty, \infty; N)$, unique up to isomorphism of the real Hilbert space N , so that

- (i) B maps $U(t)$ onto translation by t to the right.
- (ii) $BD = L_2(0, \infty; N)$.

It is a simple corollary of this theorem that such systems are Hilbertizable symplectic spaces—namely H is isomorphic to the complex Hilbert space $L_2(0, \infty; N + iN)$ and $U(t)$ has a generator with positive spectrum. The equivalence is given by the map $T : L \rightarrow \sqrt{2}P_+ \mathcal{F} B h$, where \mathcal{F} is the Fourier transform and P_+ is the projection operator from $L_2(-\infty, \infty; N + iN)$ onto $L_2(0, \infty; N + iN)$. σ is chosen to be the bilinear form that becomes the imaginary part of the inner product under T . If W is the Weyl map over $L_2(0, \infty; N + iN)$, the classical element $h \in H$ is identified with the quantum observable $R(C^\sharp Th)$. Since C^\sharp is an unbounded operator it is necessary to restrict W to the subspace of H for which

$$\int_0^\infty \langle (Th)(x), (Th)(x) \rangle x dx < \infty \quad (2)$$

—the closure of the domain of C^\sharp .

Such K-systems occur in classical scattering theory [2] and stationary stochastic processes [1, 3]. An example of such a K-system is the free electro-magnetic field, as was outlined in [2] and [4]. The classical field is a solution of

$$\frac{\partial \Omega}{\partial t}(\underline{x}, t) = A(\underline{x})\Omega(\underline{x}, t) = \begin{pmatrix} 0 & \text{curl} \\ -\text{curl} & 0 \end{pmatrix} \Omega(\underline{x}, t) \quad (3)$$

where $\Omega^T(\underline{x}, t) = (E(\underline{x}, t), H(\underline{x}, t))$ and $\text{div } E = \text{div } H = 0$. Take H to be the space of initial data of (3) with finite energy. This is a Hilbert space with $\|\Omega\|^2 = \int |\Omega(\underline{x}, 0)|^2 d\underline{x}$, and D is the space of data which are zero at $x = 0$ for all positive time. The K-system representation of this space as $L_2(-\infty, \infty; L(S_2) \oplus L(S_2))$ where $L(S_2)$ is the space of bounded operators on the sphere, is given in [2] and [4]. If f'_i is an element of the initial data space so that at time t , $\Omega_j(\underline{x}, t) = \delta_{ij} f'_i(x)$, then the quantum field operators are given by $E_i(f, t) = R(C^\sharp T f'_i)$, and $H_j(g, t) = R(C^\sharp T g'_{j+3})$. The commutation relations of the quantum system are given by the Weyl representation

$$\begin{aligned} [E_i(f, t), H_j(g, t')] &= [R(C^\sharp T f'_i), R(C^\sharp T g'_{j+3})] \\ &= i \text{Im} \langle C^\sharp T f'_i, C^\sharp T g'_{j+3} \rangle. \end{aligned} \quad (4)$$

These equations give the commutation relations normally associated with the electro-magnetic field. This method works for all K-systems, and in particular one can calculate the quantisation of the e. m. f. outside a perfect conductor.

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